Discrete-time systems: discretization, models and their properties

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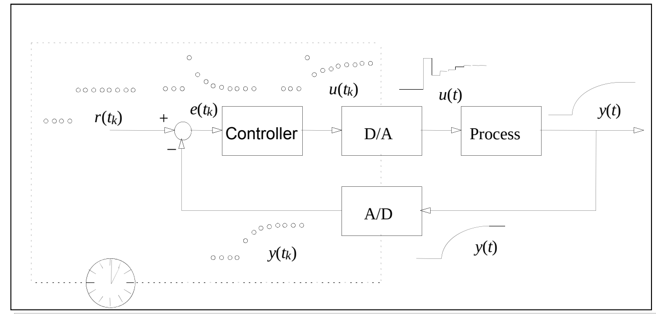
Process controlled by a digital controller



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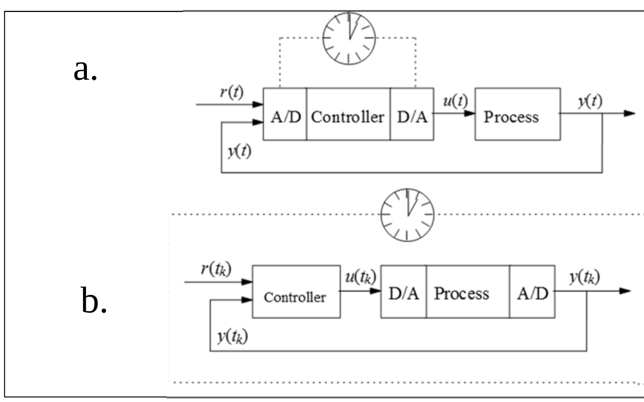
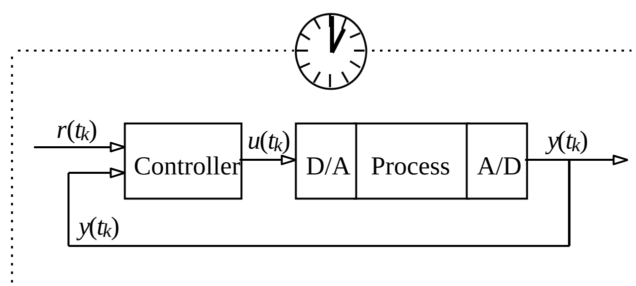
Two main design approaches: a. discretize the analog controller, b. discretize the process and do the design totally in discrete time



Let us consider the design approach b: A discrete system from the controller viewpoint



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Sampling of continuous-time signals

A continuous function *f*(*t*) $f(t) = 2e^t$

Set of integers *Z* $Z = \{\ldots, -1, 0, 1, \ldots\}$

Sampling instants $\{ t_k : k \in Z \}$ $\{0,1,2,3,\ldots\}$

Sequence $\{ f(t_k) : k \in Z \}$ $\{2, 2e, 2e^2, 2e^3, \ldots\}$

$\approx \{2.00, 5.44, 14.78, 40.17, \ldots\}$

Sampling of continuous-time signals

Periodic sampling

Sampling interval, *h* $t_k = k \cdot h$

Sampling frequency, f_s $f_s = 1/h \quad \text{(Hz)}$

Sampling angular frequency ω_s $\omega_s = 2\pi / h = 2\pi \cdot f_s \quad \text{(rad / s)}$

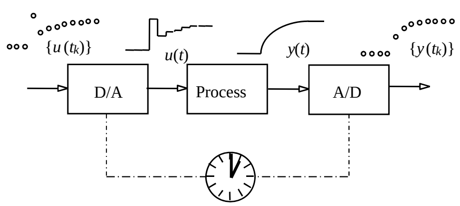
Nyquist frequency, f_N $f_N = 1/(2h) \quad \text{(Hz)}$

ω_N $\omega_N = 2\pi / (2h) = \pi / h = 2\pi \cdot f_N \quad \text{(rad/s)}$

Zero-order-hold ZOH and sampling

A continuous process is described by a linear state-space-representation.

