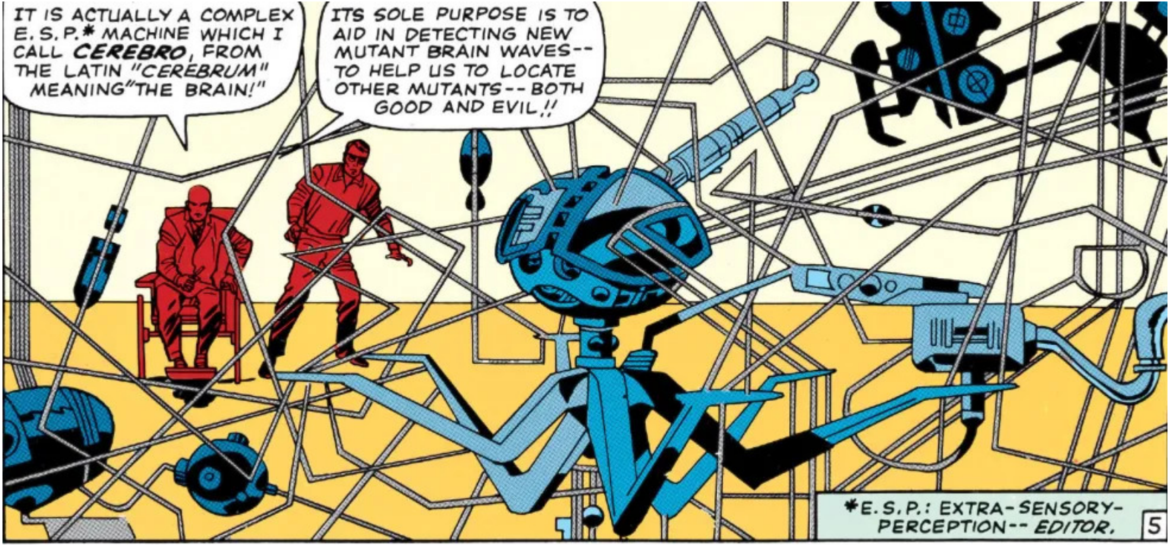
Mathematics for Economists

Mitri Kitti

Aalto University

Implicit Function Theorem

Motivation



IT IS ACTUALLY A COMPLEX E. S. P.* MACHINE WHICH I CALL CEREBRO, FROM THE LATIN "CEREBRUM" MEANING "THE BRAIN!"
ITS SOLE PURPOSE IS TO AID IN DETECTING NEW MUTANT BRAIN WAVES-- TO HELP US TO LOCATE OTHER MUTANTS-- BOTH GOOD AND EVIL!!
*E.S.P.: EXTRA-SENSORY-PERCEPTION-- EDITOR.
5

Motivation

- Most economic models analyze the relationship between endogenous and exogenous variables, e.g. GDP (endogenous) and public expenditure (exogenous) in the IS-LM model

- Sometimes, this relationship can be written as an *explicit* function

$$y = F(x_1, \dots, x_n),$$

  where y is the endogenous variable and the x_i's are exogenous
- But often the best we can do is to express y as an *implicit* function of the exogenous variables:

$$G(x_1, \dots, x_n, y) = 0 \tag{1}$$

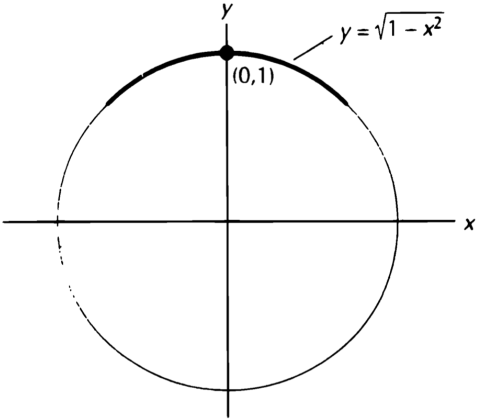
- The Implicit Function Theorem will allow us to study how changes in the exogenous variables affect y when we have an implicit function like (1)

Motivation

- **Example.** Consider a profit-maximizing firm that uses a single input z to produce a single output through the production function $f(z)$
- The unit price of output is p, and the unit price of input is w
- The firm's profit is $pf(z) - wz$, and the first order condition for profit maximization is
$$pf'(z) - w = 0 \tag{2}$$
- Equation (2) defines z as an *implicit* function of the exogenous variables w and p
- How does z change as we change w or p?

