ELEC-E8101 Digital and Optimal Control

Exercise 4
Solutions

1. The continuous time representation under consideration:

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

It is perfectly Ok to solve $e^{At} = L^{-1}\left[\left(sI - A\right)^{-1}\right]$ and then

$$\begin{cases} \Phi(h) = e^{Ah} \\ \Gamma(h) = \int_0^h e^{As} ds\, B \end{cases}$$ as usual. Do it!

The result is

$$x(kh+h) = \begin{bmatrix} \cos(h) & \sin(h) \\ -\sin(h) & \cos(h) \end{bmatrix} x(kh) + \begin{bmatrix} 1-\cos(h) \\ \sin(h) \end{bmatrix} u(kh)$$
$$y(kh) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(kh)$$

But there is another method to calculate matrix functions like exp(At) in general, which is based on the *Cayley-Hamilton theorem*. That is discussed and the problem solved in the Appendix. **NOTE: It is not absolutely necessary to study this material (Appendix).**

2. The double integrator in the differential equation form:

$$\frac{d^2 y(t)}{dt^2} = \ddot{y}(t) = u(t)$$

a. Let us choose the natural state variables

$$x_1(t) = y(t)\,,$$
$$x_2(t) = \dot{y}(t)\,.$$

By differentiation we get

$$\dot{x}_1(t) = \dot{y}(t) = x_2(t)\,,$$
$$\dot{x}_2(t) = \ddot{y}(t) = u(t)\,.$$

The state vector is

$$x(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

and the state space representation

$$\begin{cases} \dot{x}(t)=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t)+\begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t)=\begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

b.

For the double integrator

$$\Phi = e^{Ah} = L^{-1}\left\{(sI-A)^{-1}\right\}\Big|_{t=h} = L^{-1}\left\{\left(\begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}\right)^{-1}\right\}\Bigg|_{t=h} = L^{-1}\left\{\frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}\right\}\Bigg|_{t=h}$$

$$= L^{-1}\left\{\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}\right\}\Bigg|_{t=h} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\Bigg|_{t=h} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$$

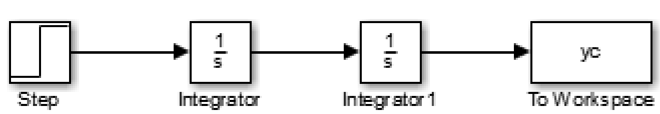
$$\Gamma = \int_0^h e^{As} ds\; B = \int_0^h \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} ds \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \int_0^h \begin{bmatrix} s \\ 1 \end{bmatrix} ds = \Big/_0^h \begin{bmatrix} \frac{1}{2}s^2 \\ s \end{bmatrix} = \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix}$$

The state-space representation is:

$$\begin{cases} x(kh+h)=\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x(kh)+\begin{bmatrix} \frac{1}{2}h^2 \\ h \end{bmatrix} u(kh) \\ y(kh)=\begin{bmatrix} 1 & 0 \end{bmatrix} x(kh) \end{cases}$$

c.

Let's first make simulations in the continuous-time. The corresponding Simulink®-model is presented below. Here a sampling time $h = 1$ is chosen. Simulation time to 0-10 is used.




```
» plot(tout,yc);hold on
```

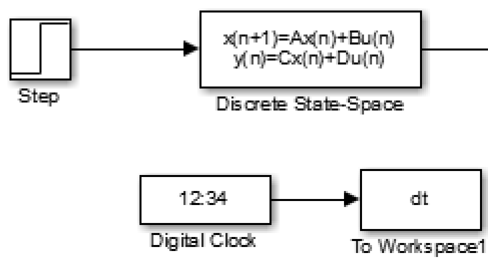
Then the sampled version. Simulink-libraries have a discrete state-space block for which can the matrices given from the command line:

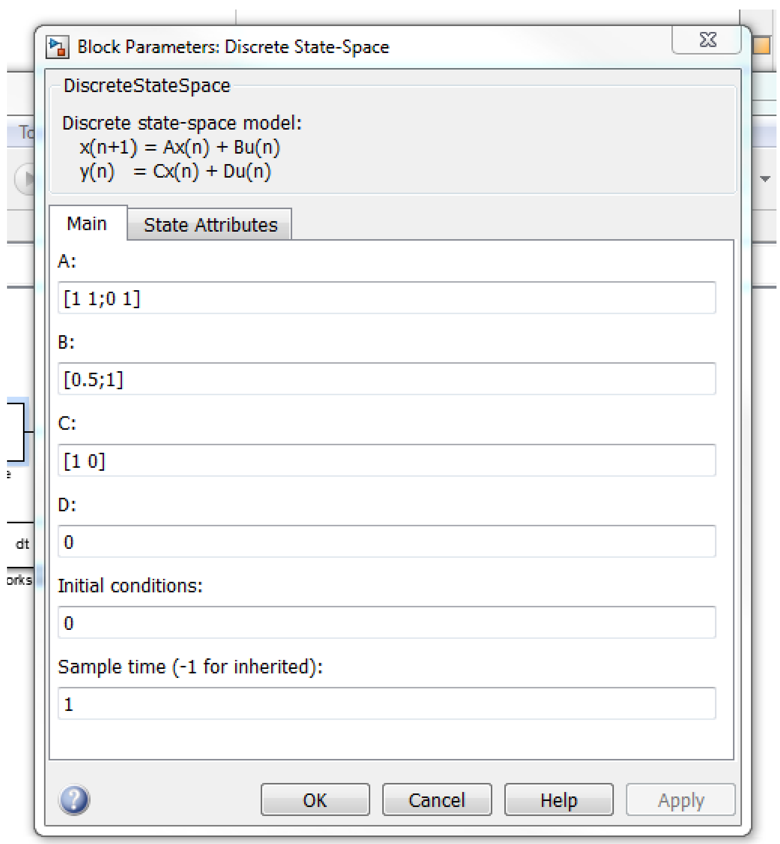
```
» phi=[1 h;0 1];

» gamma=[0.5;1]

» C=[1 0]; D=0;
```

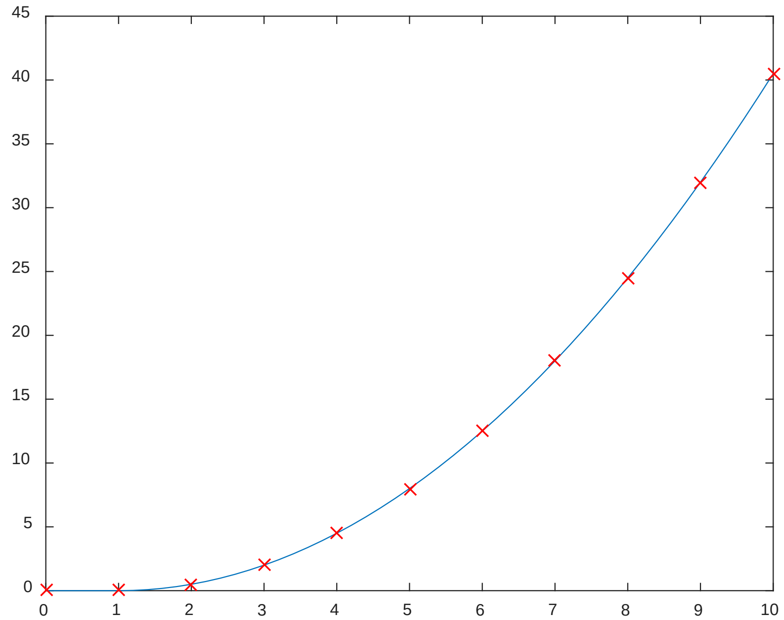
The simulink model:





Let us plot the sampled signal into the same figure as the continuous version:

```
» plot(dt,yd,'xr')
```



Responses are in agreement with each other.

3. Definitions:

stable	For every initial state, the state remains bounded (when the control signal is set to zero)
asymptotically stable	For every initial state, the state approaches origin with time (when the control signal is set to zero)
input-output stable	For zero initial state, the output is bounded for all bounded inputs.