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \qquad \begin{aligned} \dim\{\mathbf{u}\} &= r \\ \dim\{\mathbf{x}\} &= n \\ \dim\{\mathbf{y}\} &= p \end{aligned}$$



The matrix exponential

Let *A* be a square matrix, define

$$e^{At} = I + tA + \frac{1}{2}t^2A^2 + \frac{1}{6}t^3A^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i!} t^i A^i$$

which is always convergent.

From the definition $\dfrac{de^{At}}{dt} = Ae^{At} = e^{At}A$

The matrix exponential

The solution to the homogenous set of differential equations

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

is then $x(t) = e^{At}x(0) = e^{At}x_0$

The term e^{At} is called the *state transition matrix* (in this context).

The matrix exponential

Solution by using the Laplace transformation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \qquad sX(s) - x_0 = AX(s)$$

It follows

$$(sI - A)X(s) = x_0$$
$$(sI - A)^{-1}(sI - A)X(s) = (sI - A)^{-1}x_0$$
$$X(s) = (sI - A)^{-1}x_0$$

The solution is obtained by the inverse transformation

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The matrix exponential

By comparing to the previous solution it follows that

$$e^{At} = L^{-1}\left[(sI - A)^{-1}\right]$$

which is one way to solve the state-transition matrix.

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Solution of the state equation

Solution for the input-output representation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$
$$y(t) = Cx(t)$$

is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau$$

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Solution of the state equation

in which e^{At} is the above state transition matrix.

To prove the solution check the initial condition and differentiate the solution to see that the original differential equation holds.

Zero- order- hold ZOH and sampling

Solution at any time t after the sampling instant t_k

$$\mathbf{x}(t) = \mathbf{e}^{\mathbf{A}(t-t_k)}\mathbf{x}(t_k) + \int_{t_k}^{t} \mathbf{e}^{\mathbf{A}(t-s')}\mathbf{B}\mathbf{u}(s')ds'$$

$\mathbf{u}(t)$ is constant between sampling instants, ZOH

$$\mathbf{x}(t) = \mathbf{e}^{\mathbf{A}(t-t_k)}\mathbf{x}(t_k) + \int_{t_k}^{t} \mathbf{e}^{\mathbf{A}(t-s')}ds'\mathbf{B}\mathbf{u}(t_k)$$

Change the integration variable $s' = t - s$

$$\mathbf{x}(t) = \left(\mathbf{e}^{\mathbf{A}(t-t_k)}\right)\mathbf{x}(t_k) + \left(\int_{0}^{t-t_k} \mathbf{e}^{\mathbf{A}s}ds\,\mathbf{B}\right)\mathbf{u}(t_k)$$

A state transition matrix Φ and control matrix Γ are obtained (independent of **x** and **u**).

$$\mathbf{x}(t) = \mathbf{\Phi}(t-t_k)\,\mathbf{x}(t_k) + \mathbf{\Gamma}(t-t_k)\,\mathbf{u}(t_k)$$

Zero- order- hold ZOH and sampling

At the next sampling instant

$$t = t_{k+1} \quad \Rightarrow \quad \begin{cases} \mathbf{x}(t_{k+1}) = \mathbf{\Phi}(t_{k+1}-t_k)\,\mathbf{x}(t_k) + \mathbf{\Gamma}(t_{k+1}-t_k)\,\mathbf{u}(t_k) \\ \mathbf{y}(t_k) = \mathbf{C}\mathbf{x}(t_k) + \mathbf{D}\mathbf{u}(t_k) \end{cases}$$

$$\begin{cases} \mathbf{\Phi}(t_{k+1}-t_k) = \mathbf{e}^{\mathbf{A}(t_{k+1}-t_k)} \\ \mathbf{\Gamma}(t_{k+1}-t_k) = \int_{0}^{t_{k+1}-t_k} \mathbf{e}^{\mathbf{A}s}ds\,\mathbf{B} \end{cases}$$

