ELEC-E8101 Digital and Optimal Control

FORMULAE FOR THE EXAMINATION

Return the material, if you have got it from the exam supervisor. Do not write anything on this paper!

Discretization

Sampling of a continuous-time state-space representation

$$\begin{cases}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{cases} \quad \begin{matrix}\dim\{\mathbf{u}\} = r \\ \dim\{\mathbf{x}\} = n \\ \dim\{\mathbf{y}\} = p\end{matrix}$$

With periodic sampling

$$\begin{cases}\mathbf{x}(kh+h) = \mathbf{\Phi}\,\mathbf{x}(kh) + \Gamma\,\mathbf{u}(kh) \\ \mathbf{y}(kh) = \mathbf{C}\mathbf{x}(kh) + \mathbf{D}\mathbf{u}(kh)\end{cases}$$

$$\begin{cases}\mathbf{\Phi} = \mathbf{e}^{\mathbf{A}h} \\ \Gamma = \int_0^h \mathbf{e}^{\mathbf{A}s} ds\,\mathbf{B}\end{cases}$$

State transition matrix

$$e^{\mathbf{A}t} = L^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

Discretization

Sampling a system with delay

$$\begin{cases}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t-\tau) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{cases}$$

Case $\tau \leq h$

With periodic sampling

$$\begin{cases}\mathbf{x}(kh+h) = \mathbf{\Phi}\,\mathbf{x}(kh) + \mathbf{\Gamma}_0\mathbf{u}(kh) + \mathbf{\Gamma}_1\mathbf{u}(kh-h) \\ \mathbf{y}(kh) = \mathbf{C}\mathbf{x}(kh) + \mathbf{D}\mathbf{u}(kh)\end{cases}$$

$$\mathbf{\Phi} = \mathbf{e}^{\mathbf{A}h}$$

$$\mathbf{\Gamma}_1 = \mathbf{e}^{\mathbf{A}(h-\tau)} \int_0^{\tau} \mathbf{e}^{\mathbf{A}s} ds\mathbf{B}$$

$$\mathbf{\Gamma}_0 = \int_0^{h-\tau} \mathbf{e}^{\mathbf{A}s} ds\mathbf{B}$$

Characteristic equation

$$\det(\lambda\mathbf{I} - \mathbf{\Phi}) = 0$$

Mapping of the poles (eigenvalues)

$$\lambda_i(\mathbf{\Phi}) = e^{\lambda_i(\mathbf{A})h}$$

Discretization

Case $\tau > h$

$$\tau = (d-1)h + \tau'$$

$$\begin{cases}\mathbf{x}(kh+h) = \mathbf{\Phi}\,\mathbf{x}(kh) + \mathbf{\Gamma}_0\mathbf{u}(kh-(d-1)h) + \mathbf{\Gamma}_1\mathbf{u}(kh-dh) \\ \mathbf{y}(kh) = \mathbf{C}\mathbf{x}(kh) + \mathbf{D}\mathbf{u}(kh)\end{cases}$$

$$\mathbf{\Phi} = \mathbf{e}^{\mathbf{A}h}$$

$$\mathbf{\Gamma}_1 = \mathbf{e}^{\mathbf{A}(h-\tau')} \int_0^{\tau'} \mathbf{e}^{\mathbf{A}s} ds\mathbf{B}$$

$$\mathbf{\Gamma}_0 = \int_0^{h-\tau'} \mathbf{e}^{\mathbf{A}s} ds\mathbf{B}$$

Pulse-transfer function

Pulse-transfer function (no delay)

$$\mathbf{H}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{\Gamma} + \mathbf{D}$$

The ZOH-equivalent (direct discretization of the transfer function)

$$H(z) = \frac{z-1}{z} \cdot Z\left\{ L^{-1}\left\{ G(s)\frac{1}{s} \right\}\Big|_{t=kh} \right\}$$