Implicit Function

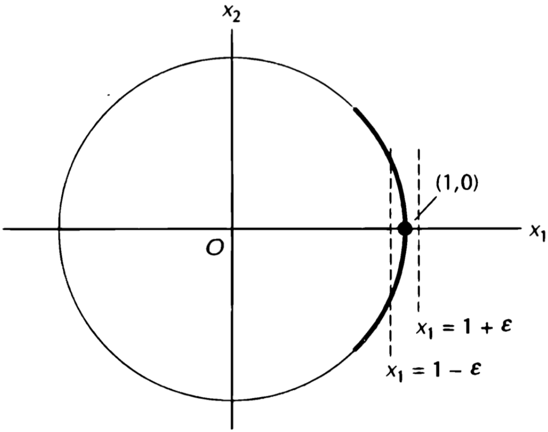
- **Example.** Suppose $x^2 + y^2 = 1$. Around the point $(0, 1)$, we can express y as an explicit function of x



The graph of $x^2 + y^2 = 1$ near the point $(0, 1)$.

Implicit Function

- However, we cannot express y as an explicit function of x around the point $(1, 0)$



The graph of $x^2 + y^2 = 1$ *near the point* (1, 0).

Implicit Function Theorem in $\mathbb{R}^2$

- We want to address the following questions

  1. Given the implicit equation $G(x, y) = c$ and a point (x_0, y_0) such that $G(x_0, y_0) = c$, does there exist a continuous function $y = y(x)$ defined on an interval I around x_0 such that:

     (a) $G(x, y(x)) = c$ for all $x \in I$

     (b) $y(x_0) = y_0$?

  2. If $y(x_0)$ exists and is differentiable, what is $y'(x_0)$?

- Note: We already know how to compute $y'(x_0)$ in (2) through the Chain Rule...

Implicit Function Theorem in $\mathbb{R}^2$

Theorem (Implicit Function Theorem in $\mathbb{R}^2$)

Let $G(x, y)$ be a C^1 function on an open ball around $(x_0, y_0) \in \mathbb{R}^2$. Suppose that $G(x_0, y_0) = c$ and consider the implicit equation $G(x, y) = c$.

If $\frac{\partial G}{\partial y}(x_0, y_0) \neq 0$, then there exists a C^1 function $y(x)$ defined on an interval $I \subset \mathbb{R}$ around x_0 such that:

1. $G(x, y(x)) = c$ *for all* $x \in I$;
2. $y(x_0) = y_0$;
3. $y'(x_0) = -\frac{\frac{\partial G}{\partial x}(x_0,y_0)}{\frac{\partial G}{\partial y}(x_0,y_0)}$.

Implicit Function Theorem in $\mathbb{R}^2$

- **Example:** Let $G : \mathbb{R}^2 \to \mathbb{R}$ be such that $G(x, y) = x^2 - 3xy + y^3 - 7$
- At $(x_0, y_0) = (4, 3)$, we have $G(x_0, y_0) = 0$
- Consider the implicit equation $G(x, y) = x^2 - 3xy + y^3 - 7 = 0$
- We have that
$$\frac{\partial G}{\partial y}(4, 3) = -3x + 3y^2\Big|_{(4,3)} = 15 \neq 0$$
- By the implicit function theorem, $G(x, y)$ defines y as a C^1 function of x around $(4, 3)$ and
$$y'(x_0) = -\frac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)} = \frac{1}{15}.$$

Implicit Function Theorem in $\mathbb{R}^2$

- **Example:** Consider the implicit equation $G(x, y) = x^2 + y^2 - 1 = 0$
- At $(x_0, y_0) = (0, 1)$, we have that $\frac{\partial G}{\partial y}(0, 1) = 2 \neq 0$. Therefore, $G(x, y)$ implicitly defines y as a function of x around this point
- However, at $(x_0, y_0) = (1, 0)$, we have that $\frac{\partial G}{\partial y}(1, 0) = 0$. Thus the Implicit Function Theorem does not hold at this point