Zero- order- hold ZOH and sampling

By a periodic sampling the equations become

$$t_k = k \cdot h \quad \Rightarrow t_{k+1} - t_k = h \quad \text{(constant)}$$

$$\begin{cases} \mathbf{x}(kh+h) = \mathbf{\Phi}(h)\,\mathbf{x}(kh) + \mathbf{\Gamma}(h)\,\mathbf{u}(kh) \\ \mathbf{y}(kh) = \mathbf{C}\mathbf{x}(kh) + \mathbf{D}\mathbf{u}(kh) \end{cases} \qquad \begin{cases} \mathbf{\Phi}(h) = \mathbf{e}^{\mathbf{A}h} \\ \mathbf{\Gamma}(h) = \int_{0}^{h} \mathbf{e}^{\mathbf{A}s}ds\,\mathbf{B} \end{cases}$$

Usually this is written in the form (h is constant)

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{\Phi}\,\mathbf{x}(k) + \mathbf{\Gamma}\,\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \end{cases}$$

How to solve for Φ and Γ ?

- Symbolically
  Eg. by using the Laplace transformation
  By symbolic programs (Maple, Mathematica, …)
- Numerically
  Eg. by the series expansion of the matrix exponential function
  By numeric software (Matlab)

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Example. Discretization by direct calculus

Sampling interval h = 0.1

$$\begin{cases} \dot{x}(t) = 2x(t) + u(t) \\ y(t) = 3x(t) \end{cases}$$ is of the form $$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

$$\begin{cases} \Phi = e^{Ah} = e^{2\cdot 0.1} = e^{0.2} \\ \Gamma = \int_0^h e^{As} ds\, B = \int_0^{0.1} e^{2s} ds = \frac{1}{2} \Big/_0^{0.1} e^{2s} = \frac{1}{2}\left(e^{2\cdot 0.1} - e^0\right) = \frac{1}{2}\left(e^{0.2} - 1\right) \end{cases}$$

$$\begin{cases} x(kh+h) = \Phi\, x(kh) + \Gamma\, u(kh) \\ y(kh) = Cx(kh) \end{cases} \quad \begin{cases} x(kh+h) = e^{0.2}\ x(kh) + 0.5(e^{0.2} - 1)\, u(kh) \\ y(kh) = 3x(kh) \end{cases}$$

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Example. Discretization by using the series expansion

State-space representation of the double integrator

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

$$\boxed{\mathbf{\Phi} = \mathbf{e}^{\mathbf{A}h} = \mathbf{I} + \mathbf{A}h + \frac{\mathbf{A}^2 h^2}{2} + \cdots}$$

$$\mathbf{\Phi} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \cdots = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{\Gamma} = \int_0^h \mathbf{e}^{\mathbf{A}s} ds\, \mathbf{B} = \int_0^h \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} ds \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \int_0^h \begin{bmatrix} s \\ 1 \end{bmatrix} ds = \Big/_0^h \begin{bmatrix} \frac{1}{2}s^2 \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{2}h^2 \\ h \end{bmatrix}$$

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Example. Discretization by the series expansion

The corresponding discrete-time model becomes

$$\begin{cases} \mathbf{x}(kh+h) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \mathbf{x}(kh) + \begin{bmatrix} \frac{1}{2}h^2 \\ h \end{bmatrix} u(kh) \\ y(kh) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(kh) \end{cases}$$

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Example. Discretization by using the Laplace transformation

State-space representation of the *DC* motor

$$\begin{cases}\dot{\mathbf{x}}(t)=\begin{bmatrix}-1&0\\1&0\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}1\\0\end{bmatrix}u(t)\\ y(t)=\begin{bmatrix}0&1\end{bmatrix}\mathbf{x}(t)\end{cases}\qquad \mathbf{\Phi} = \mathbf{e}^{\mathbf{A}h} = \mathbf{e}^{\mathbf{A}t}\big|_{t=h} = L^{-1}\{(s\mathbf{I}-\mathbf{A})^{-1}\}\big|_{t=h}$$

$$\mathbf{\Phi} = L^{-1}\left\{\begin{bmatrix}s+1&0\\-1&s\end{bmatrix}^{-1}\right\}\Bigg|_{t=h} = L^{-1}\left\{\frac{1}{s(s+1)}\begin{bmatrix}s&0\\1&s+1\end{bmatrix}\right\}\Bigg|_{t=h}$$

$$= L^{-1}\left\{\begin{bmatrix}\frac{1}{s+1}&0\\ \frac{1}{s(s+1)}&\frac{1}{s}\end{bmatrix}\right\}\Bigg|_{t=h} = \begin{bmatrix}e^{-t}&0\\1-e^{-t}&1\end{bmatrix}\Bigg|_{t=h} = \begin{bmatrix}e^{-h}&0\\1-e^{-h}&1\end{bmatrix}$$

Example. Discretization by the Laplace ….