Predictive first-order hold

Discretization of the state-space representation:

$$\Phi = \mathbf{e}^{\mathbf{A}h},\ \Gamma = \int_0^h \mathbf{e}^{\mathbf{A}s}ds\mathbf{B},\ \Gamma_1 = \int_0^h \mathbf{e}^{\mathbf{A}s}(h-s)ds\mathbf{B}$$

$$\mathrm{x}(kh+h) = \Phi\mathrm{x}(kh) + \frac{1}{h}\Gamma_1\mathrm{u}(kh+h) + \left(\Gamma - \frac{1}{h}\Gamma_1\right)\mathrm{u}(kh)$$

Pulse-transfer function:

$$\mathrm{H}(z) = \mathrm{C}(z\mathrm{I}-\Phi)^{-1}\left(\frac{1}{h}\Gamma_1 z + \Gamma - \frac{1}{h}\Gamma_1\right) + \mathrm{D}$$

Jury's test

Consider characteristic polynomial

$$A(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

With Jury's test it is possible to check whether all poles of the system are inside the unit circle i.e. the system is asymptotically stable.

$$\begin{array}{ccccc} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_n & a_{n-1} & \cdots & a_1 & a_0 \\ \hline a_0^{n-1} & a_1^{n-1} & \cdots & a_{n-1}^{n-1} & \\ a_{n-1}^{n-1} & a_{n-2}^{n-1} & \cdots & a_0^{n-1} & \\ \hline \vdots & & & & \\ a_0^0 & & & & \end{array} \left.\right\} 2n+1 \ \textit{rows total}$$

$$\alpha_n = \frac{a_n}{a_0}$$

$$\alpha_{n-1} = \frac{a_{n-1}^{n-1}}{a_0^{n-1}}$$

$$a_i^{k-1} = a_i^k - \alpha_k a_{k-i}^k$$

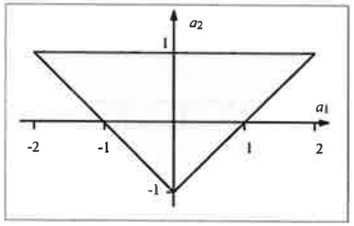
$$a_0^n = a_0,\ \ a_1^n = a_1, \ldots, a_n^n = a_n$$

Triangle rule

A system with a characteristic equation $A(z) = z^2 + a_1 z + a_2$

is stable, if the parameters a_1 ja a_2 fulfill:

$$\begin{cases} a_2 < 1 \\ a_2 > -a_1 - 1 \\ a_2 > a_1 - 1 \end{cases}$$



Gain- and phase margins

The gain- and phase margins are defined in the same way for both continuous- and for discrete-time systems.

Open loop pulse-transfer function is $H(z)$,frequency response $H(e^{i\omega h})$

ω_o is the lowest frequency where $\arg H(e^{i\omega_o h}) = -\pi$

ω_c is the lowest frequency where $\left|H(e^{i\omega_c h})\right| = 1$

Gain margin $A_{marg} = \dfrac{1}{\left|H(e^{i\omega_o h})\right|}$ Phase margin $\phi_{marg} = \pi + \arg H(e^{i\omega_c h})$

Controllability and observability matrices

Controllability matrix $\mathbf{W}_c$ and observability matrix $\mathbf{W}_o$ are defined as follows

$$\mathbf{W}_c = \left[\Gamma \;\vdots\; \Phi\Gamma \;\vdots\; \cdots \;\vdots\; \Phi^{n-1}\Gamma\right]$$

$$\mathbf{W}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\Phi \\ \vdots \\ \mathbf{C}\Phi^{n-1} \end{bmatrix}$$

Canonical forms

A system can be described with a difference equation

$$y(k+n_a) + a_1 y(k+n_a-1) + \cdots + a_{n_a} y(k) = b_0 u(k+n_b) + \cdots + b_{n_b} u(k)$$

with a pulse-transfer operator,

$$H(q) = \frac{b_0 q^{n_b} + b_1 q^{n_b-1} + \cdots + b_{n_b}}{q^{n_a} + a_1 q^{n_a-1} + \cdots + a_{n_a}}$$

with a pulse-transfer function

$$H(z) = \frac{b_0 z^{n_b} + b_1 z^{n_b-1} + \cdots + b_{n_b}}{z^{n_a} + a_1 z^{n_a-1} + \cdots + a_{n_a}}$$

or with a state-space representation.