A linear discrete system $G(z) = \frac{B(z)}{A(z)}$ is

stable	None of the poles of $A(z)$ is outside the unit circle and the poles on the unit circle are simple.
asymptotically stable	All of the poles of $A(z)$ are inside the unit circle.

Whether the poles are inside the unit circle or not, can be tested with the Jury's stability test.

a. The characteristic polynomial is

$$A(z) = z^2 - 1{,}5z + 0{,}9 \, .$$

Form the table

$$
\begin{array}{lll}
1 & -1,5 & 0,9 \\
0,9 & -1,5 & 1
\end{array}
\qquad \alpha_2 = \frac{0,9}{1} = 0,9
$$

$$--------$$

$$
\begin{array}{ll}
1-0,9*(0,9)=0,19 & -1,5-0,9*(-1,5)=-0,150 \\
-0,15 & 0,19
\end{array}
\qquad \alpha_1 = \frac{-0,15}{0,19} = -0,789
$$

$$--------$$

$$0,19-(-0,789)*(-0,15)=0,072$$

All three first elements of the odd rows (1, 3 and 5) are positive: 1, 0.19 and 0.072, which means that the poles are inside the unit circle and the system is stable.

b. The characteristic polynomial is

$$A(z) = z^3 + 5z^2 - 0,25z + 1,25$$

Form the table:

$$
\begin{array}{llll}
1 & 5 & -0,25 & 1,25 \\
1,25 & -0,25 & 5 & 1
\end{array}
\qquad \alpha_3 = \frac{1,25}{1} = 1,25
$$

$$----------------$$

$$
\begin{array}{lll}
1-1,25*1,25=-0,5625 & 5-1,25*(-0,25)=5,3125 & -0,25-1,25*5=-6,5 \\
-6,5 & 5,3125 & -0,5625
\end{array}
$$

$$\alpha_2 = \frac{-6,5}{-0,5625} = 11,56$$

$$--$$

$$
\begin{array}{ll}
-0,5625-11,56*(-6,5)=74,57 & 5,3125-11,56*5,3125=-56,1 \\
-56,1 & 74,57
\end{array}
\qquad \alpha_1 = \frac{-56,1}{74,57} = -0,752
$$

$$--------------------------------------$$

$$74,57-(-0,752)*(-56,1)=32,4$$

The first elements of the odd rows (1, 3, 5 and 7) are: 1 -0,5625 74,57 and 32,4. One of them is negative, so $A(z)$ has one root outside the unit circle and the system is not stable. That can easily be checked with Matlab: abs(roots([1 5 -0.25 1.25]))

How to form the Jury's table

- If the cofficient a_0 of the $A(z)$ is negative, multiply the polynomial with –1,
- the next odd row is acquired by subtracting the previous even row multiplied with the alpha from the previous odd row,
- an even row is always the previous odd row in reverse order.

The number of the negative first elements of the odd rows is the same as the number of the roots outside the unit circle.

4. Consider a discrete time system

$$\begin{cases} x(k+1) = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ b \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{cases}$$

where a_{11}, a_{12}, a_{22} and b are constants.

a) Consider the asymptotic stability of the above system. We remember that a discrete time system is asymptotically stable, if the eigenvalues of the system matrix Φ are inside the unit circle. The eigenvalues of a block-diagonal (and diagonal also of course) matrix are the elements on the main diagonal- Therefore, the given discrete time system is asymptotically stable if

$$|a_{11}| < 1$$
$$\text{and}$$
$$|a_{22}| < 1.$$

b) The system is BIBO-stable if any bounded input gives bounded output. The input-output relationship is given by the pulse transfer function. For the given state-space representation

$$Y(z) = C(zI - \Phi)^{-1}\Gamma U(z) = \left[\frac{1}{z - a_{11}} + \frac{ba_{12}}{(z - a_{11})(z - a_{22})} \right] U(z)$$

But this can be written as partial fractions

$$Y(z) = \cdots = \left[\frac{A}{z - a_{11}} + \frac{B}{(z - a_{22})} \right] U(z) \quad \text{for some constants } A \text{ and } B.$$

For any bounded input, the output remains clearly bounded for $|a_{11}| < 1, |a_{22}| < 1$. (Well, this is not a rigorous proof. A formal proof is pretty delicate.)

c) In the previous assignment b) we made at least somewhat clear that if the system is asymptotically stable the system is also BIBO-stable. Now, we want to show that this is not true vice versa: BIBO-stability does not necessarily imply asymptotic stability.