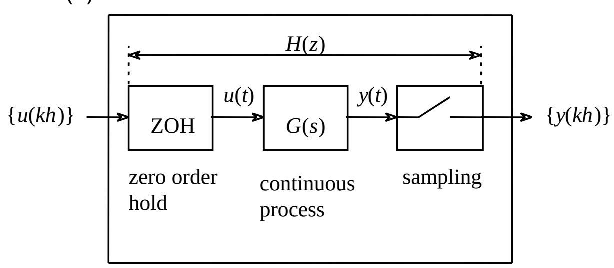
$$\mathbf{\Gamma} = \int_0^h \mathbf{e}^{\mathbf{A}s}ds\,\mathbf{B} = \int_0^h\begin{bmatrix}e^{-s}&0\\1-e^{-s}&1\end{bmatrix}ds\begin{bmatrix}1\\0\end{bmatrix} = \int_0^h\begin{bmatrix}e^{-s}\\1-e^{-s}\end{bmatrix}ds = \begin{bmatrix}1-e^{-h}\\h-1+e^{-h}\end{bmatrix}$$

The discrete-time model is obtained

$$\begin{cases}\mathbf{x}(kh+h)=\begin{bmatrix}e^{-h}&0\\1-e^{-h}&1\end{bmatrix}\mathbf{x}(kh)+\begin{bmatrix}1-e^{-h}\\h-1+e^{-h}\end{bmatrix}u(kh)\\ y(kh)=\begin{bmatrix}1&0\end{bmatrix}\mathbf{x}(kh)\end{cases}$$

Pulse transfer function, $H(z)$

A pulse transfer function $H(z)$ can also be calculated directly from $G(s)$.



Pulse transfer function, $H(z)$

After ZOH $u(t)$ consists of step functions. The output $y(t)$ is a set of step responses, which are sampled at constant intervals h.

Discrete pulse transfer function and continuous transfer function correspond to each other, if the outputs are equal at the sampling instants.
The step response of the continuous system at the sampling instants ($t = kh$)

$$y(kh) = y(t)\big|_{t=kh} = L^{-1}\{Y(s)\}\big|_{t=kh} = L^{-1}\{G(s)U(s)\}\big|_{t=kh}$$
$$= L^{-1}\left\{G(s)\frac{1}{s}\right\}\Bigg|_{t=kh}$$

Pulse transfer function, $H(z)$

The step response of the discrete system

$$y(kh) = Z^{-1}\{Y(z)\} = Z^{-1}\{H(z)U(z)\} = Z^{-1}\left\{H(z)\frac{z}{z-1}\right\}$$

Make these equal; the final result follows

$$L^{-1}\left\{G(s)\frac{1}{s}\right\}\Big|_{t=kh} = Z^{-1}\left\{H(z)\frac{z}{z-1}\right\}$$

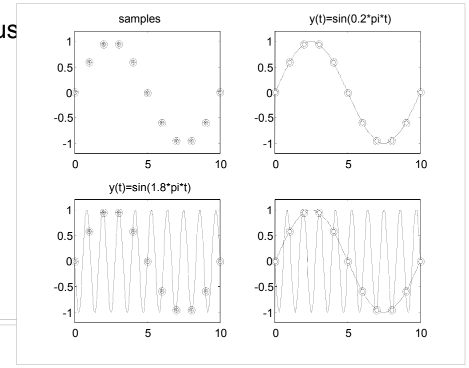
$$H(z) = \frac{z-1}{z}\cdot Z\left\{L^{-1}\left\{G(s)\frac{1}{s}\right\}\Big|_{t=kh}\right\}$$

The alias effect

Two different continuous signals can be fitted to the pulse train (h = 1)

$y_1(t) = \sin(0.2\pi\, t)$
$y_2(t) = \sin(1.8\pi\, t)$

Actually, an indefinite number of continuous signals can be fitted.

