Mathematics for Economists

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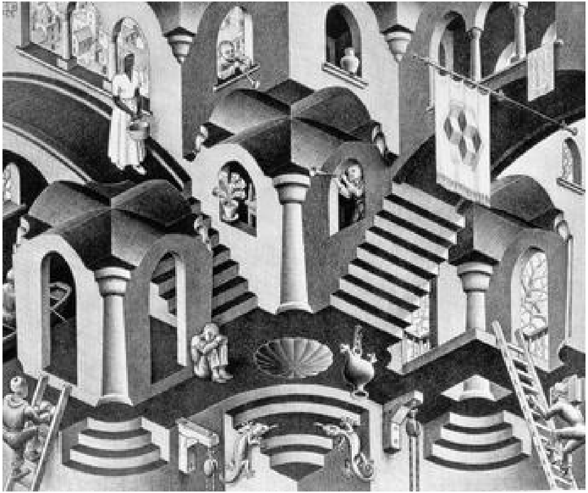
Convex Analysis

Convex and concave



Convex and concave, litograph by M.S.C Escher 1955

From local to global optima

- In the last lecture, we introduced first and second order conditions for *local* maximizers and minimizers
- If we want to look for **global** maximizers or minimizers, we have to compare local extrema with the value of the function at the "boundary" of its domain
- In general, we don't have nice necessary and sufficient conditions for global extrema
- However, global extrema are relatively easy to find for a class of functions that are commonly used in Economics: **concave** and **convex** functions

Sufficiency of First Order Conditions under Concavity

Proposition (Sufficient conditions for global extrema)

Let $f : U \to \mathbb{R}$ *and* $U \subseteq \mathbb{R}^n$ *be an* **open** *and* **convex** *set.*

1. *If* f *is a* **concave** *function, then* $\mathbf{x}^* \in U$ *is a* **global maximizer** *of* f *if and only if* $\mathbf{x}^*$ *is a critical point of* f.

2. *If* f *is a* **convex** *function, then* $\mathbf{x}^* \in U$ *is a* **global minimizer** *of* f *if and only if* $\mathbf{x}^*$ *is a critical point of* f.

Convex sets

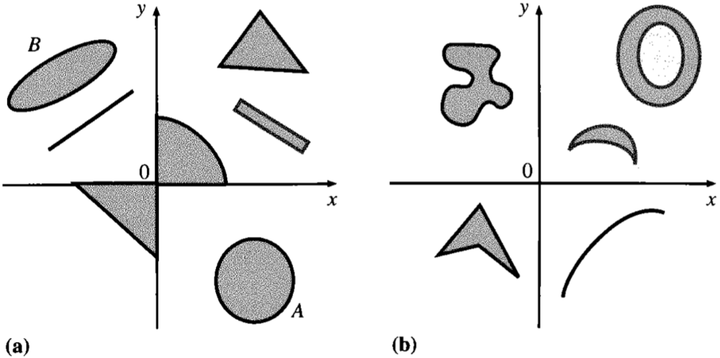
- A set is **convex** if, for every $\mathbf{x}, \mathbf{y} \in U$, and for every $t \in [0, 1]$, we have that

$$tx + (1 - t)y \in U$$

- In words, if we take any two points **x** and **y** in a convex set, the *line segment* joining **x** to **y** is entirely contained in U

Convex sets

- All the sets in panel (a) are convex, whereas all those in panel (b) are not convex



Convex sets

- Examples
  - hyperplanes $\mathbf{p} \cdot \mathbf{x} = c$, (open and closed) half spaces $\mathbf{p} \cdot \mathbf{x} \leq c$, (also $\mathbf{p} \cdot \mathbf{x} < c$, $\mathbf{p} \cdot \mathbf{x} > c$), polyhedral sets = intersections of half spaces, ellipsoids $\mathbf{x} \cdot V\mathbf{x} \leq c$ (V pos.def.), solutions of linear equations, simplex
  - Note: sets $\{\mathbf{x}\}$, $\emptyset$, and $\mathbb{R}^n$ are convex