For the given system the observability matrix (see lectures)

$$W_O = \begin{bmatrix} 1 & 0 \\ a_{11} & a_{12} \end{bmatrix}$$

has not full rank (the determinant is zero), if $a_{12} = 0$. That means that some state(s) do not show at the output at all! From the system equations that is pretty clear for the state x_2.

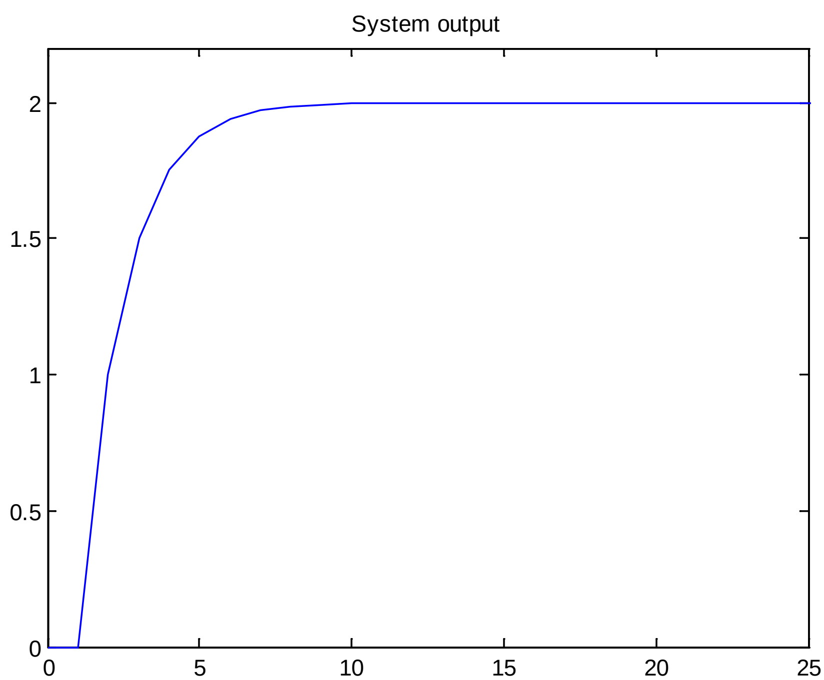
So, if a_{12} is set equal to zero the system is not observable and the system state-space representation is

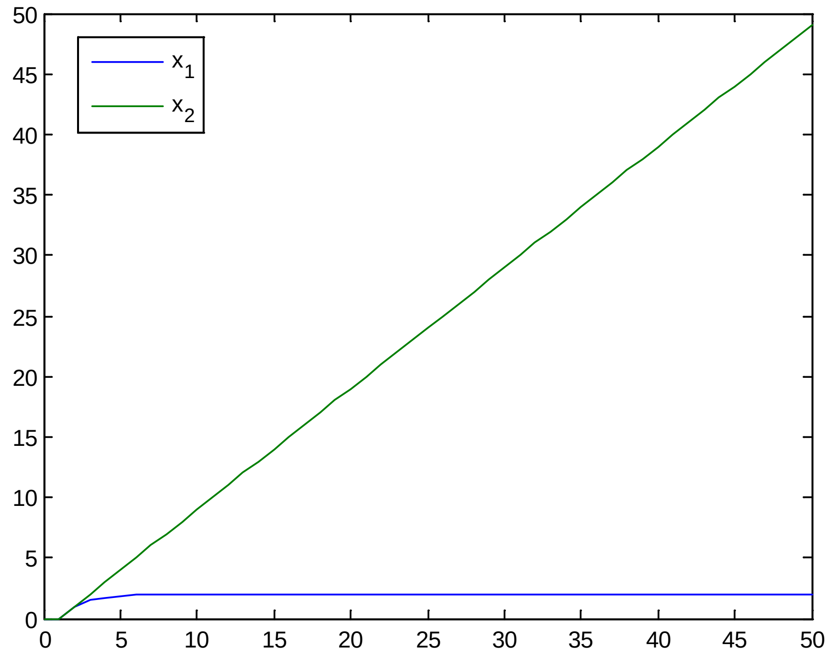
$$\begin{cases} x(k+1) = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ b \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{cases}.$$

If for example $|a_{22}| \geq 1$, $|a_{11}| < 1$ and $b \neq 0$ the system is BIBO-stable but not asymptotically stable. For a step input at u the state x_2 "blows", but $y = x_1$ does not know about this! Check the situation from the pulse transfer function also (part b).

