# Analysis of discrete-time systems



## Stability

- Stability of one solution (non-linear and/or time-varying systems)
- System stability (global property of linear systems)
- Global stability vs. local stability (non-linear systems)
- (General) stability
- Asymptotic stability
- BIBO stability (Bounded Input - Bounded Output)

Stability of linear systems:

A linear, discrete, time-invariant system is *asymptotically stable*, if and only if all the eigenvalues of the system matrix $\mathbf{\Phi}$ are inside the unit circle.



## BIBO-stability vs. Lyapunov-stability

The general solution of the state equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ is

$$\mathbf{x}(t) = \mathbf{e}^{\mathbf{A}(t-t_k)}\mathbf{x}(t_k) + \int_{t_k}^{t} \mathbf{e}^{\mathbf{A}(t-s')}\mathbf{B}\mathbf{u}(s')ds'$$

The behaviour of state **x** in the future depends on two terms: autonomous part (initial conditions) and control inputs.

Lyapunov stability concerns the autonomous part. The initial state is disturbed a bit, and it is investigated how this deviation behaves in the future; no control inputs are used.



## BIBO-stability vs. Lyapunov-stability

BIBO-stability is related to the input/output-behaviour, and it is connected to the second term of the solution. The system is *BIBO-stable* (bounded input-bounded output), if a bounded input **u** leads to a bounded output **y**.

E.g. a stock (integrator) and an ideal oscillator (harmonic oscillator) are marginally stable (generally stable) and Lyapunov stable, but not asymptotically stable nor BIBO-stable.

An inverted pendulum is unstable according to all definitions of stability.

An ideally mixed vessel (low-pass filter) is stable according to all above definitions.

## Marginal, asymptotic and general stability

$$
\begin{tikzcd}
 & \boxed{\text{System}} \arrow[dl] \arrow[dr] & \\
\boxed{\text{Stable}} \arrow[d] \arrow[dr] & & \boxed{\text{Unstable}} \\
\boxed{\text{Asymptotically stable}} & \boxed{\text{Marginally stable}} &
\end{tikzcd}
$$

Often the definitions are simplified by stating that by stability asymptotic stability is meant. (An asymptotically stable system is always BIBO-stable.)



## Stability tests

- Direct calculation of the eigenvalues of **Φ**
- Study of the characteristic polynomial
- Root locus
- Nyquist criterion
- Lyapunov's method

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## The Jury stability criterion

Consider the characteristic polynomial

$$A(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

With Jury's test it is easy to check, whether all the poles are inside the unit circle i.e. whether the system is asymptotically stable

$$
\begin{array}{ccccc}
a_0 & a_1 & \cdots & a_{n-1} & a_n \\
a_n & a_{n-1} & \cdots & a_1 & a_0 \\
\hdashline
a_0^{n-1} & a_1^{n-1} & \cdots & a_{n-1}^{n-1} & \\
a_{n-1}^{n-1} & a_{n-2}^{n-1} & \cdots & a_0^{n-1} & \\
\hdashline
\vdots & & & & \\
a_0^0 & & & &
\end{array}
$$

$$\alpha_n = \frac{a_n}{a_0}$$

$$\alpha_{n-1} = \frac{a_{n-1}^{n-1}}{a_0^{n-1}}$$

$$a_i^{k-1} = a_i^k - \alpha_k a_{k-i}^k, \quad \alpha_k = a_k^k / a_0^k$$

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## The Jury stability criterion

If $a_0 > 0$, all the roots are inside the unit circle, if and only if

$$\forall a_0^k > 0, \quad k = 0,1,\ldots,n-1$$

Consider a couple examples: The characteristic equation:

$$A(z) = 3z^2 + 2z + 1$$

$$
\begin{array}{ccc}
a_0 & a_1 & a_2 \\
a_2 & a_1 & a_0 \\
\hdashline
a_0^1 & a_1^1 & \\
a_1^1 & a_0^1 & \\
\hdashline
a_0^0 & &
\end{array}
\qquad
\begin{array}{cccc}
3 & 2 & 1 & \alpha_2 = \tfrac{1}{3} \\
1 \to (\tfrac{1}{3}) & 2 \to (\tfrac{2}{3}) & 3 \to (1) & \\
\hdashline
(3 - \tfrac{1}{3} =)\tfrac{8}{3} & (2 - \tfrac{2}{3} =)\tfrac{4}{3} & & \alpha_1 = \tfrac{1}{2} \\
\tfrac{4}{3} \to (\tfrac{2}{3}) & \tfrac{8}{3} \to (\tfrac{4}{3}) & & \\
\hdashline
(\tfrac{8}{3} - \tfrac{2}{3} =)2 & & &
\end{array}
$$

