Gradientti, divergenssi, roottori

$$\nabla F(\bar r) = \bar u_x \frac{\partial F}{\partial x} + \bar u_y \frac{\partial F}{\partial y} + \bar u_z \frac{\partial F}{\partial z}$$

$$\nabla \cdot \bar G(\bar r) = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}$$

$$\nabla \times \bar H(\bar r) = \begin{vmatrix} \bar u_x & \bar u_y & \bar u_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix}$$

$$\nabla^2 \phi(\bar r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi(x,y,z)$$

$$\nabla \times (\nabla \times \bar H) = \nabla(\nabla \cdot \bar H) - \underbrace{(\nabla \cdot \nabla)}_{\nabla^2} \bar H$$

$$\Rightarrow \nabla^2 \bar H = \nabla(\nabla \cdot \bar H) - \nabla \times (\nabla \times \bar H)$$

$$\bar F(\bar r) = \frac{\bar u_\varphi}{\rho} = \frac{1}{\rho} \bar u_\varphi$$

$$= \frac{1}{\rho} \begin{vmatrix} \bar u_\rho & \rho \bar u_\varphi & \bar u_z \\ \partial/\partial \rho & \partial/\partial \varphi & \partial/\partial z \\ 0 & \rho \cdot \frac{1}{\rho} & 0 \end{vmatrix}$$

$$= 0$$

(jos vain $\rho \neq 0$)

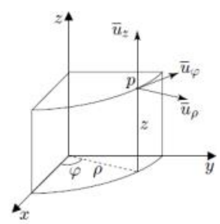
Sylinterikoordinaatisto

$$\nabla f(\bar r) = \bar u_\rho \frac{\partial}{\partial \rho} f + \bar u_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi} f + \bar u_z \frac{\partial}{\partial z} f$$

$$\nabla \times \bar f = \frac{1}{\rho} \begin{vmatrix} \bar u_\rho & \rho \bar u_\varphi & \bar u_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ f_\rho & \rho f_\varphi & f_z \end{vmatrix}$$

$$\nabla \cdot \bar f = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho f_\rho) + \frac{1}{\rho} \frac{\partial}{\partial \varphi} f_\varphi + \frac{\partial}{\partial z} f_z$$

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$



Esimerkkejä:

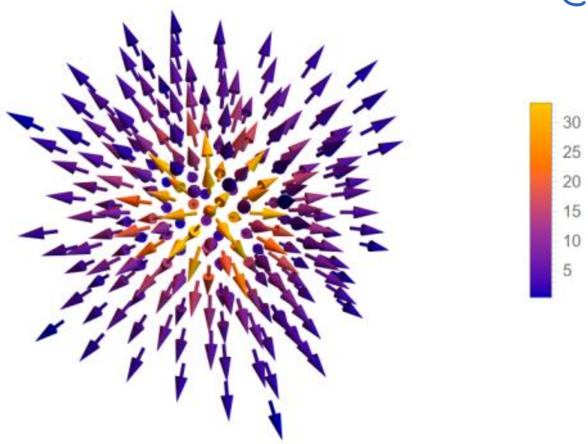
Pistevarauksen sähkökenttä

$$Q. \rightarrow \bar E(\bar r) = \frac{Q}{4\pi\varepsilon_0 r^2} \bar u_r$$

$$\begin{cases} \nabla \times \bar E = 0 \\ \nabla \cdot \bar D = \rho \end{cases}$$

TYHJIÖ:

$$\bar D(\bar r) = \varepsilon_0 \bar E(\bar r)$$



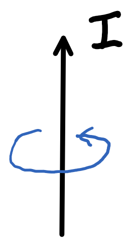
$$\left\{ \nabla\cdot\bar D = \rho \right. \qquad \bar D(\bar r) = \varepsilon_0 \bar E(\bar r)$$

ρ: $\frac{As}{m^3}$ $\qquad$ $\bar D$: $\frac{As}{m^2}$ $\qquad$ $\bar E$: V/m

$$\rho = 0 \Rightarrow \nabla\cdot\bar E = 0$$

$$\nabla\cdot\left(\frac{\bar u_r}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\overbrace{r^2\cdot\frac{1}{r^2}}^{1}\right) = 0 \qquad r\neq 0$$

