ELEC-E8101 Digital and Optimal Control

Solutions 2

1. **a.** Let us start from the corresponding transfer function

$$G(s) = \frac{1}{\tau s + 1} = \frac{Y(s)}{U(s)}$$

and state-space representation

$$\begin{cases} \dot{x}(t) = -\frac{1}{\tau}x(t) + \frac{1}{\tau}u(t) \\ y(t) = x(t) \end{cases}$$

i. The discretization formulas of a state space representation

$$\Phi = e^{Ah}$$
$$\Gamma = \int_0^h e^{As} ds B$$

$$\Phi = e^{-\frac{h}{\tau}}, \; \Gamma = \int_0^h e^{-\frac{1}{\tau}s} ds \frac{1}{\tau} = \int_0^h \frac{1}{\tau} e^{-\frac{1}{\tau}s} ds = \Big/_0^h - e^{-\frac{s}{\tau}} = \left(1 - e^{-\frac{h}{\tau}}\right)$$

$$\begin{cases} x(k+1) = \Phi x(k) + \Gamma u(k) \\ y(k) = x(k) \end{cases}$$

$$\Rightarrow \quad \begin{cases} x(k+1) = e^{-\frac{h}{\tau}} x(k) + \left(1 - e^{-\frac{h}{\tau}}\right) u(k) \\ y(k) = x(k) \end{cases}$$

Let's determine the corresponding pulse transfer function for the part *ii* of this problem

Eliminate $x(k)$: $$y(k+1) - e^{-\frac{h}{\tau}} y(k) = \left(1 - e^{-\frac{h}{\tau}}\right) u(k)$$

Z-transform (with zero inintial values): $$zY(z) - e^{-\frac{h}{\tau}} Y(z) = \left(1 - e^{-\frac{h}{\tau}}\right) U(z)$$

$$\Rightarrow \quad \left(z - e^{-\frac{h}{\tau}}\right) Y(z) = \left(1 - e^{-\frac{h}{\tau}}\right) U(z) \quad \Rightarrow \quad H(z) = \frac{Y(z)}{U(z)} = \frac{1 - e^{-\frac{h}{\tau}}}{z - e^{-\frac{h}{\tau}}}$$

ii. The discretization formula of transfer functions:

$$H(z) = \frac{z-1}{z} \cdot Z\left\{ L^{-1}\left\{ \frac{1}{s} \cdot G(s) \right\} \right\}$$

$$L^{-1}\left\{ \frac{1}{s} \cdot G(s) \right\} = L^{-1}\left\{ \frac{1}{s(\tau s + 1)} \right\} = \left(1 - e^{-\frac{t}{\tau}}\right)$$

In the case of discrete systems, the signals are defined only at sampling times.

Substitute $t = kh$ $\Rightarrow$ $L^{-1}\left\{\frac{1}{s}\cdot G(s)\right\} = 1 - e^{-\frac{h}{\tau}k} = 1 - \left(e^{-\frac{h}{\tau}}\right)^k$

$$\Rightarrow \quad Z\left\{1 - \left(e^{-\frac{h}{\tau}}\right)^k\right\} = \frac{z}{z-1} - \frac{z}{z - e^{-\frac{h}{\tau}}} \qquad \Rightarrow \qquad H(z) = \frac{z-1}{z}\cdot\left(\frac{z}{z-1} - \frac{z}{z - e^{-\frac{h}{\tau}}}\right)$$

$$\Rightarrow \quad = 1 - \frac{z-1}{z - e^{-\frac{h}{\tau}}} = \frac{z - e^{-\frac{h}{\tau}} - (z-1)}{z - e^{-\frac{h}{\tau}}} = \frac{1 - e^{-\frac{h}{\tau}}}{z - e^{-\frac{h}{\tau}}}$$

Both discretization methods give the same result.

b. *i.* The unit step response of continuous process: $y(t) = L^{-1}\{Y(s)\} = L^{-1}\left\{G(s)\cdot\frac{1}{s}\right\} = 1 - e^{-\frac{t}{\tau}}$

ii. The unit step response of the difference equation, *i.e.* $u(k) = \begin{cases} 0, & k < 0 \\ 1, & k \geq 0 \end{cases}$

The output is assumed to be zero at $k = 0$ $(y(0) = 0)$, and the state-space representation gets the following recursive form:

$$y(k+1) = e^{-\frac{h}{\tau}} y(k) + \left(1 - e^{-\frac{h}{\tau}}\right) u(k) = e^{-\frac{h}{\tau}} y(k) + 1 - e^{-\frac{h}{\tau}}$$