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## The Jury stability criterion

$$a_0 > 0, \quad a_0^1 > 0, \quad a_0^0 > 0$$

so that all roots are inside the unit circle and the system is stable (in fact the roots can easily be solved directly; $-0.3333 \pm 0.4714i$).

The Jury table is formed as follows: The last term of the first row is divided by the first term. The second row is multiplied by this factor.
The second row is then subtracted from the first row, which gives the third row. The fourth row is obtained by changing the row vector of the third row upside down. Again a factor is formed by dividing the last term in the third row by the first one. The procedure then continues as described above.



## The Jury stability criterion

The eigenvalues and roots of polynomials can most conveniently be calculated by using numerical routines and available software. The
Jury method on the other hand can easily be used in the case of a small system, when a computer is not at hand.

The real power of the Jury criterion comes into action in symbolic calculations. Stability can then be determined as a function of one or even more parameters. Consider the following example $A(z) = z^2 + a_1 z + a_2$

The characteristic polynomial:



## The Jury stability criterion

Form the Jury table

$$\begin{array}{ccc|ccc|l}
a_0 & a_1 & a_2 & 1 & a_1 & a_2 & \alpha_2 = a_2 \\
a_2 & a_1 & a_0 & a_2 & a_1 & 1 & \\
\hline
a_0^1 & a_1^1 & & 1-(a_2)^2 & a_1(1-a_2) & & \alpha_1 = \dfrac{a_1}{1+a_2} \\
a_1^1 & a_0^1 & & a_1(1-a_2) & 1-(a_2)^2 & & \\
\hline
a_0^0 & & & 1-(a_2)^2 - \dfrac{(a_1)^2(1-a_2)}{1+a_2} & & &
\end{array}$$

The system is stable,if
$$\begin{cases} 1-(a_2)^2 > 0 \\ 1-(a_2)^2 - \dfrac{(a_1)^2(1-a_2)}{1+a_2} > 0 \end{cases}$$



## The Jury stability criterion

The Jury stability criterion

$$1-(a_2)^2 - \frac{(a_1)^2(1-a_2)}{1+a_2} = (1+a_2)(1-a_2) - \frac{(a_1)^2(1-a_2)}{1+a_2}$$

$$= (1-a_2)\left((1+a_2) - \frac{(a_1)^2}{1+a_2}\right) = (1-a_2)\left(\frac{(1+a_2)^2}{1+a_2} - \frac{(a_1)^2}{1+a_2}\right)$$

$$= (1-a_2)\left(\frac{(1+a_2)^2 - (a_1)^2}{1+a_2}\right) = \frac{1-a_2}{1+a_2}\left((1+a_2)^2 - (a_1)^2\right)$$

$$= \frac{(1-a_2)(1+a_2+a_1)(1+a_2-a_1)}{1+a_2} > 0$$

## The Jury stability criterion

The stability criteria thus become

$$\begin{cases}(1-a_2)(1+a_2)>0\\ \dfrac{(1-a_2)(1+a_2+a_1)(1+a_2-a_1)}{1+a_2}>0\end{cases}$$

$$\Rightarrow \begin{cases}(1-a_2)(1+a_2)>0\\ (1+a_2+a_1)(1+a_2-a_1)>0\end{cases}$$

$$\Rightarrow \begin{cases}-1<a_2<1\\ a_2>-a_1-1\\ a_2>a_1-1\end{cases} \vee \begin{cases}-1<a_2<1\\ a_2<-a_1-1\\ a_2<a_1-1\end{cases}$$