Concave and convex functions

- Let $f : U \to \mathbb{R}$ be a function, where $U \subseteq \mathbb{R}^n$ is a convex set.
- We say that f is a **concave function** if, for all $\mathbf{x}, \mathbf{y} \in U$, and for all $t \in [0, 1]$,

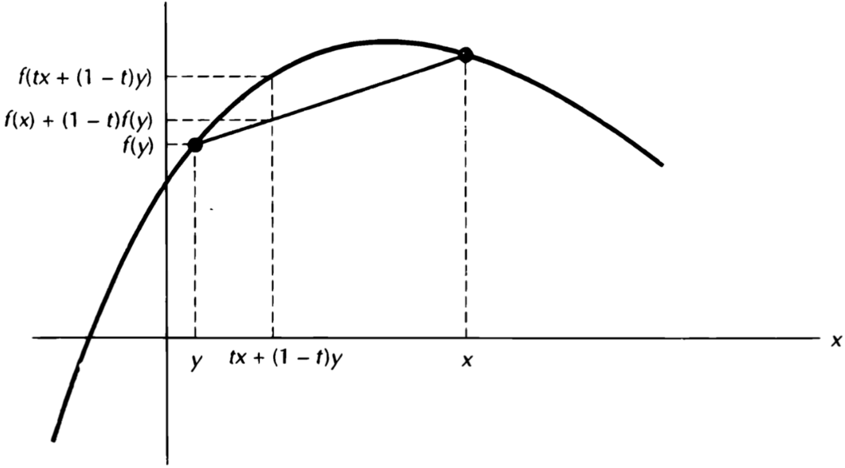
$$f(t\mathbf{x} + (1-t)\mathbf{y}) \geq tf(\mathbf{x}) + (1-t)f(\mathbf{y}).$$

- We say that f is a **convex function** if, for all $\mathbf{x}, \mathbf{y} \in U$, and for all $t \in [0, 1]$,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{y}).$$

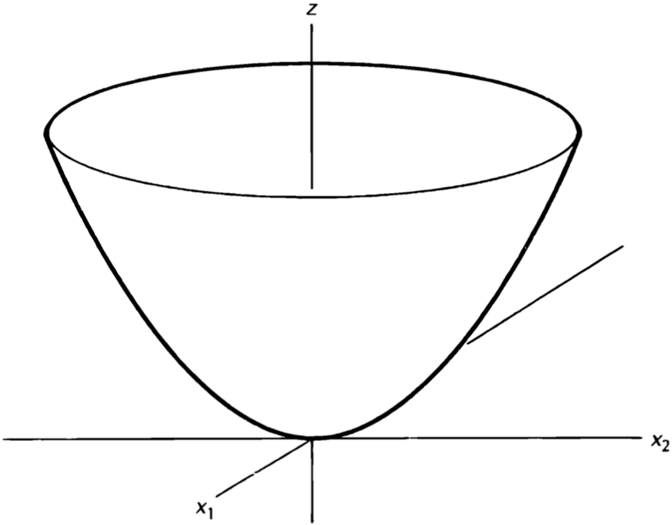
- In words, a function is concave if any secant line connecting two points on the graph of f lies *below* the graph. On the other hand, a function is convex if any secant line connecting two points on its graph lies *above* its graph

Concave and convex functions



The geometric interpretation of the definition of a concave function.

Concave and convex functions



The graph of the convex function $z = x_1^2 + x_2^2$.