Instead of solving the equation analytically (by the Z-transformation for example) let us just compute some values:

$$y(0) = 0$$

$$y(1) = e^{-\frac{h}{\tau}} y(0) + 1 - e^{-\frac{h}{\tau}} = 1 - e^{-\frac{h}{\tau}}$$

$$y(2) = e^{-\frac{h}{\tau}} y(1) + 1 - e^{-\frac{h}{\tau}} = e^{-\frac{h}{\tau}}\cdot\left(1 - e^{-\frac{h}{\tau}}\right) + 1 - e^{-\frac{h}{\tau}} = 1 - e^{-\frac{2h}{\tau}}$$

$$y(3) = e^{-\frac{h}{\tau}} y(2) + 1 - e^{-\frac{h}{\tau}} = e^{-\frac{h}{\tau}}\cdot\left(1 - e^{-\frac{2h}{\tau}}\right) + 1 - e^{-\frac{h}{\tau}} = 1 - e^{-\frac{3h}{\tau}}$$

$$y(4) = 1 - e^{-\frac{4h}{\tau}}$$

$$\vdots$$

$$y(k) = 1 - e^{-\frac{kh}{\tau}} \qquad \text{(analytical solution)}$$

iii. The step response of the pulse transfer function: $y(k) = Z^{-1}\{Y(z)\} = Z^{-1}\{H(z)U(z)\}$.

Unit step: $U(z) = \frac{z}{z-1}$

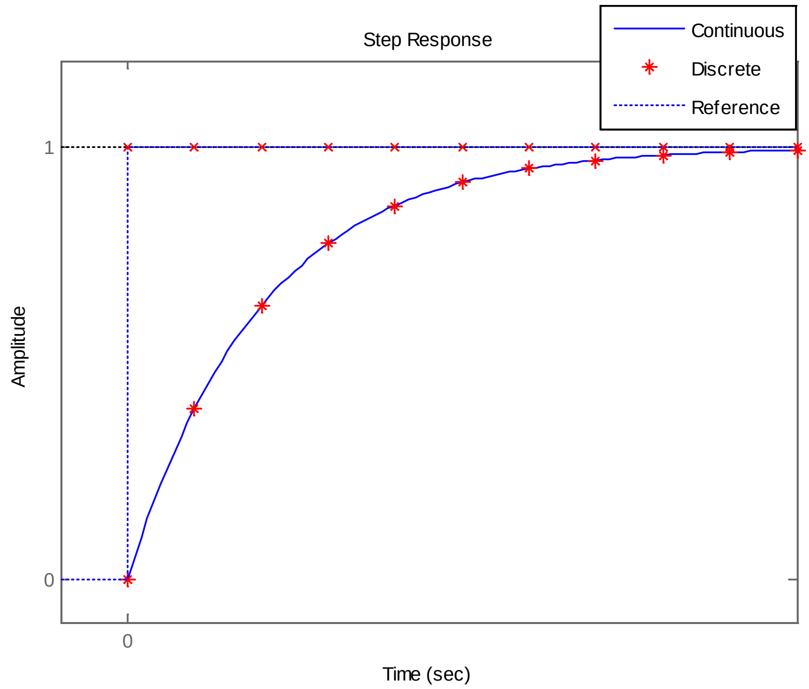
$$y(k) = Z^{-1}\left\{\frac{z\left(1-e^{-\frac{h}{\tau}}\right)}{\left(z-e^{-\frac{h}{\tau}}\right)(z-1)}\right\} = Z^{-1}\left\{z\frac{\left(1-e^{-\frac{h}{\tau}}\right)}{\left(z-e^{-\frac{h}{\tau}}\right)(z-1)}\right\}$$

Let's factor $\frac{z\left(1-e^{-\frac{h}{\tau}}\right)}{\left(z-e^{-\frac{h}{\tau}}\right)(z-1)}$:

$$\frac{z\left(1-e^{-\frac{h}{\tau}}\right)}{\left(z-e^{-\frac{h}{\tau}}\right)(z-1)} = z\left(\frac{1}{z-1}-\frac{1}{z-e^{-\frac{h}{\tau}}}\right)$$

$$y(k) = Z^{-1}\left\{\frac{z}{z-1}-\frac{z}{z-e^{-\frac{h}{\tau}}}\right\} = 1-\left(e^{-\frac{h}{\tau}}\right)^k = 1-e^{-\frac{kh}{\tau}}$$

ii. and *iii.* gave the same response and *i.* is the same during the sampling times.