The second group is not fulfilled with any values of $\alpha_1$ and $\alpha_2$

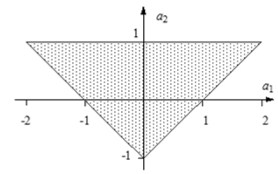


## The Jury stability criterion

System with the characteristic polynomial $A(z)=z^2+a_1z+a_2$

is stable for the values $\alpha_1$ and $\alpha_2$ such that:

$$\begin{cases}a_2<1\\ a_2>-a_1-1\\ a_2>a_1-1\end{cases}$$






## Stability in frequency domain

The frequency response of $G(s)$ is $G(i\omega)$, $\omega \in [0, \infty[$ . It can graphically be presented in the complex plane as the Nyquist curve or as amplitude/phase curves as a function of frequency (Bode diagram).

Correspondingly, for a discrete system $H(z)$ the frequency response is $H(e^{i\omega h})$, $\omega h \in [0, \pi]$ . This can also be presented graphically as a discrete Nyquist or discrete Bode diagram.

The difference is, that in the discrete case only the frequency interval $\omega h \in [-\pi, \pi[$ is considered.



## Stability in frequency domain

The continuous system $G(s)=\dfrac{1}{s^2+1.4s+1}$

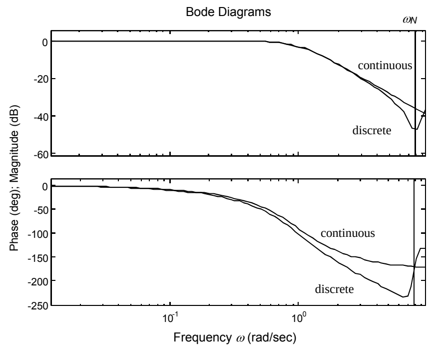
is sampled, ($h$ = 0.4), giving the discrete ZOH-equivalent

$$H(z)=\frac{0.066z+0.055}{z^2-1.450z+0.571}$$

Compare the continuous frequency response $G(i\omega)$ with the discrete one $H(e^{i\omega h})$, in the frequency range $\omega h \in [0, \pi]$ .

## Stability in frequency domain

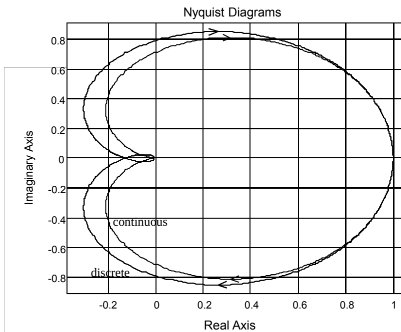
```
sys=tf(1,[1 1.4 1])
Transfer function:
       1
--------------
s^2 + 1.4 s + 1

sysd=c2d(sys,0.4)

Transfer function:
 0.06609 z + 0.05481
--------------------
z^2 - 1.45 z + 0.5712

 Sampling time: 0.4

w=logspace(-2,1,1000)';
 bode(sys,w)
 hold
 bode(sysd,w)
```






## Stability in frequency domain

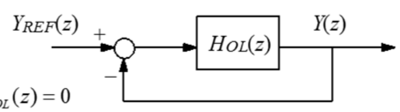
```
nyquist(sys,w)
hold
nyquist(sysd,w)
```






## Discrete Nyquist stability criterion

Discrete control system




Characteristic equation $1+H_{OL}(z)=0$

Stability can be determined by using the open loop $H_{OL}(z)$ Nyquist diagram. $H_{OL}(e^{i\omega h})$ (open loop Nyquist curve) encircles the point -1 $N$ times clockwise.

$$N = Z - P$$

in which $Z$ is the number of zeros and $P$ the number of poles of the characteristic equation **outside the unit circle**.



## Discrete Nyquist stability criterion

This fact can be applied in stability analysis. The characteristic equation (CE) has the form

$$1+\frac{num_{OL}(z)}{den_{OL}(z)} = \frac{den_{OL}(z)+num_{OL}(z)}{den_{OL}(z)} = \frac{num_{CE}(z)}{den_{CE}(z)} = 0$$

The open loop (OL) poles are the same as the poles of the characteristic equation. The **zeros** of the characteristic equation determine stability so that if the characteristic equation has zeros outside the unit circle, the closed loop system is unstable. The stability criterion is thus obtained by setting $Z$= 0 and by demanding that the Nyquist curve encircles point -1 $P$ times *counterclockwise*. ($Z$=$N$+$P$=0)

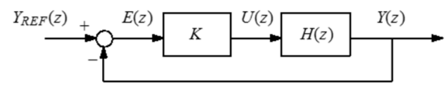
## Discrete Nyquist stability criterion

The criterion becomes simple, if the open loop pulse transfer function has no poles outside the unit circle. Then the Nyquist curve must not encircle the point –1 at all.