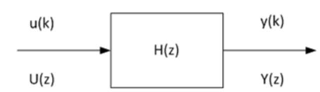


So, if we have a discrete pulse sequence only, how can we know what frequency the real continuous time signal has?

Moreover: the continous time model corresponding to a discrete model is not always unique.
(Take the harmonic oscillator $G(s) = \omega^2/(s^2+\omega^2)$,
do a realization and discretize by using $\omega = \alpha + n2\pi/h$, n = 0,1,2,...)

These examples show the **alias effect** for the first time. It is very dangerous. We will return to this later in the course.

Linear time-invariant systems: The pulse transfer function, weighting function, pulse response and convolution sum



$$Y(z) = H(z)U(z)$$ Input-output: Z-domain

$$y(k) = \sum_{i=0}^{k} h(k-i)u(i) = \sum_{i=0}^{k} h(i)u(k-i)$$

Input-output: time domain, convolution sum

The z-transform of the *weighting function h(k)* is the *pulse transfer function H(z).*
The (im)*pulse response* coincides with the weighting function from the time that the pulse enters.

Input-Output-models, I/O-models, pulse response

Discrete system: $\begin{cases} \mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma\mathbf{u}(k), \text{ initial value } x(k_0) \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \end{cases}$

Solution by direct recursive calculus starting from a given initial state $\mathbf{x}(k_0)$ up to any time k.

$$
\begin{aligned}
\mathbf{x}(k_0+1) &= \Phi\mathbf{x}(k_0) + \Gamma\mathbf{u}(k_0) \\
\mathbf{x}(k_0+2) &= \Phi\mathbf{x}(k_0+1) + \Gamma\mathbf{u}(k_0+1) \\
&= \Phi^2\mathbf{x}(k_0) + \Phi\Gamma\mathbf{u}(k_0) + \Gamma\mathbf{u}(k_0+1) \\
&\vdots \\
\mathbf{x}(k) &= \Phi^{k-k_0}\mathbf{x}(k_0) + \Phi^{k-k_0-1}\Gamma\mathbf{u}(k_0) + \cdots + \Gamma\mathbf{u}(k-1) \\
&= \Phi^{k-k_0}\mathbf{x}(k_0) + \sum_{j=k_0}^{k-1}\Phi^{k-j-1}\Gamma\mathbf{u}(j)
\end{aligned}
$$

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Input-Output-models, I/O-models, pulse response

But for a state representation the pulse response is calculated as

$$\mathbf{x}(k) = \Phi^{k-k_0}\mathbf{x}(k_0) + \sum_{j=k_0}^{k-1}\Phi^{k-j-1}\Gamma u(j) \qquad y(k) = \mathbf{C}\mathbf{x}(k) + Du(k)$$

$$y(k) = \mathbf{C}\Phi^{k-k_0}\mathbf{x}(k_0) + \sum_{j=k_0}^{k-1}\mathbf{C}\Phi^{k-j-1}\Gamma u(j) + Du(k)$$

$$h(k) = \begin{cases} 0 & k < 0 \\ D & k = 0 \\ \mathbf{C}\Phi^{k-1}\Gamma & k \ge 1 \end{cases}$$

pulse response (assume $k_0 = 0$)

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Shift-operators

Corresponding to the differential operator p used in continuous systems a (forward) shift operator q is defined for discrete systems

$$q \cdot f(k) = f(k+1)$$
$$q^{-1} \cdot f(k) = f(k-1)$$

By using the shift operator input-output relationships (difference equations) can easily be described

$$y(k+n_a) + a_1 y(k+n_a-1) + \cdots + a_{n_a} y(k) = b_0 u(k+n_b) + \cdots + b_{n_b} u(k)$$
$$(q^{n_a} + a_1 q^{n_a-1} + \cdots + a_{n_a}) y(k) = (b_0 q^{n_b} + b_1 q^{n_b-1} + \cdots + b_{n_b}) u(k)$$