Properties of concave and convex functions

- The domain of a convex or concave function is required to be a convex set
- A function f is concave if and only if $-f$ is convex
- A function $f : U \to \mathbb{R}$ is concave if and only if the set

  $$\{(\mathbf{x}, y) \in U \times \mathbb{R} : y \leq f(\mathbf{x})\} \tag{1}$$

  is convex. The set (1) is called the *hypograph* of f and is the set of points lying on or below the graph of f
- A function $f : U \to \mathbb{R}$ is convex if and only if the set

  $$\{(\mathbf{x}, y) \in U \times \mathbb{R} : y \geq f(\mathbf{x})\} \tag{2}$$

  is convex. The set (2) is called the *epigraph* of f and is the set of points lying on or above the graph of f

Upper and lower level sets

- Lower level set: $\mathcal{L}_c^-(f) = \{x \in U : f(\mathbf{x}) \leq c\}$
- Upper level set: $\mathcal{L}_c^+(f) = \{x \in U : f(\mathbf{x}) \geq c\}$
- Lower level sets of convex functions are convex sets
- Upper level sets of concave functions are convex sets

Examples of concave/convex functions

- Convex functions:

  $f(x) = ax + b,\ a, b \in \mathbb{R}$

  $f(x) = e^{ax},\ a \in \mathbb{R}$

  $f(x) = x^{\alpha},\ x \in \mathbb{R}_{++},\ \alpha \geq 1$ or $\alpha \leq 0$

- Concave functions:

  $f(x) = ax + b,\ a, b \in \mathbb{R}$

  $f(x) = \ln(x),\ x \in \mathbb{R}_{++}$

  $f(x) = x^{\alpha},\ x \in \mathbb{R}_{++},\ 0 \leq \alpha \leq 1$

Examples of concave functions

- Model of a firm
  - input vector $\mathbf{x} \in \mathbb{R}^n$
  - production function $f : \mathbb{R}^n_+ \mapsto \mathbb{R}_+ = \{y \in \mathbb{R} : y \geq 0\}$ that is concave and increasing in all its arguments
  - cost function $c : \mathbb{R}^n_+ \mapsto \mathbb{R}$, convex and increasing in all its arguments
  - price $p > 0$, profits $pf(\mathbf{x}) - c(x)$ defined for $\mathbf{x} \geq \mathbf{0}$
- Expenditure function $e(\mathbf{p}) = \min_{\mathbf{x} \in X} \mathbf{p} \cdot \mathbf{x}$ is concave
  - $\mathbf{p} \in \mathbb{R}^n$ is a vector of prices of inputs $x \in X \subset \mathbb{R}^n_+$, and X convex

Concave and convex functions

- A few properties:
  - If f and g are concave functions, then $f(\mathbf{x}) + g(\mathbf{x})$ is a concave function
  - If f is a concave function, then $af(\mathbf{x}) + b$, where $a \geq 0$, is a concave function
  - A function f is *both* concave and convex if and only if it is affine, i.e. $f(x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n + b$, where $a_1, \ldots, a_n, b \in \mathbb{R}$ are constants

Jensen's inequality

- **Jensen's inequality** is a property of convex or concave functions that is often used in microeconomics and finance in order to study an individual's attitude toward risk

- Let $f : \mathbb{R} \to \mathbb{R}$ be a concave function. Given numbers $x_1, \ldots, x_k$ in the domain of f,

$$f\left(\sum_{i=1}^{k} \lambda_i x_i\right) \geq \sum_{i=1}^{k} \lambda_i f(x_i) \tag{3}$$

  for all non-negative $\lambda_1, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i = 1$.

- If f is convex, the inequality in (3) is reversed

Strict concavity and convexity

- We say that f is a **strictly concave function** if, for all $\mathbf{x}, \mathbf{y} \in U$, $\mathbf{x} \neq \mathbf{y}$, and for all $t \in (0, 1)$,
$$f(t\mathbf{x} + (1-t)\mathbf{y}) > tf(\mathbf{x}) + (1-t)f(\mathbf{y}).$$

- We say that f is a **convex function** if, for all $\mathbf{x}, \mathbf{y} \in U$, $\mathbf{x} \neq \mathbf{y}$, and for all $t \in (0, 1)$,
$$f(t\mathbf{x} + (1-t)\mathbf{y}) < tf(\mathbf{x}) + (1-t)f(\mathbf{y}).$$

Second order test of concavity

Proposition

Let $f : U \to \mathbb{R}$ be a C^2 function, where $U \subseteq \mathbb{R}^n$ is an open and convex set.