## Discrete Nyquist stability criterion

A process is controlled with a discrete P-controller, which has the gain *K* (*h=1*)



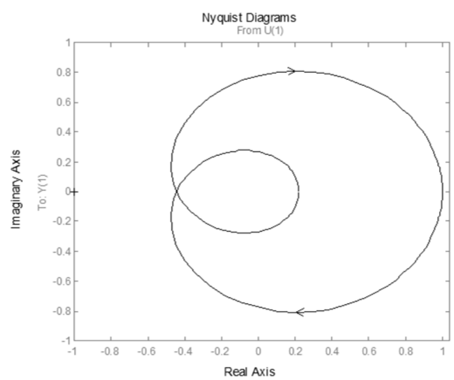

$$H(z) = \frac{0.4}{(z-0.5)(z-0.2)}$$

The discrete Nyquist diagram is constructed with Matlab

```
» sysd=zpk([],[0.2 0.5],0.4,1);

» nyquist(sysd)
```



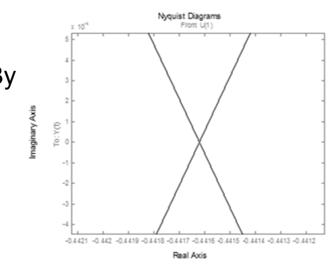
## Discrete Nyquist stability criterion






## Discrete Nyquist stability criterion

The interception point in the real axis can be found e.g. by using the *zoom*-command. By inspection, the point is approximately -0.4416




The magnitude can thus be multiplied with (1/0.4416) to reach the critical point -1.

The controlled system is stable when $K < \frac{1}{0.4416} \approx 2.26$

## Discrete Nyquist stability criterion

Stability can also be determined by direct calculus from the pulse transfer function

$$H(z) = \frac{0.4}{(z-0.5)(z-0.2)}$$

Substitute $z$ with $e^{i\omega h} = e^{i\omega} = \cos(\omega)+i\sin(\omega)$ ,(Euler formula), which gives the frequency response $H(e^{i\omega h})$

$$H(e^{i\omega}) = \frac{0.4}{(e^{i\omega}-0.5)(e^{i\omega}-0.2)}$$
$$= \frac{0.4}{(\cos\omega + i\sin\omega - 0.5)(\cos\omega + i\sin\omega - 0.2)}$$

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## Discrete Nyquist stability criterion

$$= \frac{0.4}{(\cos^2\omega - \sin^2\omega - 0.7\cos\omega + 0.1) + (2\sin\omega\cos\omega - 0.7\sin\omega)i}$$

$$= \frac{0.4}{(2\cos^2\omega - 0.7\cos\omega - 0.9) + (2\sin\omega\cos\omega - 0.7\sin\omega)i}$$

Setting the imaginary part 0 the interception point with the real axis is obtained

$$2\sin\omega\cos\omega - 0.7\sin\omega = 0 \quad\Rightarrow\quad \sin\omega(0.7 - 2\cos\omega) = 0$$
$$\Rightarrow\quad (\sin\omega = 0)\vee(0.7 - 2\cos\omega = 0) \quad\Rightarrow\quad (\sin\omega = 0)\vee(\cos\omega = \tfrac{7}{20})$$
$$\Rightarrow\quad (\omega = 0)\vee(\omega = \arccos\tfrac{7}{20})$$

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## Discrete Nyquist stability criterion

The frequency 0 describes the start point in the Nyquist curve and the frequency arccos(7/20) the interception point with the real axis. Substitute it to the frequency response function

$$H(e^{i\arccos\frac{7}{20}}) = \frac{0.4}{2(\frac{7}{20})^2 - 0.7(\frac{7}{20}) - 0.9} = \frac{-4}{9} \approx -0.444$$