Canonical forms

Controllable canonical form

$$\mathbf{x}(k+1) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n_a-1} & -a_{n_a} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mathbf{u}(k)$$

$$\mathbf{y}(k) = \begin{bmatrix} b_{n_b-n_a+1} & b_{n_b-n_a+2} & \cdots & b_{n_b-1} & b_{n_b} \end{bmatrix} \mathbf{x}(k)$$

Canonical forms

Observable canonical form

$$\mathbf{x}(k+1) = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n_a-1} & 0 & 0 & \cdots & 1 \\ -a_{n_a} & 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} b_{n_b-n_a+1} \\ b_{n_b-n_a+2} \\ \vdots \\ b_{n_b-1} \\ b_{n_b} \end{bmatrix} \mathbf{u}(k)$$

$$\mathbf{y}(k) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{x}(k)$$

State controller

State-space representation of a system

$$\begin{cases}\mathbf{x}(k+1)=\boldsymbol{\Phi}\,\mathbf{x}(k)+\Gamma\,\mathbf{u}(k)\\ \mathbf{y}(k)=\mathbf{C}\,\mathbf{x}(k)\end{cases}$$

State controller

$$\mathbf{u}(k)=-\mathbf{L}\,\mathbf{x}(k)$$

Characteristic equation of the controlled system

$$\det(z\mathbf{I}-\boldsymbol{\Phi}_{cl})=\det(z\mathbf{I}-\boldsymbol{\Phi}+\Gamma\mathbf{L})=0$$

State estimator

State estimator

$$\begin{aligned}\hat{\mathbf{x}}(k+1)&=\boldsymbol{\Phi}\hat{\mathbf{x}}(k)+\Gamma\mathbf{u}(k)+\mathbf{K}(\mathbf{y}(k)-\mathbf{C}\hat{\mathbf{x}}(k))\\&=(\boldsymbol{\Phi}-\mathbf{KC})\hat{\mathbf{x}}(k)+\Gamma\mathbf{u}(k)+\mathbf{K}\mathbf{y}(k)\end{aligned}$$

Dynamic behavior of the estimation error

$$\tilde{\mathbf{x}}(k+1)=(\boldsymbol{\Phi}-\mathbf{KC})\tilde{\mathbf{x}}(k)=\boldsymbol{\Phi}_o\tilde{\mathbf{x}}(k)$$

$$\det(z\mathbf{I}-\boldsymbol{\Phi}_o)=\det(z\mathbf{I}-\boldsymbol{\Phi}+\mathbf{KC})=0$$

NB. More accurate expression

$$\hat{\mathbf{x}}(k+1|k)=\Phi\hat{\mathbf{x}}(k|k-1)+\Gamma\mathbf{u}(k)+\mathbf{K}(\mathbf{y}(k)-\underbrace{\hat{y}(k|k-1)}_{C\hat{x}(k|k-1)})$$

Alternative state estimator (Luenberger -estimator) and servo controller

Alternative state estimator

$$\hat{\mathbf{x}}(k+1)=\boldsymbol{\Phi}\hat{\mathbf{x}}(k)+\Gamma\mathbf{u}(k)+\mathbf{K}(\mathbf{y}(k+1)-\hat{\mathbf{y}}(k+1))$$

Dynamic behavior of the estimation error

$$\tilde{\mathbf{x}}(k+1)=(\boldsymbol{\Phi}-\mathbf{KC\Phi})\tilde{\mathbf{x}}(k)=\boldsymbol{\Phi}_o\tilde{\mathbf{x}}(k)$$

The above is a reduced-order observer, if $\mathbf{CK}=\mathbf{I}$

NB. More accurate expression

$$\hat{\mathbf{x}}(k+1|k+1)=\Phi\hat{\mathbf{x}}(k|k)+\Gamma\mathbf{u}(k)+\mathbf{K}(\mathbf{y}(k+1)-C\hat{x}(k+1|k))$$