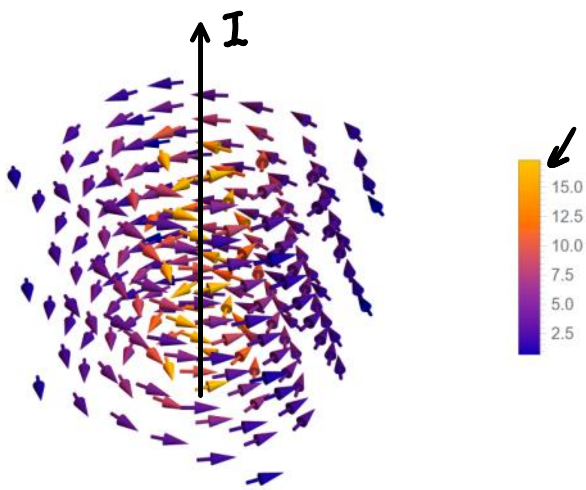
Suoran virtalangan magneettikenttä



$$\bar H(\bar r) = \frac{I}{2\pi\rho}\bar u_\varphi$$

$$\nabla\cdot\left(\frac{\bar u_\varphi}{\rho}\right) = 0$$

$$\nabla\times\left(\frac{\bar u_\varphi}{\rho}\right) = 0 \qquad (\rho\neq 0)$$

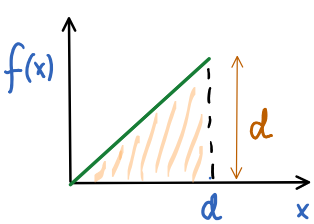


$$\nabla\cdot\bar B = 0$$

$$\nabla\times\bar H = \bar J \qquad \bar B(\bar r) = \mu_0 \bar H(\bar r)$$

$\bar J$: $\frac{A}{m^2}$ $\qquad$ $\bar B$: $\frac{Vs}{m^2}$ $\qquad$ $\bar H$: $\frac{A}{m}$

INTEGRAALILAUSEITA



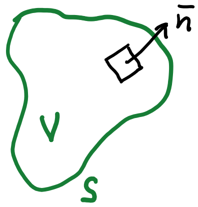
$$f(x) = x \qquad \int_0^d f(x)\,dx = \int_0^d x\,dx = \Big/_0^d \underbrace{\tfrac{1}{2}x^2}_{g(x)} = \frac{d^2}{2} - 0$$

$$g'(x) = f(x)$$

$$\int_a^b g'(x)\,dx = g(b) - g(a)$$

Gaussin lause (divergenssiteoreema, Ostrogradskin lause)

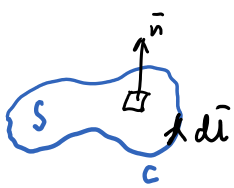
(3-ULOTTEINEN TILAVUUS V)



$$\int_V \nabla \cdot \bar{D}\, dV = \oint_S \bar{D} \cdot \underbrace{d\bar{S}}_{\bar{n}\, dS}$$

Stokesin lause

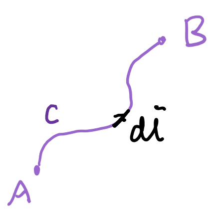
(2-ULOTTEINEN PINTA S - VOI OLLA KAAREVA)



$$\int_S \nabla \times \bar{G} \cdot d\bar{S} = \oint_C \bar{G} \cdot d\bar{l}$$

Gradienttilause

(1-ULOTTEINEN KÄYRÄ C)



$$\int_C \nabla \phi \cdot d\bar{l} = \phi(B) - \phi(A)$$

Esimerkkejä integraalilauseiden käytöstä sähkötekniikasta:

$Q_1 \qquad Q_2$

$$F \cong \frac{Q_1 Q_2}{r^2}$$

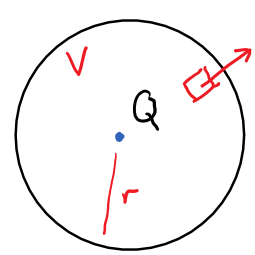
Coulombin laki

$$\bar{E}(\bar{r}) = \frac{Q}{4\pi\varepsilon_0 r^2}\,\bar{u}_r$$

$Q \cdot \; r \quad \varepsilon_0$

$$\bar{D}(\bar{r}) = \varepsilon_0 \bar{E}(\bar{r})$$

$$\nabla \cdot \bar{D} = \rho$$



$\bar{u}_r$

$$d\bar{S} = \bar{n}\, dS$$

$(r, \cancel{\theta}, \cancel{\varphi})$

$$\int_V \nabla \cdot \bar{D}\, dV = \oint_S \bar{D} \cdot d\bar{S}$$

$$= \int_V \rho\, dV$$

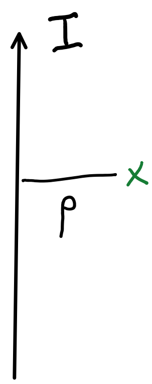
$$\bar{D}(\bar{r}) = \bar{D}(r) = \bar{u}_r D(r)$$

$$= \int_V \ldots \qquad \bar D(\bar r) = D(r) = \bar u_r D(r)$$

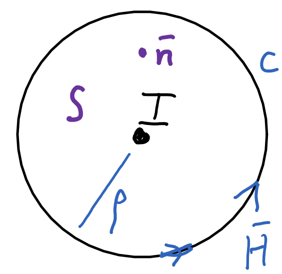
$$= Q \qquad = \oint_S D(r)\,dS = D(r)\oint_S dS = D(r)\,4\pi r^2$$

$$\bar D(\bar r) = \bar u_r \frac{Q}{4\pi r^2} \quad \Rightarrow \quad \bar E(\bar r) = \bar u_r \frac{Q}{4\pi\varepsilon_0 r^2}$$