The interception point is -0.444. The gain of the controller $K$ can be multiplied by the factor (1/0.444) in order the crossing at point –1 to take place. The controlled system is stable,when

$$K < \tfrac{9}{4} = 2.25$$

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## Symbolic frequency response calculated with the MATLAB symbolic toolbox

```
f='(4/10)/((z-(1/2))*(z-(2/10)))'
z='cos(w)+i*sin(w)'
g=subs(f,z)
g =

(4/10)/((cos(w)+i*sin(w)-(1/2))*(cos(w)+i*sin(w)-(2/10)))

g2=simplify(g)
g2 =

4/(20*cos(w)^2+20*i*cos(w)*sin(w)-7*cos(w)-9-7*i*sin(w))
```

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## Gain and Phase margins

Definitions of gain and phase margins are identical to those of the continuous time systems.

Open loop pulse transfer function $H(z)$

$\omega_o$ is the lowest frequency, for which $\arg H(e^{i\omega_o h}) = -\pi$

$\omega_c$ is the lowest frequency, for which $|H(e^{i\omega_c h})| = 1$

Gain margin $A_{marg} = \dfrac{1}{|H(e^{i\omega_o h})|}$ Phase margin $\phi_{marg} = \pi + \arg H(e^{i\omega_c h})$

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## Reachability, observability

Principal questions:
* How can any state be transferred into any other state ?
* How can a state be determined from observations ?

Consider the state-space realization $\begin{cases} \mathbf{x}(k+1) = \mathbf{\Phi}\,\mathbf{x}(k) + \mathbf{\Gamma}\,\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \end{cases}, \quad \mathbf{x}(0) = \mathbf{x}_0$

The solution at the time instant $n$ ($n$ is the dimension of the system or in other words the number of state components) is

$$\mathbf{x}(n) = \mathbf{\Phi}^n\,\mathbf{x}_0 + \mathbf{\Phi}^{n-1}\mathbf{\Gamma}\,\mathbf{u}(0) + \cdots + \mathbf{\Gamma}\,\mathbf{u}(n-1) = \mathbf{\Phi}^n\,\mathbf{x}_0 + \mathbf{W}_c\mathbf{U}$$

$$\mathbf{W}_c = \left[\mathbf{\Gamma} \;\vdots\; \mathbf{\Phi\Gamma} \;\vdots\; \cdots \;\vdots\; \mathbf{\Phi}^{n-1}\mathbf{\Gamma}\right]$$

$$\mathbf{U} = \left[\mathbf{u}^T(n-1) \;\vdots\; \mathbf{u}^T(n-2) \;\vdots\; \cdots \;\vdots\; \mathbf{u}^T(0)\right]^T$$

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## Reachability, observability

If the rank of the matrix ***W***c is $n$, then $n$ linear equations are obtained, from which the controls ***U*** can be calculated, which drive the system into any desired final state.

The system is *controllable*, if it is possible to find a control sequence, which drives the system to the origin from any state in a finite time interval.

The system is *reachable*, if it is possible to find a control sequence, which drives the system from any state to any state in a finite time interval. The system is *stabilizable*, if non-controllable states are asymptotically stable.

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## Reachability, observability

Reachability is a stronger property than controllability. For example, if $\mathbf{\Phi}^n = 0$, the state will go to the origin without any controls, so that the system is controllable while not necessarily reachable.
The system is reachable, if and only if the rank of ***W***c is $n$ (***W***c is the *controllability matrix*)

Ex. System $\mathbf{x}(k+1) = \mathbf{\Phi}\,\mathbf{x}(k) + \mathbf{\Gamma}\,\mathbf{u}(k), \quad \mathbf{x}(0) = [2 \quad -1]^T$

is planned to be driven to i. origin $\boldsymbol{x} = [0 \quad 0]^T$
ii. state $\boldsymbol{x} = [0 \quad 3]^T$
iii. state $\boldsymbol{x} = [-1 \quad 2]^T$