Servo controller $\mathbf{u}(k)=\mathbf{L}_c\mathbf{y}_{REF}(k)-\mathbf{L}\mathbf{x}(k)$

Deterministic polynomial control

Process $A(q)y(k)=B(q)u(k)$

General polynomial controller

$$R(q)u(k)=T(q)y_{REF}(k)-S(q)y(k)$$

Controlled system

$$\big(A(q)R(q)+B(q)S(q)\big)y(k)=\big(B(q)T(q)\big)y_{REF}(k)$$

Adding an integrator

$$R=(q-1)\bar{R}$$

Deterministic polynomial control

Controlled system

$$A_{cl}(q)y(k) = B_{cl}(q)y_{REF}(k)$$

$$\begin{cases} A_{cl}(q) = A_c(q)A_o(q) = A(q)R(q) + B(q)S(q) \\ T(q) = t_0 A_o(q) \end{cases}$$

$\deg A = \deg A_c = n$ $\quad$ $\deg R = \deg S = \deg T = \deg A_o = n-1$

If integrator is added into the controller

$$\deg A = \deg A_c = \deg A_o = \deg R = \deg S = \deg T = n$$

Elimination of zeros and poles

Undesired stable poles and zeros can be eliminated

$$\begin{cases} A(q) = A^+(q)A^-(q) \\ B(q) = B^+(q)B^-(q) \end{cases}$$

$$\begin{cases} R(q) = B^+(q)\bar{R}(q) \\ S(q) = A^+(q)\bar{S}(q) \\ T(q) = A^+(q)\bar{T}(q) \end{cases}$$

$$\bar{A}_{cl}(q) = \bar{A}_c(q)\bar{A}_o(q) = A^-(q)\bar{R}(q) + B^-(q)\bar{S}(q)$$

Deterministic polynomial control, servo problem

Reference model

$$H_m(q) = \frac{B_m(q)}{A_m(q)}$$

Complete model following

$$B_m(q) = \bar{B}_m(q)B^-$$

Controller polynomials

$$\begin{cases} R = A_m B^+ \bar{R} \\ S = A_m A^+ \bar{S} \\ T = \bar{B}_m \bar{A}_0 \bar{A}_c A^+ \end{cases}$$

Approximations of differential equations

Each derivative operator p in the differential equation is replaced with the corresponding function of the shift operator q.

Backward difference approximation

$$p \approx \frac{1-q^{-1}}{h} = \frac{q-1}{qh}$$

Euler's method

$$p \approx \frac{1-q^{-1}}{q^{-1}h} = \frac{q-1}{h}$$

Tustin approximation

$$p \approx \frac{2}{h}\cdot\frac{1-q^{-1}}{1+q^{-1}} = \frac{2}{h}\cdot\frac{q-1}{q+1}$$

Approximations of transfer functions

When an approximation of a transfer function is written (pulse-transfer function), variables s and z are used instead of operators p and q.

Backward difference approximation
$$H_{bd}(z) \approx G\left(\frac{1-z^{-1}}{h}\right) = G\left(\frac{z-1}{zh}\right)$$

Euler's method
$$H_e(z) \approx G\left(\frac{1-z^{-1}}{hz^{-1}}\right) = G\left(\frac{z-1}{h}\right)$$

Tustin approximation
$$H_t(z) \approx G\left(\frac{2}{h}\cdot\frac{1-z^{-1}}{1+z^{-1}}\right) = G\left(\frac{2}{h}\cdot\frac{z-1}{z+1}\right)$$

Tustin approximation with prewarping
$$H_{t,pw}(z) = G\left(\frac{\omega_1}{\tan(\omega_1 h/2)}\cdot\frac{z-1}{z+1}\right)$$