2. Let's consider the first order process of

$$\dot{x} = ax + bu$$
$$y = cx$$

a, b, c, u, x, and y are scalars with $c, b > 0$

The process can be presented also as a transfer function. The formula: $G(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$ can be written as: $G(s) = c(s-a)^{-1}b = \frac{cb}{s-a}$ in the scalar case.

a. Let us consider stability.

$$G(s) = \frac{cb}{s-a} \Rightarrow \text{ the process has a pole: } s_{P1} = a$$

The process is stable, when the pole is in the left half of the complex plane $\Rightarrow \quad a \le 0$

b. Discretize the process. The discretization formulas for the scalar systems are:

$$\Phi = e^{Ah} = e^{ah}$$
$$\Gamma = \int_0^h e^{As} ds B = b\int_0^h e^{as} ds = \frac{b}{a} \Big/_0^h \left(e^{as}\right) = \frac{b}{a}\left(e^{ah} - 1\right)$$

The discrete state-space representation:

$$\begin{cases} x(k+1) = e^{ah} x(k) + \frac{b}{a}\left(e^{ah} - 1\right)u(k) \\ y(k) = cx(k) \end{cases} \qquad \begin{cases} x(k+1) = \Phi x(k) + \Gamma u(k) \\ y(k) = Cx(k) \end{cases}$$

The discretized process can also be presented with the pulse transfer function:

$$H(z) = \mathbf{C}\left(z\mathbf{I} - \Phi\right)^{-1}\Gamma = \frac{\frac{cb}{a}\left(e^{ah} - 1\right)}{z - e^{ah}}$$

c. Let us consider the stability of

$$H(z) = \frac{\frac{cb}{a}\left(e^{ah} - 1\right)}{z - e^{ah}} \quad \Rightarrow \quad \text{process has the pole:} \quad z_{P1} = e^{ah}$$

The process is stable if the pole is inside the unit circle

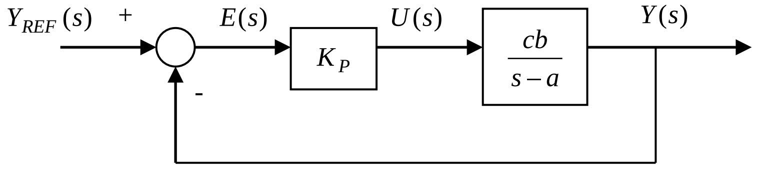
$\Rightarrow \quad \left|e^{ah}\right| \le 1 \qquad \Rightarrow \qquad -1 \le e^{ah} \le 1 \qquad e^{ah}$ is always positive

$\Rightarrow \quad e^{ah} \le 1 \qquad \Rightarrow \qquad \ln\left(e^{ah}\right) \le \ln(1) \qquad \Rightarrow \qquad ah \le 0$

sampling time h is always positive $\quad \Rightarrow \quad a \le 0$ (however, it was assumed that a is non-zero).

The stability region of the discretized process is the same as the stability region of the continuous process. Discretization does not effect the stability of the uncontrolled process.

d. The process is controlled with a continuous time *P*-controller:



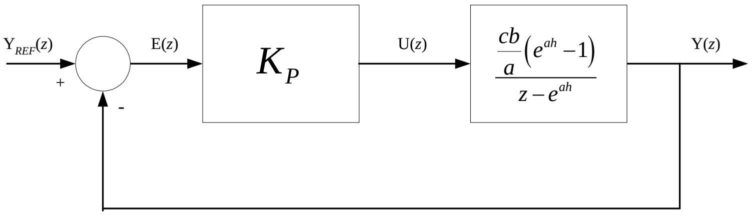
The transfer function of the controlled system:

$$G_{TOT}(s) = \frac{Y(s)}{Y_{REF}(s)} = \frac{K_P \frac{cb}{s-a}}{1 + K_P \frac{cb}{s-a}} = \frac{K_P cb}{s - a + K_P cb} = \frac{K_P cb}{s - \left(a - K_P cb\right)}$$

The controlled system has the pole: $s_{P1} = a - K_P cb$

System is stable, when $a - K_P cb \leq 0$ $\Rightarrow K_P \geq \dfrac{a}{cb}$ since $c, b \geq 0$

e. The process is controlled with a discrete time *P*-controller:



The pulse transfer function of the controlled system:

$$H_{TOT}(z) = \frac{Y(z)}{Y_{REF}(z)} = \frac{K_P \dfrac{\frac{cb}{a}\left(e^{ah}-1\right)}{z-e^{ah}}}{1 + K_P \dfrac{\frac{cb}{a}\left(e^{ah}-1\right)}{z-e^{ah}}} = \frac{K_P \frac{cb}{a}\left(e^{ah}-1\right)}{z - e^{ah} + K_P \frac{cb}{a}\left(e^{ah}-1\right)}$$

$\Rightarrow$ The controlled system has the pole: $z_{P1} = e^{ah} - K_P \dfrac{cb}{a}\left(e^{ah}-1\right)$

The system is stable, while

$$\left| e^{ah} - K_P \frac{cb}{a}\left(e^{ah}-1\right) \right| \leq 1 \qquad \Rightarrow \qquad -1 \leq e^{ah} - K_P \frac{cb}{a}\left(e^{ah}-1\right) \leq 1$$

$$\Rightarrow \quad -\left(1+e^{ah}\right) \leq -K_P \frac{cb}{a}\left(e^{ah}-1\right) \leq \left(1-e^{ah}\right) \qquad \Rightarrow \qquad \frac{a}{cb}\left(\frac{1-e^{ah}}{1-e^{ah}}\right) \leq K_P \leq \frac{a}{cb}\left(\frac{1+e^{ah}}{e^{ah}-1}\right)$$

$$\Rightarrow \quad \frac{a}{cb} \leq K_P \leq \frac{a}{cb}\left(\frac{e^{ah}+1}{e^{ah}-1}\right)$$ (remember. $c, b, \dfrac{e^{ah}-1}{a} \geq 0$, *a* can be positive or negative)

f. If the sampling time *h* approaches to zero:

$$\lim_{h \to 0}\left(\frac{e^{ah}+1}{e^{ah}-1}\right) = \infty$$

The stability region of the discrete system approaches:

$$\frac{a}{cb} \leq K_P < \infty \qquad \Leftarrow \qquad \frac{a}{cb} \leq K_P$$

which is the same as the stability region of the continuous system.

$\Leftarrow$

3. $y(k+2)-1{,}3y(k+1)+0{,}4y(k)=u(k+1)-0{,}4u(k)$

a. Let's use the q-operator:

$$q^2y(k)-1{,}3qy(k)+0{,}4y(k)=qu(k)-0{,}4u(k)$$
$$q\{q\{y(k)\}-1{,}3y(k)-u(k)\}=-0{,}4y(k)-0{,}4u(k)$$

$$\begin{cases} x_1(k)=y(k) \\ x_2(k)=x_1(k+1)-1{,}3x_1(k)-u(k) \\ x_2(k+1)=-0{,}4x_1(k)-0{,}4u(k) \end{cases} \qquad \begin{cases} x_1(k+1)=1{,}3x_1(k)+x_2(k)+u(k) \\ x_2(k+1)=-0{,}4x_1(k)-0{,}4u(k) \\ y(k)=x_1(k) \end{cases}$$

$$\begin{cases} \mathbf{x}(k+1)=\begin{bmatrix} 1{,}3 & 1 \\ -0{,}4 & 0 \end{bmatrix}\mathbf{x}(k)+\begin{bmatrix} 1 \\ -0{,}4 \end{bmatrix}u(k) \\ y(k)=\begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(k) \end{cases}$$

b. The pulse transfer function can be determined from the state-space representation with $H(z)=\mathbf{C}(z\mathbf{I}-\Phi)^{-1}\Gamma$ or from the difference equation by Z-transformation (with setting the initial values to zero).

i. From state-space representation:

$$H(z)=\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} z-1{,}3 & -1 \\ 0{,}4 & z \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -0{,}4 \end{bmatrix}=\frac{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} z & 1 \\ -0{,}4 & z-1{,}3 \end{bmatrix}\begin{bmatrix} 1 \\ -0{,}4 \end{bmatrix}}{z(z-1{,}3)+0{,}4}$$
$$=\frac{z-0{,}4}{z^2-1{,}3z+0{,}4}=\frac{z-0{,}4}{(z-0{,}8)(z-0{,}5)}$$

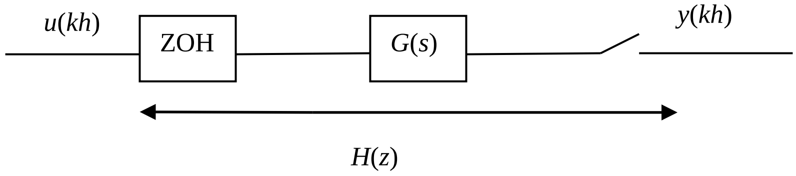
The poles of the pulse transformation, $z_{P1}=0{,}8$ and $z_{P2}=0{,}5$, are in the unit disc.