Implicit Function Theorem in $\mathbb{R}^2$

- **Example:** In microeconomics, we can invoke the Implicit Function Theorem to derive the Marginal Rate of Substitution (In the last lecture, we used the total differential to do that)

- Suppose $u : \mathbb{R}^2_+ \to \mathbb{R}$ is a C^1 utility function

- The implicit equation $u(x, y) = c$ identifies the indifference curve that gives total utility c

- At (x_0, y_0), if the marginal utility of y is different from zero, we can use the Implicit Function Theorem to write

$$y'(x_0) = -\frac{\frac{\partial u}{\partial x}(x_0, y_0)}{\frac{\partial u}{\partial y}(x_0, y_0)},$$

which is the Marginal Rate of Substitution at (x_0, y_0)

Implicit Function Theorem: a Real Valued Implicit Function

Theorem (Implicit Function Theorem)

Let $G(x_1, \ldots, x_k, y)$ be a C^1 function around the point $(x_1^*, \ldots, x_k^*, y^*)$. Suppose further that $(x_1^*, \ldots, x_k^*, y^*)$ satisfies

$$G(x_1^*, \ldots, x_k^*, y^*) = c \quad \text{and} \quad \frac{\partial G}{\partial y}(x_1^*, \ldots, x_k^*, y^*) \neq 0.$$

Then there is a C^1 function $y = y(x_1, \ldots, x_k)$ defined on an open ball B around $(x_1^*, \ldots, x_k^*)$ such that

1. $G(x_1, \ldots, x_k, y(x_1, \ldots, x_k)) = c$ *for all* $(x_1, \ldots, x_k) \in B$;
2. $y(x_1^*, \ldots, x_k^*) = y^*$;
3. *for each* i,
$$\frac{\partial y}{\partial x_i}(x_1^*, \ldots, x_k^*) = -\frac{\frac{\partial G}{\partial x_i}(x_1^*, \ldots, x_k^*, y^*)}{\frac{\partial G}{\partial y}(x_1^*, \ldots, x_k^*, y^*)}.$$

Implicit Function Theorem: Example

- **Example:** Consider a profit-maximizing firm that uses a single input z to produce a single output through the production function $f(z)$

- The first-order condition for profit maximization is

$$pf'(z) - w = 0, \tag{3}$$

  where p and w are prices

- How does the optimal quantity of input z depend on prices p and w?

- The derivative of the implicit equation (3) w.r.t. z is

$$pf''(z),$$

  which we assume is strictly negative (i.e. f is strictly concave)

Implicit Function Theorem: Example

- By the Implicit Function Theorem, we have

$$\frac{\partial z}{\partial w}(p, w) = \frac{1}{pf''(z)} < 0$$

and

$$\frac{\partial z}{\partial p}(p, w) = -\frac{f'(z)}{pf''(z)} > 0$$

Linear implicit function theorem

Theorem

- *Assume that* $A \in \mathbb{R}^{m\times(n+m)}$, *and* $A = (A_x, A_y)$, *with* $A_x \in \mathbb{R}^{m\times n}$, $A_y \in \mathbb{R}^{m\times m}$ *such that* $A(\mathbf{x}, \mathbf{y}) = A_x\mathbf{x} + A_y\mathbf{y}$
- *If* A_y *is invertible we obtain from* $A(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ *the function* $\mathbf{y}(\mathbf{x}) = -A_y^{-1}A_x\mathbf{x}$ *(i.e.,* $D\mathbf{y}(\mathbf{x}) = -A_y^{-1}A_x$*)*
- *If we write* $A_x dx + A_y dy = 0$ *we obtain* $dy = -A_y^{-1}A_x dx$, *when only one exogenous variable is changed, then we can use Cramer's rule to find* dy

Implicit function theorem: Example

IS-LM model

$$\begin{aligned}(1-b)Y + i_1 r &= a + i_0 + G - bT \\ c_1 Y - c_2 r &= M^s\end{aligned}$$

Multiplier matrix [(Y, r) as endogenous]

$$A = \begin{pmatrix} 1-b & i_1 \\ c_1 & -c_2 \end{pmatrix}$$

... but what are the exogenous variables? For example what is the solution as a function of G?