Discrete approximation of a continuous controller

Discrete approximation of a continuous-time PID-controller in the form of a pulse-transfer function

$$U(s) = G_{PID}(s)E(s) = K\left(1+\frac{1}{T_I}\cdot\frac{1}{s}+T_D\cdot s\right)E(s)$$
$$U(z) = H_{PID}(z)E(z) = K\left(1+\frac{1}{\frac{T_I}{h}}\cdot\frac{1}{z-1}+\frac{T_D}{h}\cdot\frac{z-1}{z}\right)E(z)$$

Disturbance models

Definition of a stochastic variable x by a density function p

$$\int_{-\infty}^{\infty} p(x)dx = 1 \qquad E\{x\} = \int_{-\infty}^{\infty} x\cdot p(x)dx$$

$$\operatorname{var}\{x\} = E\{(x-E\{x\})^2\} = \int_{-\infty}^{\infty}(x-E\{x\})^2\cdot p(x)dx$$

For vectors

$$\mathbf{m}(t) = E\{\mathbf{x}(t)\}$$
$$\operatorname{var}\{\mathbf{x}(t)\} = E\left\{(\mathbf{x}(t)-\mathbf{m}(t))(\mathbf{x}(t)-\mathbf{m}(t))^T\right\}$$
$$= E\left\{(\mathbf{x}(t)-E\{\mathbf{x}(t)\})(\mathbf{x}(t)-E\{\mathbf{x}(t)\})^T\right\}$$

Properties of stochastic variables

The basic properties of expectation value and variance are

(a is constant, x and y are stochastic variables)

$$E\{ax\} = aE\{x\}$$
$$E\{x+y\} = E\{x\}+E\{y\}$$
$$E\{a\} = a$$
$$\operatorname{var}\{ax\} = a^2\operatorname{var}\{x\}$$
$$\operatorname{var}\{a\} = 0$$

Additionally, if x and y are independent of each other

$$E\{xy\} = E\{x\}E\{y\}$$
$$\operatorname{var}\{x+y\} = \operatorname{var}\{x\}+\operatorname{var}\{y\}$$

Covariances and spectral densities

Covariance functions (autocovariance and cross-covariance)

$$\mathbf{r}_x(\tau) = \mathbf{r}_{xx}(\tau) = \operatorname{cov}\{\mathbf{x}(t+\tau), \mathbf{x}(t)\} = E\left\{(\mathbf{x}(t+\tau) - \mathbf{m}(t+\tau))(\mathbf{x}(t) - \mathbf{m}(t))^T\right\}$$

$$\mathbf{r}_{xy}(\tau) = \operatorname{cov}\{\mathbf{x}(t+\tau), \mathbf{y}(t)\} = E\left\{(\mathbf{y}(t+\tau) - \mathbf{m}_y(t+\tau))(\mathbf{x}(t) - \mathbf{m}(t))^T\right\}$$

Spectral densities (autospectral- and cross-spectral density)

$$\boldsymbol{\phi}_{xx}(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\mathbf{r}_{xx}(k)e^{-ik\omega} \qquad \mathbf{r}_{xx}(k) = \int_{-\pi}^{\pi} e^{-ik\omega}\boldsymbol{\phi}_{xx}(\omega)d\omega$$

$$\boldsymbol{\phi}_{xy}(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\mathbf{r}_{xy}(k)e^{-ik\omega} \qquad \mathbf{r}_{xy}(k) = \int_{-\pi}^{\pi} e^{-ik\omega}\boldsymbol{\phi}_{xy}(\omega)d\omega$$

Variance determined from autocorrelation $\operatorname{var}\{\mathbf{x}(t)\} = \mathbf{r}_{xx}(0)$

Covariance of sampled data and white noise

Covariance of a finite data set

$$E\{\mathbf{x}\} \approx \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}(i) \Rightarrow$$

$$\mathbf{r}_{xy}(\tau) \approx \frac{1}{N}\sum_{i=1}^{N}(\mathbf{x}(i) - \mathbf{m}_x(i))(\mathbf{y}(i) - \mathbf{m}_y(i))^T$$

White noise

$$r(\tau) = \begin{cases}\sigma^2 & \tau = 0 \\ 0 & \tau \neq 0\end{cases} \qquad \phi(\omega) = \frac{\sigma^2}{2\pi}$$