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Shift-operators

Difference equations can be described by polynomials

$$A(z) = z^{n_a} + a_1 z^{n_a-1} + \cdots + a_{n_a}$$
$$B(z) = b_0 z^{n_b} + b_1 z^{n_b-1} + \cdots + b_{n_b}$$
$$A(q)y(k) = B(q)u(k)$$

A backward shift operator q^{-1} can also be used

$$y(k) + a_1 y(k-1) + \cdots + a_{n_a} y(k-n_a) = b_0 u(k-d) + \cdots + b_{n_b} u(k-d-n_b)$$
$$d = n_a - n_b$$

$$(1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}) y(k) = (b_0 + b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b}) u(k-d)$$

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Shift-operators

$$A^*(z) = 1 + a_1 z + \cdots + a_{n_a} z^{n_a} \qquad A^*(q^{-1})y(k) = B^*(q^{-1})u(k-d)$$
$$B^*(z) = b_0 + b_1 z + \cdots + b_{n_b} z^{n_b} \qquad A^*(q^{-1})y(k) = q^{-d} \cdot B^*(q^{-1})u(k)$$

Resiprocal polynomials A^*, B^*

Polynomial representations are used, because in some control design methods they are more natural to use than state-space representations.

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Pulse-transfer operator, *H*(*q*)

The pulse transfer operator is an I/O-representation obtained by eliminating internal variables. E.g. from state representation

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{\Phi x}(k) + \mathbf{\Gamma u}(k) \\ \mathbf{y}(k) = \mathbf{Cx}(k) + \mathbf{Du}(k) \end{cases} \qquad \begin{cases} q \cdot \mathbf{x}(k) = \mathbf{\Phi x}(k) + \mathbf{\Gamma u}(k) \\ \mathbf{y}(k) = \mathbf{Cx}(k) + \mathbf{Du}(k) \end{cases}$$

$$\begin{cases} (q\mathbf{I} - \mathbf{\Phi})\mathbf{x}(k) = \mathbf{\Gamma u}(k) \\ \mathbf{y}(k) = \mathbf{Cx}(k) + \mathbf{Du}(k) \end{cases} \qquad \begin{cases} \mathbf{x}(k) = (q\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{\Gamma u}(k) \\ \mathbf{y}(k) = \mathbf{Cx}(k) + \mathbf{Du}(k) \end{cases}$$

$$\mathbf{y}(k) = \left(\mathbf{C}(q\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{\Gamma} + \mathbf{D}\right)\mathbf{u}(k) = \mathbf{H}(q)\mathbf{u}(k)$$

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Pulse-transfer operator, *H*(*q*)

From polynomial representations

$$A(q)y(k) = B(q)u(k) \quad H(q) = \frac{B(q)}{A(q)}$$

or

$$A^*(q^{-1})y(k) = q^{-d} \cdot B^*(q^{-1})u(k) \quad H^*(q^{-1}) = \frac{q^{-d}B^*(q^{-1})}{A^*(q^{-1})}$$

Representation by *reciprocal polynomials*

$$H^*(q^{-1}) = H(q)$$

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Pulse-transfer operator, *H(q)*

Consider a simple example of a pulse transfer function.

$$\begin{cases} x(k+1) = 0.5x(k) + 0.5u(k) \\ y(k) = 2x(k) \end{cases}$$

We obtain

$$H(q) = C(q - \Phi)^{-1}\Gamma + D = \frac{2 \cdot 0.5}{q - 0.5} = \frac{1}{q - 0.5}$$

Starting from IO-difference equation the calculation is equally easy

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Pulse-transfer operator, *H*(*q*)

$$\begin{cases} x(k) = \frac{1}{2} y(k) \\ x(k+1) = 0.5x(k) + 0.5u(k) \end{cases} \Rightarrow \quad 0.5y(k+1) = 0.25y(k) + 0.5u(k)$$