In the simulation below,

$$a_{11} = 0.5$$
$$a_{22} = 1$$
$$b = 1$$





50
45
40
35
30
25
20
15
10
5
0
0
5
10
15
20
25
30
35
40
45
50
x_1
x_2

APPENDIX

Material related to problem 1 for the calculation of matrix functions by utilizing the *Cayley-Hamilton* theorem.

Extra material for those interested. Not asked in examinations.

At first, let us get to know the Cayley-Hamilton's theorem.

Earlier, we have calculated matrix function e^A by its definition

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

or by using the equation

$$e^{At} = L^{-1}[(sI - A)^{-1}].$$

which was derived by using the solution of the homogenous differential equation system (se lecture slides).

But, how can you calculate a matrix function, for example ln(A), if you can't give this kind of interpretation. The Cayley-Hamilton theorem is a good tool for these kinds of calculations.

Assume A is a *nxn* square matrix, whose characteristic equation is

$$a(\lambda) = \det\left[\lambda I - A\right] = \lambda^n + \alpha_1\lambda^{n-1} + \alpha_2\lambda^{n-2} + \cdots + \alpha_n = 0\,.$$

Cayley-Hamilton's theorem says that every matrix fulfills its own characteristic equation i.e.

$$a(A) = A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots + \alpha_n I = 0\,,$$

where *I* is the identity matrix of the same dimension as *A*. The theorem is not proved here.

Example:

Assume $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$. Verify Cayley-Hamilton's theorem for this matrix.

$\det\left[sI - A\right] = s^2 + 4s + 3$. Then

```
A=[0 1;-3 -4];

>> A^2+4*A+3*eye(2)
```

ans =

```
    0     0
    0     0
```
Ok!

Next, assume that A is square matrix and $f(\lambda)$ is a function of scalar variable λ. If *f* is a polynomial function

$$f(\lambda) = \beta_0\lambda^n + \beta_1\lambda^{n-1} + \beta_2\lambda^{n-2} + \cdots + \beta_n$$

the corresponding matrix polynomial *f*(*A*) is

$$f(A) = \beta_0 A^n + \beta_1 A^{n-1} + \beta_2 A^{n-2} + \cdots + \beta_n I$$

The eigenvalues of matrix *f*(A) can be calculated using the following theorem:

> Assume *f*(λ) is scalar function, λ_i is a eigenvalue of the matrix *A* and the e_i is the corresponding eigenvector.
>
> Then
>
> $$f(A)e_i = f(\lambda_i)e_i$$

This means that the eigenvalues of the matrix *f*(*A*) are $f(\lambda_i)$ and the corresponding eigenvectors e_i. The proof is omitted, although it is quite easy (apply the definition of eigenvalues and eigenvectors repeatedly on the function *f*(*A*)).

Assume that a function *f* is defined as an infinite power series

$$f(\lambda) = \sum_{i=0}^{\infty} c_i \lambda^i$$

which converges when $|\lambda| < R$. It can be shown that the corresponding matrix function

$$f(A) = \sum_{i=0}^{\infty} c_i A^i$$

converges, if for all eigenvalues of the matrix A it holds $|\lambda_i| < R$.

From the Cayley-Hamilton's theorem it can be deduced that for all functions *f* (degree *n*) there exists a polynomial *p* with a smaller degree *n-1* such that

$$f(A) = p(A) = \gamma_0 A^{n-1} + \gamma_1 A^{n-2} + \cdots + \gamma_{n-1} I$$

Example:

Assume $f(A) = A^3 + 2A^2 + A + 5I$, where $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$. Define a polynomial *p*, whose degree is smaller than 2 and $f(A) = p(A)$.