Noise process in state-space form

For a stochastic process

$$\begin{cases}\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{v}(k) \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k)\end{cases}$$

For the mean value

$$\begin{cases}\mathbf{m}(k+1) = \mathbf{\Phi}\mathbf{m}(k), \quad \mathbf{m}(0) = \mathbf{m}_0 \\ \mathbf{m}_y(k) = \mathbf{C}\mathbf{m}(k)\end{cases}$$

For the covariances

$$\begin{cases}\mathbf{r}_{xx}(k+\tau, k) = \mathbf{\Phi}^{\tau}\mathbf{P}(k) \quad , \quad \tau \geq 0 \\ \mathbf{r}_{yy}(k+\tau, k) = \mathbf{C}\mathbf{r}_{xx}(k+\tau, k)\mathbf{C}^T \\ \mathbf{r}_{yx}(k+\tau, k) = \mathbf{C}\mathbf{r}_{xx}(k+\tau, k) \\ \mathbf{P}(k+1) = \mathbf{\Phi}\mathbf{P}(k)\mathbf{\Phi}^T + \mathbf{R}_1 \quad , \quad \mathbf{P}(0) = \mathbf{R}_0\end{cases}$$

$$\mathbf{P}(k) = \operatorname{cov}\{\mathbf{x}(k), \mathbf{x}(k)\} = E\{\tilde{\mathbf{x}}(k)\tilde{\mathbf{x}}^T(k)\}$$

Noise process in I/O form

A stochastic process can be described with a filter with the pulse-transfer function $H(z)$

Mean $m_y = H(1)m_u$

Spectral density $\phi_y(\omega) = H(e^{i\omega})\phi_u(\omega)H^T(e^{-i\omega})$

Cross-spectral density $\phi_{yu}(\omega) = H(e^{i\omega})\phi_u(\omega)$

About I/O models

ARMAX model

$$A(q)y(k) = B(q)u(k) + C(q)e(k)$$

$$A^*(q^{-1})y(k) = B^*(q^{-1})q^{-d}u(k) + C^*(q^{-1})e(k)$$

Optimization criteria of I/O models

$$J_{mv} = E\{y^2(k)\}$$

$$J_{lq} = E\{y^2(k) + \rho u^2(k)\}$$

Optimal *m*-step predictor

Optimal *m*-step predictor

$$\hat{y}(k+m \mid k) = \frac{qG(q)}{C(q)} y(k)$$

For polynomials F and G holds

$$q^{m-1}C(q) \equiv A(q)F(q) + G(q) \qquad \begin{cases} F(q) = q^{m-1} + f_1 q^{m-2} + \cdots + f_{m-1} \\ G(q) = g_0 q^{n-1} + g_1 q^{n-2} + \cdots + g_{n-1} \end{cases}$$

Estimation error and its variance

$$\tilde{y}(k+m|k) = F(q)e(k+1) \qquad \sigma_{\tilde{y}}^2 = \left(1 + f_1^2 + \cdots + f_{m-1}^2\right)\sigma_e^2$$

Minimum variance controller

Minimum variance controller for systems with stable inverses

$$u(k) = -\frac{G(q)}{B(q)F(q)} y(k)$$

For polynomials F and G holds

$$q^{d-1}C(q) \equiv A(q)F(q) + G(q) \qquad \begin{cases} F(q) = q^{d-1} + f_1 q^{d-2} + \cdots + f_{d-1} \\ G(q) = g_0 q^{n-1} + g_1 q^{n-2} + \cdots + g_{n-1} \end{cases}$$

Output signal and its variance

$$y(k) = e(k) + f_1 e(k-1) + \cdots + f_{d-1} e(k-d+1)$$

$$\sigma_y^2 = \left(1 + f_1^2 + \cdots + f_{d-1}^2\right)\sigma_e^2$$

Tables

Continuous-time transfer function G(s) and its dicrete-time ZOH-equivalent H(q)