$\Leftarrow$ The system is stable.

ii. From difference equation: $y(k+2)-1{,}3y(k+1)+0{,}4y(k)=u(k+1)-0{,}4u(k)$

$$z^2Y(z)-1{,}3zY(z)+0{,}4Y(z)=zU(z)-0{,}4U(z)$$
$$\Leftarrow \quad (z^2-1{,}3z+0{,}4)Y(z)=(z-0{,}4)U(z)$$
$$\Leftarrow \quad H(z)=\frac{Y(z)}{U(z)}=\frac{z-0{,}4}{z^2-1{,}3z+0{,}4}$$

* **4.** The block diagram of the system:



In this case the pulse transfer function is

$$H(z^{-1}) = \frac{0.2z^{-1}}{1-0.8z^{-1}}$$

or

$$H(z) = \frac{0.2}{z-0.8}.$$

Impulse response:

$$u(k) = \delta(k) = \begin{cases} 1, k=0 \\ 0, k \neq 0 \end{cases} \text{ (z-transformation) } \Rightarrow U(z) = 1.$$

$$Y(z) = \frac{0.2}{z-0.8} \cdot 1 = 0.2z^{-1}\frac{z}{z-0.8}$$

$$\Rightarrow y(k) = \begin{cases} 0.2 \cdot 0.8^{k-1} = 0.25 \cdot 0.8^{k}, \; k = 1, 2, 3, \dots \\ 0, \; k = 0 \end{cases}$$

Step response:

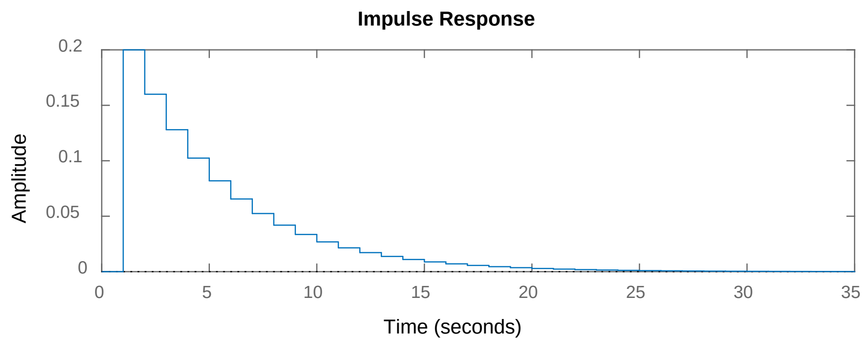
$$u(k) = 1 \Rightarrow U(z) = \frac{z}{z-1}$$

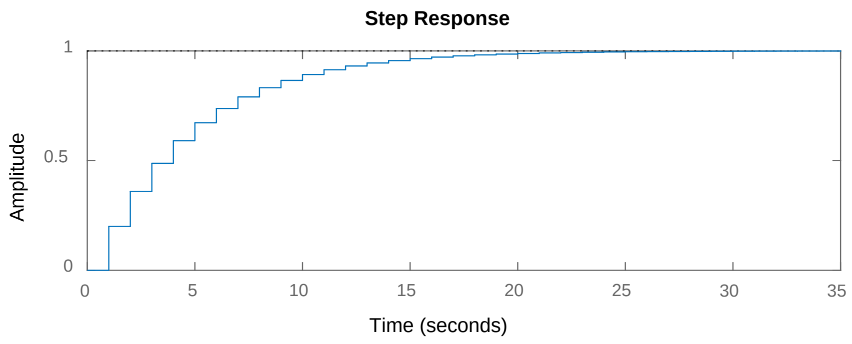
$$\Rightarrow Y(z) = \frac{0.2}{z-0.8} \cdot \frac{z}{z-1} = 0.2z\left(\frac{A}{z-0.8} + \frac{B}{z-1}\right)$$

Partial fractions $\Rightarrow A = -5, B = 5$

$$\Rightarrow Y(z) = \frac{-z}{z-0.8} + \frac{z}{z-1}$$

$$\Rightarrow y(k) = 1 - 0.8^{k}$$





The Matlab code:

```
H=tf(0.2,[1 -0.8],1);
subplot(211)
impulse(H)
subplot(212)
step(H)
```