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## Reachability, observability

$$\mathbf{\Phi} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{\Gamma} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Consider the reachability

$$\mathbf{W}_c = [\mathbf{\Gamma} \mid \mathbf{\Phi\Gamma}] = \left[ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \middle| \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] = \left[ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For a square matrix the rank can be investigated by the determinant $\det \mathbf{W}_c = 1 \neq 0$ ,the rank is full, so the system is reachable and thus controllable also

$$\operatorname{rank} \mathbf{W}_c = 2 = n$$

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## Reachability, observability

The original state can be driven to any other state with ($n$ = 2) steps at the maximum.

$$\mathbf{x}(0) = [2 \quad -1]^T$$

$$\mathbf{x}(1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x}(0) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(0) = \begin{bmatrix} u(0) \\ x_1(0) \end{bmatrix} = \begin{bmatrix} u(0) \\ 2 \end{bmatrix}$$

$$\mathbf{x}(2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x}(1) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(1) = \begin{bmatrix} u(1) \\ x_1(1) \end{bmatrix} = \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

i. $\boldsymbol{x}(2) = [0 \quad 0]^T$ => $\boldsymbol{U} = [u(0) \quad u(1)]^T = [0 \quad 0]^T$
ii. $\boldsymbol{x}(2) = [0 \quad 3]^T$ => $\boldsymbol{U} = [u(0) \quad u(1)]^T = [3 \quad 0]^T$
iii. $\boldsymbol{x}(2) = [-1 \quad 2]^T$ => $\boldsymbol{U} = [u(0) \quad u(1)]^T = [2 \quad -1]^T$

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## Reachability, observability

$$\mathbf{\Phi} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{\Gamma} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Consider reachability

$$\mathbf{W}_c = [\mathbf{\Gamma} \mid \mathbf{\Phi\Gamma}] = \left[ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \middle| \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \left[ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

the rank is not full, so that the system is not reachable. Nothing can be said about controllability by this analysis.

$$\det \mathbf{W}_c = 0 \qquad \operatorname{rank} \mathbf{W}_c \neq n$$

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## Reachability, observability

Let us check how the state behaves $\mathbf{x}(0) = [2 \quad -1]^T$

$$\mathbf{x}(1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x}(0) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(0) = \begin{bmatrix} 0 \\ x_1(0) + u(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 + u(0) \end{bmatrix}$$

$$\mathbf{x}(2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x}(1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(1) = \begin{bmatrix} 0 \\ u(1) \end{bmatrix}$$

The first state cannot be influenced so that it is not reachable. The origin can be reached and the system is controllable. The last state (iii) cannot be reached by any control sequence.

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## Reachability, observability

On the other hand, both the origin (i) and the second state (ii) can be reached even with one control step

i. $\boldsymbol{x}(1) = [0 \;\; 0]^T$ => $\boldsymbol{U} = u(0) = -2$
ii. $\boldsymbol{x}(1) = [0 \;\; 3]^T$ => $\boldsymbol{U} = u(0) = 1$
iii. $\boldsymbol{x}(1) = [-1 \;\; 2]^T$ => Not reachable

If the aim is to reach the desired states only after two steps, the first control step can be arbitrary

i. $\boldsymbol{x}(2) = [0 \;\; 0]^T$ => $\boldsymbol{U} = [u(0)\; u(1)]^T = [* \;\; 0]^T$
ii. $\boldsymbol{x}(2) = [0 \;\; 3]^T$ => $\boldsymbol{U} = [u(0)\; u(1)]^T = [* \;\; 3]^T$
iii. $\boldsymbol{x}(2) = [-1 \;\; 2]^T$ => Not reachable



## Reachability, observability

Consider $\begin{cases} \mathbf{x}(k+1) = \mathbf{\Phi}\,\mathbf{x}(k) + \mathbf{\Gamma}\,\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \end{cases}, \quad \mathbf{x}(0) = \mathbf{x}_0$

Let the output signal $\mathbf{y}$ and control signal $\mathbf{u}$ be known from previous time instants. Based on this the aim is to find $\mathbf{x}_0$. Consider the solution for $\mathbf{y}$.