Solution: The characteristic polynomial of matrix A is $a(\lambda) = \lambda^2 + 2\lambda + 1$. Thus, from Cayley-Hamilton

$$a(A) = A^2 + 2A + I = 0$$

and $A^2 = -(2A + I)$. In that case $A^3 = AA^2 = -A(2A + I) = -2A^2 - A$. Substituting A^2 to the equation, we get $A^3 = -2[-2A - I] - A = 3A + 2I$. Now, these can be substituted to the original equation leading to

$$p(A) = (3A + 2I) + 2(-2A - I) + A + 5I = 5I$$

So $f(A) = p(A) = 5I$, whose degree is 0.

By using the eigenvalue theorem

$$f(\lambda_i) = p(\lambda_i), \quad i = 1, 2, \ldots, n$$

which is enough to determine λ_i , i =0, 1 ,..., n-1 if the all eigenvalues are distinct. If some eigenvalue has the degree m the following conditions are needed

$$\begin{gathered} f^{(1)}(\lambda_i) = p^{(1)}(\lambda_i) \\ \vdots \\ f^{(m-1)}(\lambda_i) = p^{(m-1)}(\lambda_i) \end{gathered}$$

where $f^{(i)}$ is the i^{th} derivative.

Then, back to the exercise

a) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Determine e^{Ah} .

The matrix

$$\mathrm{A}h = \begin{bmatrix} 0 & h \\ -h & 0 \end{bmatrix}$$

has the characteristic equation:

$$\det(\lambda \mathrm{I} - \mathrm{A}h) = \lambda^2 + h^2 = 0.$$

$\Rightarrow$ A's eigenvalues: $\lambda = \pm\, ih$.

From the previous calculation, there exist two constants α_0 and α_1 such that

$$e^{Ah} = f(A) = p(A) = \alpha_0 Ah + \alpha_1 I$$

The eigenvalues of matrix Ah are $\pm ih$. Therefore

$$
\begin{aligned}
f(\lambda_i) &= p(\lambda_i) \\
e^{ih} &= \alpha_0 ih + \alpha_1 \\
e^{-ih} &= -\alpha_0 ih + \alpha_1
\end{aligned}
$$

The solutions for this are

$$
\begin{aligned}
\alpha_0 &= \frac{1}{2ih}\left(e^{ih} - e^{-ih}\right) = \frac{\sin h}{h} \\
\alpha_1 &= \frac{1}{2}\left(e^{ih} + e^{-ih}\right) = \cos h
\end{aligned}
$$

and we get

$$
e^{Ah} = \alpha_0 A\mathrm{h} + \alpha_1 I = \sin h \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \cos h \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos h & \sin h \\ \text{-}\sin h & \cos h \end{bmatrix}
$$

which is the same system matrix obtained previously in problem 1.

Let us, just for fun, look at another example for calculating the logarithm function *ln* Φ

for $\Phi = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$

The characteristic equation of matrix Φ is $\left(\lambda - 1\right)^2 = 0$ and we see that 1 is a double eigenvalue. Now the matrix function can be rewritten as

$$
\ln \Phi = f(\Phi) = p(\Phi) = \alpha_0 \Phi + \alpha_1 I
$$

where the constants can be determined from equations

$$
\begin{cases}
f(\lambda_i) = p(\lambda_i) \\
\dfrac{\partial}{\partial \lambda} f(\lambda)\Big|_{\lambda = \lambda_i} = \dfrac{\partial}{\partial \lambda} p(\lambda)\Big|_{\lambda = \lambda_i}
\end{cases}
$$

$$
\begin{cases}
\ln 1 = \alpha_0 + \alpha_1 \\
\dfrac{\partial}{\partial \lambda}(\ln \lambda)\Big|_{\lambda = 1} = \alpha_0
\end{cases}
$$

Thus,

$$
\begin{aligned}
0 &= \alpha_0 + \alpha_1 \\
1 &= \alpha_0
\end{aligned}
$$

and the matrix function is

$$
\ln \Phi = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix}
$$