Notes - Theoretical exercises 1

November 21, 2021

Exercise 3.1

Remember that in order to show that stochastic process $(x_t)_{t\in T}$ is stationary one needs to check:

- $\mathbb{E}(x_t) = \mu \quad \forall t \in T$,
- $\mathrm{Var}(x_t) = \sigma^2 < \infty \quad \forall t \in T$ and
- $\mathrm{Cov}(x_t, x_s) = \mathrm{Cov}(x_{t-s}, x_0) \quad \forall t, s \in T$

Notice that we assume $\mathbb{E}(u) = 0$ and $\mathrm{Var}(u) = \sigma^2$. Then we also have

$$\mathrm{Var}(u) = \mathbb{E}(u^2) - \underbrace{\mathbb{E}(u)^2}_{=0} = 0$$
$$\Rightarrow \mathbb{E}(u^2) = \sigma^2.$$

Similarly to above, one can check that $\mathbb{E}(v^2) = \sigma^2$.

Exercise 3.2

Remember that

$$\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X, Y).$$

See also that

$$\mathrm{Cov}(x_{t-1}, \varepsilon_t) = \underbrace{\mathbb{E}(x_{t-1}\varepsilon_t)}_{=0} - \underbrace{\mathbb{E}(x_{t-1})}_{=\mu}\underbrace{\mathbb{E}(\varepsilon_t)}_{=0} = 0.$$

Then

$$\begin{aligned}
&\mathrm{Var}(x_t) = \mathrm{Var}(\phi_1 x_{t-1} + \varepsilon_t) \\
&= \phi_1^2 \mathrm{Var}(x_{t-1}) + 2\phi_1 \mathrm{Cov}(x_{t-1}, \varepsilon_t) + \mathrm{Var}(\varepsilon_t) \\
&\overset{ii)}{=} \phi_1^2 \mathrm{Var}(x_t) + \hat{\sigma}^2.
\end{aligned}$$

So all in all we got,

$$\begin{aligned}
&\mathrm{Var}(x_t) = \phi_1^2 \mathrm{Var}(x_t) + \hat{\sigma}^2 \\
&\Rightarrow \mathrm{Var}(x_t) = \frac{\hat{\sigma}^2}{1 - \phi_1^2}.
\end{aligned}$$

Hints for homework

Exercise 3.3 Check conditions

- $\mathbb{E}(x_t) = \mu \quad \forall t \in T$,
- $\mathrm{Var}(x_t) = \sigma^2 < \infty \quad \forall t \in T$ and
- $\mathrm{Cov}(x_t, x_s) = \mathrm{Cov}(x_{t-s}, x_0) \quad \forall t, s \in T$.

Exercise 3.4

a) Autocorrelation $\rho(\tau)$ for lag τ is defined as

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)},$$

where $\gamma(\tau)$ is the autocovariance function at lag τ.

b)

- Note that even though expected value is linear, one cannot always change order of *infinite* sum and expected value.
- However, by hint one can change order of expected value and infinite sum in this exercise. This is because by exercise 3.2, weak stationarity of AR(1) process implies that $|\phi| < 1$.
- First calculate $\mathbb{E}(x_t)$ and remember that $\mathrm{Cov}(x_t, x_s) = \mathbb{E}(x_t x_s) - \mathbb{E}(x_t)\,\mathbb{E}(x_s)$.

- Lastly, it is useful to remember that

$$\mathbb{E}(\varepsilon_i \varepsilon_j) = \begin{cases} 0, & i \neq j \\ \sigma^2, & i = j. \end{cases}$$