$$\mathbf{y}(0) = \mathbf{C}\mathbf{x}(0) = \mathbf{C}\mathbf{x}_0$$

$$\mathbf{y}(1) = \mathbf{C}\mathbf{x}(1) = \mathbf{C}\big(\mathbf{\Phi}\mathbf{x}_0 + \mathbf{\Gamma}\,\mathbf{u}(0)\big) = \mathbf{C\Phi}\mathbf{x}_0 + \mathbf{C\Gamma}\,\mathbf{u}(0)$$

$$\mathbf{y}(2) = \mathbf{C}\mathbf{x}(2) = \mathbf{C}\big(\mathbf{\Phi}\mathbf{x}(1) + \mathbf{\Gamma}\,\mathbf{u}(1)\big) = \mathbf{C\Phi}^2\mathbf{x}_0 + \mathbf{C\Phi\Gamma}\,\mathbf{u}(0) + \mathbf{C\Gamma}\,\mathbf{u}(1)$$

$$\vdots$$

$$\mathbf{y}(n-1) = \mathbf{C\Phi}^{n-1}\mathbf{x}_0 + \mathbf{C}\sum_{i=1}^{n-1}\mathbf{\Phi}^{n-1-i}\mathbf{\Gamma}\,\mathbf{u}(i-1)$$



## Reachability, observability

Because the control $\mathbf{u}$ is known at time instants $k = 0 \ldots n-1$, the determination of $\mathbf{x}_0$ does not depend on the weighted sum of controls. The formula of $\mathbf{y}$ can be divided into two parts; one depending on the initial condition $\mathbf{y}_x$ and one depending on controls $\mathbf{y}_u$.

$$\mathbf{y}(n-1) = \mathbf{y}_x(n-1) + \mathbf{y}_u(n-1) = \mathbf{C\Phi}^{n-1}\mathbf{x}_0 + \mathbf{C}\sum_{i=1}^{n-1}\mathbf{\Phi}^{n-1-i}\mathbf{\Gamma}\,\mathbf{u}(i-1)$$

The initial condition is found if $\mathbf{y}_x$ can be solved at different time instants.
Putting the equations at different time instants together gives

$$\mathbf{y}_x(n-1) = \mathbf{C\Phi}^{n-1}\mathbf{x}_0$$



## Reachability, observability

$$\mathbf{Y}_x = \begin{bmatrix} \mathbf{y}(0) \\ \mathbf{y}(1) \\ \vdots \\ \mathbf{y}_x(n-1) \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C\Phi} \\ \vdots \\ \mathbf{C\Phi}^{n-1} \end{bmatrix} \mathbf{x}_0 = \mathbf{W}_o\mathbf{x}_0$$

The initial condition $\mathbf{x}_0$ can be calculated,if $\mathbf{W}_o$ is of full rank.

The system is *observable*, if and only if the rank of $\mathbf{W}_o$ is *n.*

The system is *detectable*, if non-observable states are stable.

## Canonical forms

The same system can be described by the difference equation

$$y(k+n_a)+a_1 y(k+n_a-1)+\dots+a_{n_a} y(k)=b_0 u(k+n_b)+\dots+b_{n_b} u(k)$$

or by the pulse transfer operator or by the pulse transfer function

$$H(q)=\frac{b_0 q^{n_b}+b_1 q^{n_b-1}+\dots+b_{n_b}}{q^{n_a}+a_1 q^{n_a-1}+\dots+a_{n_a}} \quad H(z)=\frac{b_0 z^{n_b}+b_1 z^{n_b-1}+\dots+b_{n_b}}{z^{n_a}+a_1 z^{n_a-1}+\dots+a_{n_a}}$$

or by the state-space representation. The last alternative is not unique, since it is possible to form an indefinite number of state-space representations, which give the same input-output behaviour (e.g. diagonal form or Jordan form)



## Canonical forms

With respect to reachability and observability the most important forms are the *controllable canonical form* and the *observable canonical form*.
Controllable canonical form:

$$\mathbf{x}(k+1)=\begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n_a-1} & -a_{n_a} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}\mathbf{x}(k)+\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\mathbf{u}(k)$$

$$\mathbf{y}(k)=\begin{bmatrix} b_{n_b-n_a+1} & b_{n_b-n_a+2} & \cdots & b_{n_b-1} & b_{n_b} \end{bmatrix}\mathbf{x}(k)$$