$G(s)$	$H(q)=\dfrac{b_1q^{n-1}+b_2q^{n-2}+\cdots+b_n}{q^n+a_1q^{n-1}+\cdots+a_n}$
$\dfrac{1}{s}$	$\dfrac{h}{q-1}$
$\dfrac{1}{s^2}$	$\dfrac{h^2(q+1)}{2(q-1)^2}$
$\dfrac{1}{s^m}$	$\dfrac{q-1}{q}\lim_{a\to 0}\dfrac{(-1)^m}{m!}\dfrac{\partial^m}{\partial a^m}\left(\dfrac{q}{q-e^{-ah}}\right)$
e^{-sh}	q^{-1}
$\dfrac{a}{s+a}$	$\dfrac{1-\exp(-ah)}{q-\exp(-ah)}$
$\dfrac{a}{s(s+a)}$	$b_1=\dfrac{1}{a}\left(ah-1+e^{-ah}\right)$ $b_2=\dfrac{1}{a}\left(1-e^{-ah}-ahe^{-ah}\right)$ $a_1=-\left(1+e^{-ah}\right)$ $a_2=e^{-ah}$
$\dfrac{a^2}{(s+a)^2}$	$b_1=1-e^{-ah}(1+ah)$ $b_2=e^{-ah}\left(e^{-ah}+ah-1\right)$ $a_1=-2e^{-ah}$ $a_2=e^{-2ah}$
$\dfrac{s}{(s+a)^2}$	$\dfrac{(q-1)he^{-ah}}{\left(q-e^{-ah}\right)^2}$
$\dfrac{ab}{(s+a)(s+b)}, a\neq b$	$b_1=\dfrac{b\left(1-e^{-ah}\right)-a\left(1-e^{-bh}\right)}{b-a}$ $b_2=\dfrac{a\left(1-e^{-bh}\right)e^{-ah}-b\left(1-e^{-ah}\right)e^{-bh}}{b-a}$ $a_1=-\left(e^{-ah}+e^{-bh}\right)$ $a_2=e^{-(a+b)h}$

Tables...

$G(s)$	$H(q)$	
$\dfrac{(s+c)}{(s+a)(s+b)}, a\neq b$	$b_1=\dfrac{e^{-bh}-e^{-ah}+\left(1-e^{-bh}\right)c/b-\left(1-e^{-ah}\right)c/a}{a-b}$ $b_2=\dfrac{c}{ab}e^{-(a+b)h}+\dfrac{b-c}{b(a-b)}e^{-ah}+\dfrac{c-a}{a(a-b)}e^{-bh}$ $a_1=-e^{-ah}-e^{-bh}$	$a_2=e^{-(a+b)h}$
$\dfrac{\omega_0^2}{s^2+2\xi\omega_0 s+\omega_0^2}$	$b_1=1-\alpha\left(\beta+\dfrac{\xi\omega_0}{\omega}\gamma\right)$	$\omega=\omega_0\sqrt{1-\xi^2},\ \xi<1$
	$b_2=\alpha^2+\alpha\left(\dfrac{\xi\omega_0}{\omega}\gamma-\beta\right)$	$\alpha=e^{-\xi\omega_0 h}$
	$a_1=-2\alpha\beta$	$\beta=\cos(\omega h)$
	$a_2=\alpha^2$	$\gamma=\sin(\omega h)$
$\dfrac{s}{s^2+2\xi\omega_0 s+\omega_0^2}$	$b_1=\dfrac{1}{\omega}e^{-\xi\omega_0 h}\sin(\omega h)$	$b_2=-b_1$
	$a_1=-2e^{-\xi\omega_0 h}\cos(\omega h)$	$a_2=e^{-2\xi\omega_0 h}$
	$\omega=\omega_0\sqrt{1-\xi^2}$	
$\dfrac{a^2}{s^2+a^2}$	$b_1=1-\cos(ah)$	$b_2=1-\cos(ah)$
	$a_1=-2\cos(ah)$	$a_2=1$
$\dfrac{s}{s^2+a^2}$	$b_1=\dfrac{1}{a}\sin(ah)$	$b_2=-\dfrac{1}{a}\sin(ah)$
	$a_1=-2\cos(ah)$	$a_2=1$
$\dfrac{a}{s^2(s+a)}$	$b_1=\dfrac{1-\alpha}{a^2}+h\left(\dfrac{h}{2}-\dfrac{1}{a}\right)$, $\alpha=e^{-ah}$	
	$b_2=(1-\alpha)\left(\dfrac{h^2}{2}-\dfrac{2}{a^2}\right)+\dfrac{h}{a}(1+\alpha)$	
	$b_3=-\left[\dfrac{1}{a^2}(\alpha-1)+\alpha h\left(\dfrac{h}{2}+\dfrac{1}{a}\right)\right]$	
	$a_1=-(\alpha+2)$ $a_2=2\alpha+1$ $a_3=-\alpha$	

