Ilmonen/ Shafik/ Pere/ Mellin

3. Theoretical exercises

Demo exercises

3.1 Let u and v be random variables with the following properties,

$$\begin{aligned} \mathbb{E}(u) &= \mathbb{E}(v) = 0 \\ \mathrm{Var}(u) &= \mathrm{Var}(v) = \sigma^2 < \infty \\ \mathrm{Cov}(u, v) &= \mathbb{E}(uv) = 0 \end{aligned}$$

Let $\lambda \in \mathbb{R}$ be a nonrandom constant. Show that the process,

$$x_t = u \cdot \cos(\lambda t) + v \cdot \sin(\lambda t), \quad t \in T,$$

is stationary.

Solution.

i) The expectation of x_t is,

$$\begin{aligned} \mathbb{E}(x_t) &= \mathbb{E}(u \cdot \cos(\lambda t) + v \cdot \sin(\lambda t)) \\ &= E(u) \cos(\lambda t) + \mathbb{E}(v) \sin(\lambda t) \\ &= 0 \cdot \cos(\lambda t) + 0 \cdot \sin(\lambda t) \\ &= 0, \quad \text{for all } t \in T. \end{aligned}$$

ii) The variance of x_t is,

$$\begin{aligned} \mathrm{Var}(x_t) &= \mathbb{E}(x_t^2) \\ &= \mathbb{E}(u^2 \cos^2(\lambda t) + v^2 \sin^2(\lambda t) + 2uv \cos(\lambda t) \sin(\lambda t)) \\ &= \mathbb{E}(u^2) \cos^2(\lambda t) + \mathbb{E}(v^2) \sin^2(\lambda t) + 2\mathbb{E}(uv) \cos(\lambda t) \sin(\lambda t) \\ &= \sigma^2 \cos^2(\lambda t) + \sigma^2 \sin^2(\lambda t) \\ &= \sigma^2, \quad \text{for all } t \in T. \end{aligned}$$

iii) The autocovariance of the process x_t is given by,

$$\begin{aligned} \mathrm{Cov}(x_t, x_{t-\tau}) &= \mathbb{E}(x_t x_{t-\tau}) \\ &= \mathbb{E}(u^2 \cos(\lambda t) \cos(\lambda(t-\tau)) + v^2 \sin(\lambda t) \sin(\lambda(t-\tau)) \\ &+ uv \cos(\lambda t) \sin(\lambda(t-\tau)) + uv \sin(\lambda t) \cos(\lambda(t-\tau))] \\ &= \mathbb{E}(u^2) \cos(\lambda t) \cos(\lambda(t-\tau)) + \mathbb{E}(v^2) \sin(\lambda t) \sin(\lambda(t-\tau)) \\ &+ \mathbb{E}(uv) \cos(\lambda t) \sin(\lambda(t-\tau)) + \mathbb{E}(uv) \sin(\lambda t) \cos(\lambda(t-\tau)) \\ &= \sigma^2 \cos(\lambda t) \cos(\lambda(t-\tau)) + \sigma^2 \sin(\lambda t) \sin(\lambda(t-\tau)) \\ &= \sigma^2 \cos(\lambda t - \lambda(t-\tau)) \\ &= \sigma^2 \cos(\lambda \tau), \quad \text{for all } t \in T. \end{aligned}$$

Note that $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$. By i), ii) and iii), it follows that x_t is stationary.

3.2 Consider the following stationary AR(1) process,

$$x_t - \phi_1 x_{t-1} = \varepsilon_t, \quad \varepsilon_t \sim \text{i.i.d.}(0, \sigma^2). \tag{1}$$

Show that, in the case of AR(1)-processes, the assumption of weak stationarity implies that $|\phi_1| < 1$. In the proof, assume that,

$$\mathbb{E}\left[x_{t-1}\varepsilon_t\right] = 0, \quad \text{for every } t \in T. \tag{2}$$

Solution. A stochastic process x_t is weakly stationary if the following conditions are satisfied.

(i) $\mathbb{E}(x_t) = \mu, \quad \forall t \in T,$

(ii) $\text{Var}(x_t) = \sigma^2 < \infty, \quad \forall t \in T,$

(iii) $\text{Cov}(x_t, x_{t-\tau}) = \gamma_\tau, \quad \forall t, \tau \in T.$

By condition (ii), the variance of the process is not time dependent. Then, by Assumption (2),

$$\text{Var}(x_t) = \text{Var}(\phi_1 x_{t-1} + \varepsilon_t) = \phi_1^2 \text{Var}(x_{t-1}) + \text{Var}(\varepsilon_t) = \phi_1^2 \text{Var}(x_t) + \sigma^2, \tag{3}$$

since $\varepsilon_t \sim \text{i.i.d.}(0, \sigma^2)$. By solving the equation above for $\text{Var}(x_t)$, we obtain,

$$\text{Var}(x_t) = \frac{\sigma^2}{1 - \phi_1^2}.$$

Condition (ii) gives us that the variance is finite. Furthermore, by definition, variance cannot be negative. Thus, $|\phi_1| < 1$.

Homework

3.3 Show that MA(1) process

$$x_t = \varepsilon_t + \theta\varepsilon_{t-1}, \qquad \varepsilon_t \sim \text{i.i.d.}(0, \sigma^2),$$

is always stationary.

3.4 a) Derive the autocorrelation function of a MA(1) process.

b) Derive the autocorrelation function of a stationary AR(1) process:

$$x_t = \phi x_{t-1} + \varepsilon_t, \quad (\varepsilon_t)_{t\in\mathbb{Z}} \sim \text{i.i.d.}(0, \sigma^2).$$

Use the MA(∞) representation,

$$x_t = \Psi(L)\varepsilon_t = \sum_{i=0}^{\infty} \phi^i L^i \varepsilon_t,$$

where the series $\sum_{i=0}^{\infty} \phi^i$ converges absolutely.

Hint: If $\sum_{i=0}^{\infty} a_i$ converges absolutely, and $\mathbb{E}[|x_i|]$ is an index invariant finite constant, then $\mathbb{E}[\sum_{i=0}^{\infty} a_i x_i] = \sum_{i=0}^{\infty} a_i \mathbb{E}[x_i]$.