## Canonical forms

The observable canonical form:

$$\mathbf{x}(k+1)=\begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n_a-1} & 0 & 0 & \cdots & 1 \\ -a_{n_a} & 0 & 0 & \cdots & 0 \end{bmatrix}\mathbf{x}(k)+\begin{bmatrix} b_{n_b-n_a+1} \\ b_{n_b-n_a+2} \\ \vdots \\ b_{n_b-1} \\ b_{n_b} \end{bmatrix}\mathbf{u}(k)$$

$$\mathbf{y}(k)=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}\mathbf{x}(k)$$

Both of these have an alternative version, in which the states have been chosen in the reverse order.



## Canonical forms

The above forms are as such valid only for strictly proper systems, but by small modifications they can be modified also in the case that the **D**-matrix is non-zero. However, the state-space representation must always be *causal*, i.e. $n_a \geq n_b$.

If $n_b$ is much smaller than $n_a$ ,the formulas will point to coefficients of the *B*-polynomial, which do not exist (e.g. $b_{-1}$ , $b_{-2}$ , …). These can always be set to zero.

$$\dots+\underbrace{b_{-2}}_{0}u(k+n_b+2)+\underbrace{b_{-1}}_{0}u(k+n_b+1)+b_0 u(k+n_b)+\dots+b_{n_b}u(k)$$

## Canonical forms

Develop controllable canonical forms for the given pulse transfer functions:

$$H_a(z)=\frac{2z+1}{z^2+2z+1},\quad n_b=1,\ n_a=2$$

$$a_1=2, a_2=1, b_0=2, b_1=1$$

$$\mathbf{x}(k+1)=\begin{bmatrix}-2 & -1\\ 1 & 0\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}1\\0\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix}2 & 1\end{bmatrix}\mathbf{x}(k)$$

$$H_b(z)=\frac{1}{z^2+2z+1},\quad n_b=0,\ n_a=2$$

$$a_1=2, a_2=1, b_0=1$$

$$\mathbf{x}(k+1)=\begin{bmatrix}-2 & -1\\ 1 & 0\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}1\\0\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix}0 & 1\end{bmatrix}\mathbf{x}(k)$$



## Non-reachable and/or non-observable systems

There may be several reasons, why a discrete system is not reachable or observable:

* The original continuous system (which is then sampled) is not reachable or observable.

* Hidden oscillations (sampling frequency too low)

* Pole-zero cancellation. Reachability is lost, if sampling leads to a system with a common pole and zero. The sampling interval must be changed.



## Analysis of simple control loops

Control problems:

* Regulator problem
  Setpoint is constant

* Combination of regulator and servo problems
  e.g. several but rare step changes in the setpoint.

* Servo problem
  The changing setpoint trajectory must be followed.



## Analysis of simple control loops

Classification of disturbances:

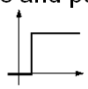
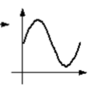
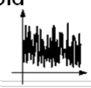
* Load disturbance
  Influence on control variables, often stepwise and change the long term average (low frequencies)

* Measurement noise
  Often high-frequency noise caused by measurement devices.

* Parameter changes
  System parameters change with time.

## Analysis of simple control loops

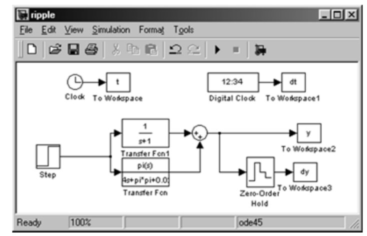
Typical disturbance models, which are used in system analysis

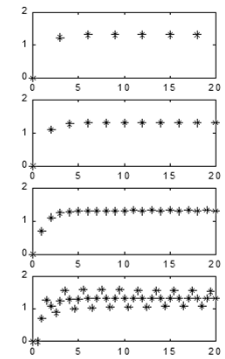
* Impulse and pulse 
* Step 
* Ramp 
* Sinusoid 
* Noise 



## Hidden oscillations

The following model is simulated and the response is sampled with different sampling intervals ($h$ = 3, 2, 1, 0.5)







## Hidden oscillations

What is really happening in the system, can be seen from the figures.

This is an example of hidden oscillations or ripple.