Suoran tasavirtalangan magneettikenttä



$$\bar H(\bar r) = \bar u_\varphi \frac{I}{2\pi\rho}$$



$$d\bar l = \bar u_\varphi\, \rho\, d\varphi$$

$$\nabla\times\bar H = \bar J$$

$$\int_S \nabla\times\bar H\cdot \underbrace{d\bar S}_{\bar n\,dS} = \oint_C \bar H\cdot d\bar l$$

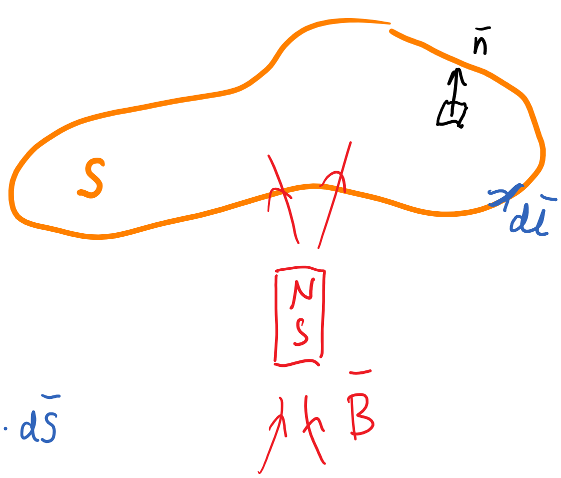
$$\bar n \uparrow \bar u_z$$

$$= \int_0^{2\pi} \frac{I}{2\pi\cancel{\rho}}\, \overbrace{\bar u_\varphi\cdot\bar u_\varphi}^{1}\, \cancel{\rho}\, d\varphi = \frac{I}{2\pi}\int_0^{2\pi} d\varphi = I$$

$$= \int_S \bar J\cdot d\bar S = I$$

Faradayn laki

$$\nabla\times\bar E = -\frac{\partial\bar B}{\partial t}$$

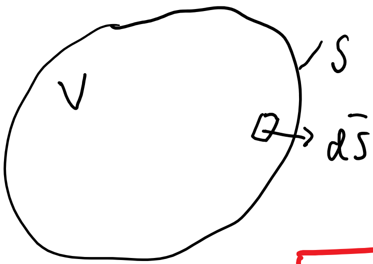


$$\int_S \nabla\times\bar E\cdot d\bar S = \oint \bar E\cdot d\bar l = \text{smv}$$

$$= -\int \frac{\partial\bar B}{\partial t}\cdot d\bar S = -\frac{\partial}{\partial t}\int \bar B\cdot d\bar S$$

$$= -\int \frac{\partial \bar{B}}{\partial t} \cdot d\bar{S} = -\frac{\partial}{\partial t} \int \bar{B} \cdot d\bar{S} \qquad \partial_t B$$

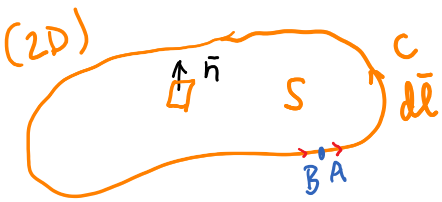
Nollakaavat



$$\int_V \underbrace{\nabla \cdot \nabla \times \bar{H}}_{=0} \, dV = \oint_S \nabla \times \bar{H} \cdot d\bar{S}$$

$$= \oint_C \bar{H} \cdot d\bar{l} = 0$$

$$\Rightarrow \boxed{\nabla \cdot \nabla \times \bar{H} = 0}$$



$$\int_S \underbrace{\nabla \times \nabla F}_{0} \cdot d\bar{S} = \oint_C \nabla F \cdot d\bar{l} = F_B - F_A = 0$$

$$\Rightarrow \boxed{\nabla \times \nabla F = 0}$$

Seuraauksia: potentiaalit

ϕ SKALAARIPOTENTIAALI

$$\bar{E} = -\nabla \phi \qquad \Rightarrow \qquad \nabla \times \bar{E} = -\nabla \times \nabla \phi = 0$$

$\bar{A}$ VEKTORIPOTENTIAALI

$$\bar{B} = \nabla \times \bar{A} \qquad \Rightarrow \qquad \nabla \cdot \bar{B} = \nabla \cdot \nabla \times \bar{A} = 0$$

$$\nabla \cdot \nabla \times \bar{A} = 0$$

$$\nabla \times \nabla \phi = 0$$

$$\nabla \times (\nabla \times \bar{E}) = \nabla (\nabla \cdot \bar{E}) - \underbrace{(\nabla \cdot \nabla)}_{\nabla^2} \bar{E}$$

$\nabla \times \nabla\phi = 0$

∇^2

LAPLACE

STATIIKKA

$\nabla \times \bar{E} = 0$

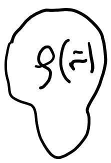
$\nabla \cdot \bar{D} = \rho$

$\nabla \cdot \varepsilon_0 \bar{E} = \rho$

$-\nabla\phi$

$\Rightarrow -\varepsilon_0 \nabla\cdot\nabla\phi = \rho$

$\Rightarrow \nabla^2 \phi(\bar{r}) = -\dfrac{\rho(\bar{r})}{\varepsilon_0}$



x

$\rho = 0$

$\nabla^2 \phi = 0$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = 0$$