Form *A*- and *B*-polynomials

$$(0.5q - 0.25)y(k) = 0.5u(k) \qquad (q-0.5)y(k) = u(k)$$

$$A(q)y(k) = B(q)u(k) \qquad A(q) = q - 0.5 \qquad B(q) = 1$$

The pulse transfer operator is $H(q) = \dfrac{B(q)}{A(q)} = \dfrac{1}{q-0.5}$

Pulse-transfer operator, *H*(*q*)

Let us calculate the same with the q^{-1}-operator

$$0.5y(k+1) = 0.25y(k) + 0.5u(k) \qquad 0.5y(k) = 0.25y(k-1) + 0.5u(k-1)$$

$$(0.5 - 0.25q^{-1})y(k) = 0.5q^{-1}u(k) \qquad (1 - 0.5q^{-1})y(k) = 1 \cdot u(k-1)$$

$$A^*(q^{-1})y(k) = q^{-d} \cdot B^*(q^{-1})u(k) \qquad A^*(q^{-1}) = 1 - 0.5q^{-1} \quad B^*(q^{-1}) = 1 \quad d = 1$$

The result becomes

$$H^*(q^{-1}) = \frac{q^{-d}B^*(q^{-1})}{A^*(q^{-1})} = \frac{q^{-1} \cdot 1}{1 - 0.5q^{-1}} \qquad \left(\frac{q^{-1}}{1-0.5q^{-1}} = \frac{1}{q-0.5} \right)$$

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Pulse-transfer operator, *H*(*q*)

All methods lead to the same pulse transfer operator

$$\left(\frac{q^{-1}}{1-0.5q^{-1}} = \frac{1}{q-0.5} \right)$$

$$H^*(q^{-1}) = H(q)$$

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Pulse-transfer operator, *H*(*q*)

- The characteristic polynomial of the system is the denominator *A*(*q*) of the pulse transfer operator.

- The *poles* of a discrete system are the zeros of the characteristic polynomial.

- The *zeros* of a discrete system are the zeros of the numerator polynomial *B*(*q*).

- More delay means more *pole excess d.*
- The *order (dimension)* of the system is the same as the dimension of the state-space representation or the number of poles.

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Poles and zeros

Some basics from matrix calculus. For the pulse transfer operator we can write

$$H(q) = C(qI-\Phi)^{-1}\Gamma + D = \frac{C\cdot \mathrm{adj}(qI-\Phi)\Gamma + D\cdot \det(qI-\Phi)}{\det(qI-\Phi)}$$

where 'adj' means the adjungate matrix. For the matrix Φ the eigenvalues λ_i and eigenvectors e_i are defined as follows

$$\Phi e_i = \lambda_i e_i \Rightarrow (\lambda_i I - \Phi)e_i = 0$$

There exist non-zero eigenvectors when $\det(\lambda_i I - \Phi) = 0$

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Poles and zeros

The poles are the roots of the characteristic polynomial. They Belong to the set of eigenvalues of the system matrix Φ.

Consider the polynomial

$$f(\lambda) = \alpha_0\lambda^m + \alpha_1\lambda^{m-1} + \cdots + \alpha_m$$

The corresponding function for the square matrix *A* is defined as

$$f(A) = \alpha_0 A^m + \alpha_1 A^{m-1} + \cdots + \alpha_m I$$

Result: Let the eigenvalues of *A* be λ_i and the corresponding eigenvectors e_i. It holds

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Poles and zeros

$$f(A)e_i = f(\lambda_i)e_i$$

meaning that $f(\lambda_i)$ is the eigenvalue of $f(A)$ and the Corresponding eigenvector is e_i.

The proof is straightforward by writing the above formula and using repeatedly the definitions of eigenvalue and eigenvector $\Rightarrow$ Exercise.

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Poles and zeros

The poles of a discrete system are the zeros of the denominator of $H(z)$. The zeros are the zeros of the numerator. The poles are also eigenvalues of the the system matrix $\mathbf{\Phi}$.

From the location of poles in the complex plane stability, oscillations and speed of the system can be deduced.