Tables...

Definition: $F(z) = Z\{f(t)\} = \sum_{k=0}^{\infty} f(kh) z^{-k}$	
z-transform	Function of time
$F(z)$	$f(t)$
$C_1F_2(z) + C_2F_2(z)$	$C_1f_2(t) + C_2f_2(t)$
$z^{-n}F(z)$	$q^{-n}f(t)$
$z^n\left(F(z) - \sum_{j=0}^{n-1} f(jh)z^{-j}\right)$	$q^n f(t)$
$F_1(z)F_2(z)$	$\sum_{n=0}^{k} f_1(n) f_2(k-n)$

If the limits for $f(kh)$ and $F(z)$ exist, for them holds

$$\lim_{k\to\infty}\{f(kh)\} = \lim_{z\to 1}\{(1-z^{-1})F(z)\} \qquad f(0) = \lim_{z\to\infty} F(z)$$

z-transform	Function of time
1	$\delta(k)$
$\frac{z}{z-1}$	$1,\ k \geq 0.$
$\frac{hz}{(z-1)^2}$	kh
$\frac{h^2 z(z+1)}{(z-1)^3}$	$(kh)^2$
$\frac{z}{z-a}$	a^k
$\frac{z\sin(\omega h)}{z^2 - 2z\cos(\omega h) + 1}$	$\sin(\omega kh)$

Laplace-transform	Function of time	
1	$\delta(t)$	M1
$1/s$	1	M2
$1/s^2$	t	M3
$1/s^{n+1}$	$t^n/n!$	M4
$\frac{1}{s+a}$	e^{-at}	M5
$\frac{1}{(s+a)^2}$	te^{-at}	M6
$\frac{1}{(s+a)^{n+1}}$	$\frac{t^n e^{-at}}{n!}$	M7
$\frac{1}{s(s+a)}$	$\frac{1}{a}\left(1-e^{-at}\right)$	M8
$\frac{1}{(s+a)(s+b)}$	$\frac{1}{a-b}\left(e^{-bt} - e^{-at}\right)$	M9
$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab} + \frac{1}{ab(b-a)}\left(ae^{-bt} - be^{-at}\right)$	M10
$\frac{a}{s^2+a^2}$	$\sin(at)$	M11
$\frac{s}{s^2+a^2}$	$\cos(at)$	M12
$\frac{a}{(s+b)^2+a^2}$	$e^{-bt}\sin(at)$	M13
$\frac{s+b}{(s+b)^2+a^2}$	$e^{-bt}\cos(at)$	M14
$\frac{s+a}{s+b}$	$\delta(t)+(a-b)e^{-bt}$	M15

LQR optimal control, with the gain to change over time based on the cost

$$J = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} \left\{ x_k^T Q x_k + u_k^T R u_k \right\}$$

which gives:

$$L_k = (\Gamma^T S_{k+1} \Gamma + R)^{-1} \Gamma^T S_{k+1} \Phi$$

$$u_k^* = -L_k x_k$$

$$S_k = (\Phi - \Gamma L_k)^T S_{k+1} (\Phi - \Gamma L_k) + Q + L_k^T R L_k$$

$$J_k^* = \frac{1}{2} x_k^T S_k x_k$$