CS–E4500 Advanced Course in Algorithms

Week 01 – Tutorial

1. Suppose that we independently roll two standard six-sided dice. Let X_1 be the number that shows on the first die, X_2 the number on the second die, and X the sum of the numbers on the two dice.

   (a) What is $\mathrm{E}[X \mid X_1 \text{ is even}]$?

   (b) What is $\mathrm{E}[X \mid X_1 = X_2]$?

   (c) What is $\mathrm{E}[X_1 \mid X = 9]$?

   (d) What is $\mathrm{E}[X_1 - X_2 \mid X = k]$ for k in the range $[2, 12]$?

   Solution.
   (a)

$$\begin{aligned}\mathrm{E}[X \mid X_1 \in \{2,4,6\}] &= \mathrm{E}[X_1 + X_2 \mid X_1 \in \{2,4,6\}] \\ &= \mathrm{E}[X_1 \mid X_1 \in \{2,4,6\}] + \mathrm{E}[X_2 \mid X_1 \in \{2,4,6\}] \\ &= 4 + 3.5 = 7.5\end{aligned}$$

   (b)

$$\mathrm{E}[X \mid X_1 = X_2] = \sum_{i=1}^{6} 2j/6 = 7$$

   (c) There are only 4 combinations resulting in $X = 9$. Over these combinations, X_1 takes values $3, 4, 5, 6$. Hence,

$$\mathrm{E}[X_1 \mid X = 9] = \sum_{i=3}^{6} \mathrm{P}(X_1 = i \mid X = 9) = 1/4 \cdot (3 + 4 + 5 + 6) = 4.5\ .$$

   (d)

$$\mathrm{E}[X_1 - X_2 \mid X = k] = \mathrm{E}[X_1 \mid X = k] - \mathrm{E}[X_2 \mid X = k] = 0$$

2. We flip a fair coin ten times. Find the probability of the following events.

   (a) The number of heads and the number of tails are equal.

   (b) There are more heads than tails.

   (c) The ith flip and the $(11 - i)$th flip are the same for $i = 1, \ldots, 5$.

   (d) We flip at least four consecutive heads.

   Solution.
   (a) $\binom{10}{5}\left(\frac{1}{2}\right)^{10} \approx 0.246$, since there are $\binom{10}{5}$ ways to choose five heads out of ten flips.

   (b) $\sum_{i=6}^{10} \binom{10}{i}\left(\frac{1}{2}\right)^{10} \approx 0.377$, since we account for the number of heads being $6, 7, 8, 9, 10$.

   (c) $\left(\frac{1}{2}\right)^{5} \approx 0.03$, since the first 5 flips fix the sequence, and the last 5 have to match it.

(d) Clearly $\mathrm{P}(\geq 4 \text{ consecutive heads}) = 1 - \mathrm{P}(< 4 \text{ consecutive heads})$. Notice that there are four sequences that do not lead to four consecutive heads: $P(T) = 1/2$, $P(HT) = 1/2^2$, $P(HHT) = 1/2^3$, $P(HHHT) = 1/2^4$. Therefore, we can set up a recursion for k flips where P_k is the probability of not observing four consecutive heads in k flips. Note the base cases being $P_0 = P_1 = P_2 = P_3 = 1$. Hence, we have

$$P_k = 1/2 \cdot P_{k-1} + 1/4 \cdot P_{k-2} + 1/8 \cdot P_{k-3} + 1/16 \cdot P_{k-4} \,.$$

Solving the recurrence yields $\mathrm{P}(\geq 4 \text{ consecutive heads}) = 0.2451$.

3. The following problem is known as the Monty Hall problem, after the host of the game show "Let's Make a Deal". There are three curtains. Behind one curtain is a new car, and behind the other two are goats. The game is played as follows. The contestant chooses the curtain that she thinks the car is behind. Monty then opens one of the other curtains to show a goat. (Monty may have more than one goat to choose from; in this case, assume he chooses which goat to show uniformly at random.) The contestant can then stay with the curtain she originally chose or switch to the other unopened curtain. After that, the location of the car is revealed, and the contestant wins the car or the remaining goat. Should the contestant switch curtains or not, or does it make no difference?

Solution. Let's assume you pick door 1 and then Monty shows you the goat behind door 2. Let event A be that the car is behind door number 1, and let event B be that Monty opens up door 2 to show the goat.

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(B \mid A)\mathrm{P}(A)}{\mathrm{P}(B)} = \frac{1/2 \cdot 1/3}{1/3 \cdot 1/2 + 1/3 \cdot 0 + 1/3 \cdot 1} = 1/3 \,.$$

As Monty has opened door 2, you know the car is either behind door 1 (your choice) or door 3. The probability of the car being behind door 1 is $1/3$. This means that the probability of the car being behind door 3 is $1 - 1/3 = 2/3$. And that is why you switch.