ELEC-E8101 Digital and Optimal Control

Exercise 3

1. The continuous time state-space representation of a two tank system is:

$$\frac{dx(t)}{dt} = \begin{bmatrix} -0.0197 & 0 \\ 0.0178 & -0.0129 \end{bmatrix} x(t) + \begin{bmatrix} 0.0263 \\ 0 \end{bmatrix} u(t)$$

.

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)$$

a) Sample the system with the sampling period $h = 12$.

b) Verify that the pulse transfer operator for the system is

$$H(q) = \frac{0.030q + 0.026}{q^2 - 1.65q + 0.68}.$$

2. Determine the pulse transfer function, pulse transfer operator and pulse response of system

$$y(k) + 0{,}5y(k - 1) = u(k - 1).$$

3. Let's consider the following simplified model of DC-motor

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s+1)}$$

and the corresponding discrete time state-space representation

$$\mathbf{x}(kh+h) = \begin{bmatrix} e^{-h} & 0 \\ 1-e^{-h} & 1 \end{bmatrix} \mathbf{x}(kh) + \begin{bmatrix} 1-e^{-h} \\ h-1+e^{-h} \end{bmatrix} u(kh)$$

$$y(kh) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(kh).$$

Determine

a) pulse transfer function,

b) pulse response

c) corresponding difference equation,

d) the poles and zeros of the pulse transfer function as a function of sampling time.

* **4.** Transform

$$G(s) = \frac{6}{(s+2)(s+3)}$$

into pulse transfer function using the tables of transformations. Assume ZOH and unit sampling time.

Verify that the final value of the unit step response in the discrete case is the same as in continuous case (use the final value theorem).

Tables

Continuous-time transfer function G(s) and its
dicrete-time ZOH-equivalent H(q)

$G(s)$	$H(q)=\dfrac{b_1q^{n-1}+b_2q^{n-2}+\cdots+b_n}{q^n+a_1q^{n-1}+\cdots+a_n}$
$\dfrac{1}{s}$	$\dfrac{h}{q-1}$
$\dfrac{1}{s^2}$	$\dfrac{h^2(q+1)}{2(q-1)^2}$
$\dfrac{1}{s^m}$	$\dfrac{q-1}{q}\lim_{a\to 0}\dfrac{(-1)^m}{m!}\dfrac{\partial^m}{\partial a^m}\left(\dfrac{q}{q-e^{-ah}}\right)$
e^{-sh}	q^{-1}
$\dfrac{a}{s+a}$	$\dfrac{1-\exp(-ah)}{q-\exp(-ah)}$
$\dfrac{a}{s(s+a)}$	$b_1=\dfrac{1}{a}\left(ah-1+e^{-ah}\right)$ $b_2=\dfrac{1}{a}\left(1-e^{-ah}-ahe^{-ah}\right)$ $a_1=-\left(1+e^{-ah}\right)$ $a_2=e^{-ah}$
$\dfrac{a^2}{(s+a)^2}$	$b_1=1-e^{-ah}(1+ah)$ $b_2=e^{-ah}\left(e^{-ah}+ah-1\right)$ $a_1=-2e^{-ah}$ $a_2=e^{-2ah}$
$\dfrac{s}{(s+a)^2}$	$\dfrac{(q-1)he^{-ah}}{\left(q-e^{-ah}\right)^2}$
$\dfrac{ab}{(s+a)(s+b)};a\neq b$	$b_1=\dfrac{b\left(1-e^{-ah}\right)-a\left(1-e^{-bh}\right)}{b-a}$ $b_2=\dfrac{a\left(1-e^{-bh}\right)e^{-ah}-b\left(1-e^{-ah}\right)e^{-bh}}{b-a}$ $a_1=-\left(e^{-ah}+e^{-bh}\right)$ $a_2=e^{-(a+b)h}$

Tables...

$G(s)$	$H(q)$
$\frac{(s+c)}{(s+a)(s+b)}, a \neq b$	$b_1 = \frac{e^{-bh} - e^{-ah} + (1-e^{-bh})c/b - (1-e^{-ah})c/a}{a-b}$ $b_2 = \frac{c}{ab}e^{-(a+b)h} + \frac{b-c}{b(a-b)}e^{-ah} + \frac{c-a}{a(a-b)}e^{-bh}$ $a_1 = -e^{-ah} - e^{-bh}$ $\quad a_2 = e^{-(a+b)h}$
$\frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2}$	$b_1 = 1 - \alpha\left(\beta + \frac{\xi\omega_0}{\omega}\gamma\right)$ $\quad \omega = \omega_0\sqrt{1-\xi^2},\ \xi < 1$ $b_2 = \alpha^2 + \alpha\left(\frac{\xi\omega_0}{\omega}\gamma - \beta\right)$ $\quad \alpha = e^{-\xi\omega_0 h}$ $a_1 = -2\alpha\beta$ $\quad \beta = \cos(\omega h)$ $a_2 = \alpha^2$ $\quad \gamma = \sin(\omega h)$
$\frac{s}{s^2 + 2\xi\omega_0 s + \omega_0^2}$	$b_1 = \frac{1}{\omega}e^{-\xi\omega_0 h}\sin(\omega h)$ $\quad b_2 = -b_1$ $a_1 = -2e^{-\xi\omega_0 h}\cos(\omega h)$ $\quad a_2 = e^{-2\xi\omega_0 h}$ $\omega = \omega_0\sqrt{1-\xi^2}$
$\frac{a^2}{s^2 + a^2}$	$b_1 = 1 - \cos(ah)$ $\quad b_2 = 1 - \cos(ah)$ $a_1 = -2\cos(ah)$ $\quad a_2 = 1$
$\frac{s}{s^2 + a^2}$	$b_1 = \frac{1}{a}\sin(ah)$ $\quad b_2 = -\frac{1}{a}\sin(ah)$ $a_1 = -2\cos(ah)$ $\quad a_2 = 1$
$\frac{a}{s^2(s+a)}$	$b_1 = \frac{1-\alpha}{a^2} + h\left(\frac{h}{2} - \frac{1}{a}\right), \quad \alpha = e^{-ah}$ $b_2 = (1-\alpha)\left(\frac{h^2}{2} - \frac{2}{a^2}\right) + \frac{h}{a}(1+\alpha)$ $b_3 = -\left[\frac{1}{a^2}(\alpha - 1) + \alpha h\left(\frac{h}{2} + \frac{1}{a}\right)\right]$ $a_1 = -(\alpha+2)$ $\quad a_2 = 2\alpha + 1$ $\quad a_3 = -\alpha$