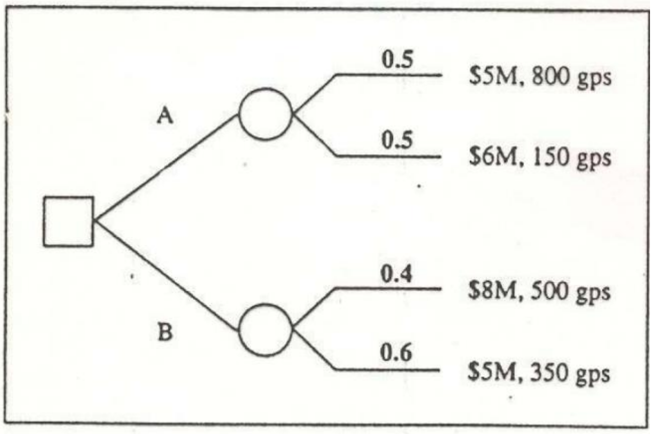
# Exercise 2 – Solutions

**20.9.2022**

## #1 Utility assessment and expected utility

a)

The decision tree is shown in below.




b)

To start with, the best and the worst possible outcomes correspond to utilities 1 and 0, respectively: U($5M,800gps) = 1 and U($6M,150gps) = 0. Furthermore, the indifferences between the introduced games (=equalities between the alternatives' expected utilities) impose constraints:

1* U($5M,350gps) = 0.35 * U($5M,800gps) + 0.65 * U($6M,150gps)=0.35 and

1* U($8M,500gps) = 0.70 * U($5M,800gps) + 0.30 * U($6M,150gps)=0.7.

The expected utility of alternative A is

**E(U(A))** = 0.5 * U($5M,800gps) + 0.5 * U($6M,150gps) = 0.5 * 1 + 0.5 * 0

= **0.50**

And that of B:

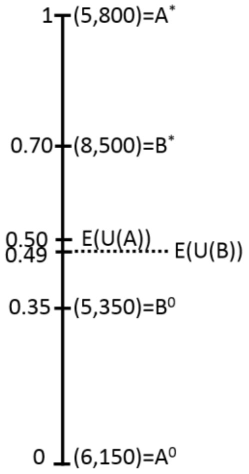
**E(U(B))** = 0.4 * U($8M,500gps) + 0.6 * U($5M,350gps) = 0.4 * 0.7 + 0.6 * 0.35

= **0.49**

Thus, A is better in terms of expected utility.

c)

The outcomes $A^*$=(5,800) and $A^0$=(6,150) of option A are more dispersed on the utility scale than the outcomes $B^*$=(8,500) and $B^0$=(5,350) of B. With this regard, A seems riskier than B.




Mathematically, the dispersion of the utility outcomes of the alternatives can be examined for example through variance. Applying

$$Var(y) = E((y - E(y))^2)$$

to the utilities gives

**Var(U(A))** $= 0.5*(U(\$6M,150gps) - E(U(A)))^2 + 0.5*(U(\$5M,800gps) - E(U(A)))^2$

$= (0 - 0.5)^2 * 0.5 + (1 - 0.5)^2 * 0.5$

$= \mathbf{0.25}$

**Var(U(B))** $= 0.6*(U(\$5M,350gps) - E(U(B)))^2 + 0.4*(U(\$8M,500gps) - E(U(B)))^2$

$= 0.6*(0.35 - 0.49)^2 + 0.4*(0.7 - 0.49)^2$

$= \mathbf{0.0294}$.

The variance of the utility of B is significantly smaller than that of A.

However, according to utility theory, the DM's risk attitude is already accounted for in the utility function through assessments of simple games.

**NOTE:** Because the projects' expected utilities differ only a little, a small change in the risk attitude/answers to the elicitation questions suffices to change the decision recommendation. For example, if you would have been able to decide the probabilities in Figures 1 (a) and 1 (b) of the exercise sheet yourself for real, maybe E(U(A)) would have become smaller than E(U(B)).

d)