Implicit function theorem: General Form

Theorem

$G : \mathbb{R}^{n+m} \mapsto \mathbb{R}^m$ *continuously differentiable on* $B_\varepsilon(\mathbf{x}^0, \mathbf{y}^0)$ *for some* $\varepsilon > 0$ *and* $G(\mathbf{x}^0, \mathbf{y}^0) = \mathbf{0}$, $\det(D_y G(\mathbf{x}^0, \mathbf{y}^0)) \neq 0$, *then there is* $\delta > 0$ *and function* $\mathbf{y}(\mathbf{x}) \in C^1(B_\delta(\mathbf{x}^0))$ *such that*

1. $G(\mathbf{x}, \mathbf{y}(\mathbf{x})) = \mathbf{0}$
2. $\mathbf{y}(\mathbf{x}^0) = \mathbf{y}^0$
3. $D_x \mathbf{y}(\mathbf{x}^0) = -(D_y G(\mathbf{x}^0, \mathbf{y}^0))^{-1} D_x G(\mathbf{x}^0, \mathbf{y}^0)$

Example 1

- $G(\mathbf{x},\mathbf{y}) = \begin{pmatrix} G_1(\mathbf{x},\mathbf{y}) \\ G_2(\mathbf{x},\mathbf{y}) \end{pmatrix}$, $G_1(\mathbf{x},\mathbf{y}) = y_1 y_2^2 - x_1 x_2 + x_2 - 7$, $G_2(\mathbf{x},\mathbf{y}) = y_1 - x_1/y_2 + x_2 - 5$
- Let us take $(\mathbf{x}^0,\mathbf{y}^0) = (-2,2,1,1)$ $(x_1^0 = -2, x_2^0 = 2, y_1^0 = 1, y_2^0 = 1)$

$$D_y G(\mathbf{x}^0,\mathbf{y}^0) = \begin{pmatrix} \frac{\partial G_1(\mathbf{x}^0,\mathbf{y}^0)}{\partial y_1} & \frac{\partial G_1(\mathbf{x}^0,\mathbf{y}^0)}{\partial y_2} \\ \frac{\partial G_2(\mathbf{x}^0,\mathbf{y}^0)}{\partial y_1} & \frac{\partial G_2(\mathbf{x}^0,\mathbf{y}^0)}{\partial y_2} \end{pmatrix}$$

$$= \begin{pmatrix} (y_2^0)^2 & 2y_1^0 y_2^0 \\ 1 & x_1^0/(y_2^0)^2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$$

$$D_x G(\mathbf{x}^0,\mathbf{y}^0) = \begin{pmatrix} \frac{\partial G_1(\mathbf{x}^0,\mathbf{y}^0)}{\partial x_1} & \frac{\partial G_1(\mathbf{x}^0,\mathbf{y}^0)}{\partial x_2} \\ \frac{\partial G_2(\mathbf{x}^0,\mathbf{y}^0)}{\partial x_1} & \frac{\partial G_2(\mathbf{x}^0,\mathbf{y}^0)}{\partial x_2} \end{pmatrix}$$

$$= \begin{pmatrix} -(x_2^0) & 1 - x_2^0 \\ -1/y_2^0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix}$$

Example 1

- How does change in x_1 affect the endogenous variables $\mathbf{y}$?

$$D_y G(\mathbf{x}^0, \mathbf{y}^0) \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} + D_{x_1} G(\mathbf{x}^0, \mathbf{y}^0) dx_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we get

$$\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \end{pmatrix} dx_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by Cramer's rule

$$dy_1 = \frac{\det \begin{pmatrix} 2 & 2 \\ 1 & -2 \end{pmatrix}}{\det \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}} dx_1 = \frac{1}{2} dx_1$$

$$dy_2 = \frac{\det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}}{\det \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}} dx_1 = \frac{1}{4} dx_1$$

