ELEC-E8116 Model-based control systems
/exercises 2

Problem 1

Prove that the solution of the state equation $\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0$ is

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Problem 2

Derive the general solution for the discrete-time difference equation

$$x(t+1) = Ax(t) + Bu(t), \quad x(t_0) = x_0$$

Problem 3

The *trace* of a square matrix is defined as the sum of the elements in the main diagonal. Let *A* and *B* be square matrices of equal dimensions and *C* and *D* such matrices that *CD* and *DC* are both properly defined. Prove

a. $tr(A+B) = tr(A)+tr(B)$
b. $tr(CD) = tr(DC)$

Problem 4

Square matrices *B* and *A* of equal dimensions are called *similar*, if there exists an invertible square matrix *T* such that

$$B = TAT^{-1}$$

Prove that similar matrices have the same eigenvalues, the same *trace* and the same determinant.

Problem 5

Consider the MISO-system

$$y(t) = \frac{p+2}{p^2+2p+1}u_1(t) + \frac{1}{p^2+3p+2}u_2(t)$$

Form a realization (state-space representation).

Problem 6

Consider the following optimization problem. Let

$$\dot{x}(t) = u(t), \quad x(0) = 1$$

and find an optimal control *u*, which minimizes the criterion

$$J = \int_0^{\infty} \left[x^2(t) + u^2(t) \right] dt$$

Prove that the solution can be presented in state feedback form $u^*(t) = -x(t)$. What is the optimal control as a function of time? What is the optimal cost?

Hint: Prove first the identity

$$\int_0^T \left[x^2(t) + \dot{x}^2(t) \right] dt = x^2(0) - x^2(T) + \int_0^T \left[x(t) + \dot{x}(t) \right]^2 dt$$