Let $C_A$ and $C_B$ denote the statements of the colleague about how A and B will turn out. Thus, it now applies $P(C_A = A^*|A^*) = 1$, $P(C_A = A^0|A^0) = 1$, $P(C_B = B^*|B^*) = 1$, $P(C_B = B^0|B^0) = 1$. Does this mean that also $P(A^* |C_A = A^*) = 1$? Yes, with Bayes' theorem, it is obtained, e.g.,

$$P(A^*|C_A = A^*) = \frac{P(C_A = A^*|A^*)P(A^*)}{P(C_A = A^*)} = 1$$

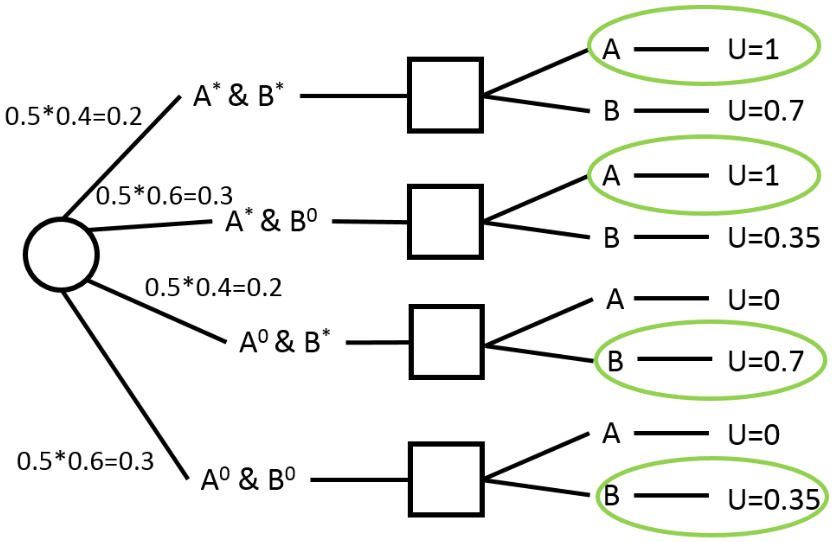
because $P(C_A = A^*|A^0) = 0$ and

$$P(C_A = A^*) = P(C_A = A^*|A^*)P(A^*) + P(C_A = A^*|A^0)P(A^0) = P(C_A = A^*|A^*)P(A^*).$$

Note also that now $P(C_A = A^*) = P(A^*)$ because $P(C_A = A^*|A^*) = 1$.

Similarly, it applies $P(C_A = A^0) = P(A^0)$, $P(C_B = B^*) = P(B^*)$ and $P(C_B = B^0) = P(B^0)$.

Therefore, if the decision between A and B can be made only after talking with the colleague, the situation corresponds to having access to perfect information in the sense that you will know the state of the world before making the decision. This situation be portrayed with a reversed decision tree shown below.




Based on the reversed decision tree, one can solve that the expected utility with perfect information (EUwPI) is

**EUwPI**=0.2*1+0.3*1+0.2*0.7+0.3*0.35=**0.745**

In part b), it was solved that the expected utility without perfect information (EUwoPI) was EUwoPI=E(U(A))=0.50. Thus, the expected utility of perfect information (EUPI) is now

**EUPI**=EUwPI-EUwoPI=0.745-0.50=**0.245**

This is the extra amount of utility that one can expect to gain if perfect information becomes available.

**Important note:** Since the exercise states that the colleague can **predict** with 100 % accuracy the success and failure of both projects under consideration, it is possible to make the conclusion $\mathrm{P}(\mathrm{A}^* \,|\mathrm{C}_\mathrm{A} = \mathrm{A}^*) = 1$ directly. This does not change the results of the part d, since for any events $\mathrm{A}$ and $\mathrm{B}$ that have $\mathrm{P}(\mathrm{A}|\mathrm{B}) = 1$ and $\mathrm{P}(\mathrm{A}|\mathrm{B}^c) = 0$ we have

$$\mathbf{P}(\mathbf{B}|\mathbf{A}) = \frac{\mathrm{P}(\mathrm{A}|\mathrm{B})\ \mathrm{P}(\mathrm{B})}{\mathbf{P}(\mathbf{A})} = \frac{\mathrm{P}(\mathrm{A}|\mathrm{B})\ \mathrm{P}(\mathrm{B})}{\mathrm{P}(\mathrm{A}|\mathrm{B})\mathrm{P}(\mathrm{B}) + \mathrm{P}(\mathrm{A}|\mathrm{B}^c)\mathrm{P}(\mathrm{B}^c)} = \frac{\mathrm{P}(\mathrm{A}|\mathrm{B})\ \mathrm{P}(\mathrm{B})}{\mathbf{P}(\mathbf{B})} = \mathbf{P}(\mathbf{A}|\mathbf{B})$$

So, in any case $\mathrm{P}(\mathrm{B}|\mathrm{A}) = \mathrm{P}(\mathrm{A}|\mathrm{B})$ and $\mathrm{P}(\mathrm{A}) = \mathrm{P}(\mathrm{B})$.