Example 2

- Assume that $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ is continuously differentiable such that $\det(DF(\mathbf{x})) \neq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, consider the equation $F(\mathbf{x}) = \mathbf{b}$
- Set $G(\mathbf{x}, \mathbf{b}) = F(\mathbf{x}) - \mathbf{b}$
- Apply the implicit function theorem, what do you get?
- Implicit function theorem tells us that there is $\mathbf{x}(\mathbf{b})$ around any $\mathbf{b} \in \mathbb{R}^n$ such that $F(\mathbf{x}(\mathbf{b})) = \mathbf{b}$ and $D\mathbf{x}(\mathbf{b}) = [DF(\mathbf{x}(\mathbf{b}))]^{-1}$
- This result is known as the inverse function theorem
  - note that $\mathbf{x}(\mathbf{b})$ is (a local) inverse function

Homogeneous Functions

- Homogeneous functions are an important class of functions studied in economics
- Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a function. For any scalar k, we say that f is **homogeneous of degree *k*** if

$$f(tx_1, \ldots, tx_n) = t^k f(x_1, \ldots, x_n) \text{ for all } (x_1, \ldots, x_n) \in \mathbb{R}^n_+ \text{ and all } t > 0.$$

Homogeneous Functions

- **Example:** Let $f(x, y) = x^2 y^3$
- For any $t > 0$ we have

$$f(tx, ty) = (tx)^2 (ty)^3 = t^5 (x^2 y^3) = t^5 f(x, y)$$

- Hence f is homogeneous of degree 5
- An example of a *non-homogeneous* function is $g(x, y) = x^2 + y^3$

Homogeneous Functions

- Homogeneous functions are closely related to the concept of *returns to scale* in economics
- Suppose f is a production function. Then f has
  - Constant returns to scale if $f(tx_1, \dots, tx_n) = tf(x_1, \dots, x_n)$ for all $t > 0$
  - Decreasing returns to scale if $f(tx_1, \dots, tx_n) < tf(x_1, \dots, x_n)$ for all $t > 1$
  - Increasing returns to scale if $f(tx_1, \dots, tx_n) > tf(x_1, \dots, x_n)$ for all $t > 1$

Homogeneous Functions

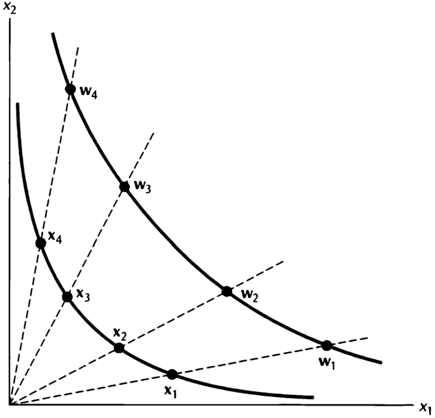
- Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a C^1 function homogeneous of degree k. Then its first order partial derivatives are homogeneous of degree $k - 1$.

- To prove this result, take the following definition of homogeneity of degree k and then use the chain rule to differentiate both sides w.r.t. any x_i:

$$f(tx_1, \ldots, tx_n) = t^k f(x_1, \ldots, x_n)$$

Homogeneous Functions

- Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a C^1 function homogeneous of degree k. Then the tangent planes to the level sets of f have constant slope along each ray from the origin
- For utility (production) functions, this says that the Marginal Rate of (Technical) Substitution is constant along each ray from the origin



Homogeneous Functions

- **Euler's theorem.** Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a C^1 function homogeneous of degree k. Then, for all $\boldsymbol{x} \in \mathbb{R}^n_+$,

$$x_1 \frac{\partial f}{\partial x_1}(\boldsymbol{x}) + x_2 \frac{\partial f}{\partial x_2}(\boldsymbol{x}) + \cdots + x_n \frac{\partial f}{\partial x_n}(\boldsymbol{x}) = kf(\boldsymbol{x}).$$

- Conversely, if f is such that

$$x_1 \frac{\partial f}{\partial x_1}(\boldsymbol{x}) + x_2 \frac{\partial f}{\partial x_2}(\boldsymbol{x}) + \cdots + x_n \frac{\partial f}{\partial x_n}(\boldsymbol{x}) = kf(\boldsymbol{x}).$$

  for all $\boldsymbol{x} \in \mathbb{R}^n_+$, then f is homogeneous of degree k.

- A couple of properties:
  - The product of homogeneous functions is homogeneous
  - The sum of two functions that are homogeneous of different degrees is not homogeneous

Homogeneous Functions

- **Exercise:** Look at all the production functions listed in the slides from Lecture 5. Are they homogeneous? If so, of what degree?