The poles of a continuous *n*:th order system are mapped to the discrete system poles according to:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases} \quad \text{poles: } \lambda_i(\mathbf{A}), \quad i = 1, \ldots, n$$

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Poles and zeros

Discrete system
$$\begin{cases} \mathbf{x}(kh+h) = \mathbf{\Phi}\,\mathbf{x}(kh) + \mathbf{\Gamma}\,\mathbf{u}(kh) \\ \mathbf{y}(kh) = \mathbf{C}\mathbf{x}(kh) \end{cases}$$

poles: $\lambda_i(\mathbf{\Phi}), \quad i = 1, \ldots, n$

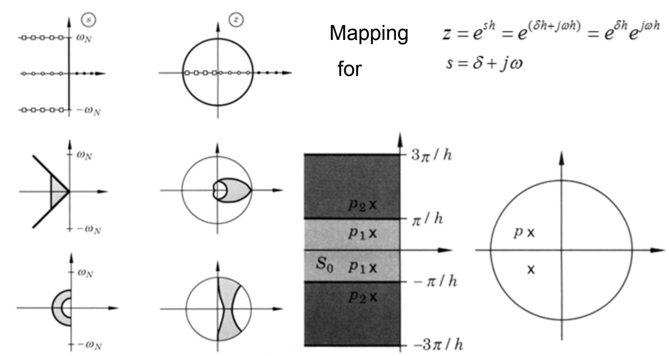
$\mathbf{\Phi} = \mathbf{e}^{\mathbf{A}h}$ => $\lambda_i(\mathbf{\Phi}) = \mathbf{e}^{\lambda_i(\mathbf{A})h}$

A simple relationship for the mapping of zeros does **not** hold. Even the number of zeros does not necessarily remain invariant. The mapping of zeros is a complicated issue.

Poles and zeros

Mapping $z = e^{sh} = e^{(\delta h + j\omega h)} = e^{\delta h} e^{j\omega h}$

for $s = \delta + j\omega$



Unstable inverse, non-minimum phase systems

•A continuous time system is *non-minimum phase*, if has zeros on the right half plane (RHP) or if it contains a delay.

•A discrete system has an unstable inverse, if it has zeros outside the unit circle.

•Zeros are not mapped in a similar way as poles, so a minimum phase continuous system may have a discrete counterpart with an unstable inverse and a non-minimum phase continuous system may have a discrete counterpart with a stable inverse.

Selection of the sampling rate

- The proper choice of the sampling interval is very important. Too low sampling frequency may lose so much information that the control performance deteriorates and the system dynamics is lost.
- Too high sampling rate increases the burden of the processor; also, it may lead to discrete representation with bad numerical properties.
- For oscillating systems the sampling interval is often tied to the frequency of the dominating oscillation. For damped systems the sampling interval is usually chosen to be in relation to the time constant.

Selection of the sampling rate

N_r means the amount of samples during the *rise time*.

$$N_r = \frac{T_r}{h}$$

For a self-oscillating system (2nd order, damping ratio ς and natural frequency ω_0) the rise time is:

$$T_r = \omega_0^{-1} e^{\frac{\varphi}{\tan\varphi}}, \quad \varsigma = \cos\varphi$$

A usual sampling rate is $N_r = \frac{T_r}{h} \approx 4 \cdots 10$ which leads to a

"Rule of Thumb" $\omega_0 h = 0.2 \ldots 0.6$

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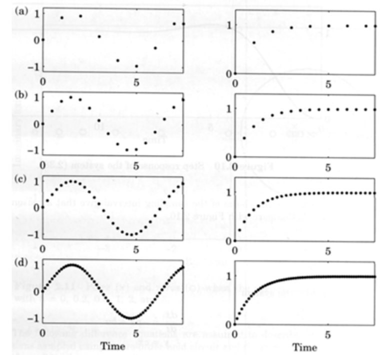
Selection of the sampling rate

Sampling examples for a sinusoidal and exponential signal.

a. $N_r = 1$
b. $N_r = 2$
c. $N_r = 4$
d. $N_r = 8$



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