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Differential and Integral Calculus 1

Exercises, Week 3

Work on Warm-up 1–4 during the exercise sessions of Week 3.
Submit on MyCourses your solutions for Homework 1 and 2 by Sunday, September 25th.

Warm-up 1: Give an example or explain why an example does not exist:

1. A function $f\colon \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $f'(x_0) = f''(x_0) = 0$ and $f'''(x_0) \neq 0$.
2. A function $f\colon \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $f'(x_0) = f''(x_0) = 0$ and f has a local minimum at x_0.
3. The "triangle inequality for integrals" states that
$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx \qquad \text{(where } a \leq b\text{)}.$$
Find a function (and some $a \leq b$) for which the inequality is strict and find a function (and some $a \leq b$) for which the two sides are equal.

Warm-up 2: Compute the definite integral
$$\int_0^2 (x^2 + x + 2)\,dx$$
as a limit of Riemann sums, checking that the upper and lower Riemann sums $U(f, P_n)$ and $L(f, P_n)$ converge to the same limit, where P_n is the partition of $[0, 2]$ into n subintervals of length $2/n$. Then compute the same integral also by using the Fundamental Theorem of Calculus.

Warm-up 3: Compute the following indefinite—remember the integration constant!—and definite integrals. If you use substitutions, integration by parts or other special methods, be explicit.

$$(a)\quad \int \frac{3}{\sqrt{4 - 9x^2}}\,dx \qquad\qquad (d)\quad \int_0^1 e^x \sin(x)\,dx$$
$$(b)\quad \int \frac{x - 2}{x^2 - x}\,dx \qquad\qquad (e)\quad \int \sin(x)^3 \cos(x)^5\,dx$$
$$(c)\quad \int \frac{\ln x}{x}\,dx \qquad\qquad (f)\quad \int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx$$

Warm-up 4: An *elementary function* is a function written in terms of polynomials, exponentials, logarithms, trigonometric functions, and fractions or other combinations of these.

> **Theorem (Liouville).** Let f be a non-constant function. Then the function $x \mapsto e^{f(x)}$ has an elementary antiderivative if and only if there exist two polynomials P and Q such that $\left(\frac{P}{Q}\right)' + \frac{P}{Q} f' = 1$.

Show that e^{-x^2} does not have an elementary antiderivative.

Hint: Assume by contradiction that e^{-x^2} has an elementary antiderivative. Do some manupulations and show that Q has to be a constant. Then find a contradiction by a degree argument. (This is really an exercise in algebra, not calculus.)

Submit on MyCourses your solutions **only** for the following two problems. Explain the reasoning behind your solutions, do not just return the final result. If you make use of important results from the lectures, state what they are.

If you use substitutions, integration by parts or other special methods, be explicit.

Homework 1: Compute the definite integral

$$\int_0^3 (4x^3 + x^2 + 1)\, dx$$

as a limit of Riemann sums, checking that the upper and lower Riemann sums $U(f, P_n)$ and $L(f, P_n)$ converge to the same limit, where P_n is the partition of $[0, 3]$ into n subintervals of length $3/n$. Then compute the same integral also by using the Fundamental Theorem of Calculus.

Hint: You can take for granted $1^3 + 2^3 + \cdots + (N-1)^3 + N^3 = \sum_{i=1}^{N} i^3 = \dfrac{N^2(N+1)^2}{4}$ and other similar formulas given in the lectures. [2 points]

Homework 2: Compute the following definite or indefinite integrals. Remember the integration constants for indefinite integrals and remember that improper definite integrals are defined as limits of proper ones.

$$(a)\quad \int e^{2x} \sin(3x)\, dx \qquad\qquad (c)\quad \int_e^{e^2} \frac{1}{x \ln x}\, dx$$

$$(b)\quad \int \frac{1}{x^2 + x - 2}\, dx \qquad\qquad (d)\quad \int_{-1}^{1} \frac{1}{(x+1)^{2/3}}\, dx$$

[2 points]