1. *f is a* **concave function** *if and only if the Hessian $D^2 f(\mathbf{x})$ is* **negative semidefinite** *for all $x \in U$.*

2. *f is a* **convex function** *if and only if the Hessian $D^2 f(\mathbf{x})$ is* **positive semidefinite** *for all $x \in U$.*

Note 1 : for strict concavity/convexity semidefinitiness is replaced by definiteness

Note 2: the semidefiniteness of $D^2 f(\mathbf{x})$ must be checked *for all* $\mathbf{x} \in U$

Recall that if a matrix is positive (respectively, negative) definite, then it is also positive (respectively, negative) *semi*definite

Second order test of concavity

- **Example.** Consider the Cobb-Douglas production function $f : \mathbb{R}^2_+ \to \mathbb{R}$ given by $f(x, y) = x^a y^b$, with $a, b > 0$
- When is f concave?
- The Hessian of f is

$$D^2 f(x,y) = \begin{pmatrix} a(a-1)x^{a-2}y^b & abx^{a-1}y^{b-1} \\ abx^{a-1}y^{b-1} & b(b-1)x^a y^{b-2} \end{pmatrix}$$

- The Hessian has three principal submatrices/minors

Second order test of concavity

- **Example (cont'd).** $D^2 f(x, y)$ is negative semidefinite if and only if the following two conditions are met:

  1. The two first order principal minors $a(a-1)x^{a-2}y^b$ and $b(b-1)x^a y^{b-2}$ are non-positive. This happens when $a, b \leq 1$
  2. The second order principal minor $\det(D^2 f(x, y))$ is non-negative, which happens when $a + b \leq 1$

- Recalling that $a, b > 0$ by assumption, we can conclude that f is concave if and only if

$$0 < a < 1, \quad 0 < b < 1, \quad \text{and} \quad a + b \leq 1.$$

- Notice that f is concave if and only if returns to scale are constant or decreasing

Second order test of concavity

- **Example (cont'd).** When is f convex?

- $D^2 f(x, y)$ is positive semidefinite if and only if the following two conditions are met:

  1. The two first order principal minors $a(a-1)x^{a-2}y^b$ and $b(b-1)x^a y^{b-2}$ are non-negative. This happens when $a, b \geq 1$

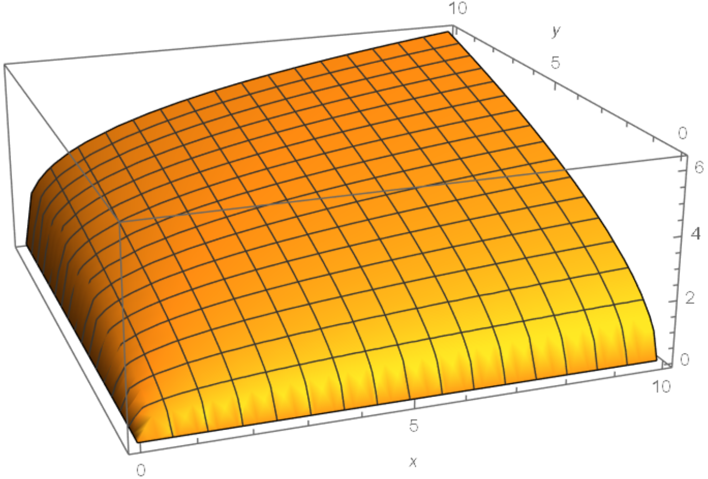
  2. The second order principal minor $\det(D^2 f(x, y))$ is non-negative, which happens when $a + b \leq 1$

- We clearly have that the two conditions $a, b \geq 1$ and $a + b \leq 1$ cannot hold simultaneously

- Thus we conclude that f is:
  - concave when $a + b \leq 1$
  - neither concave nor convex when $a + b > 1$

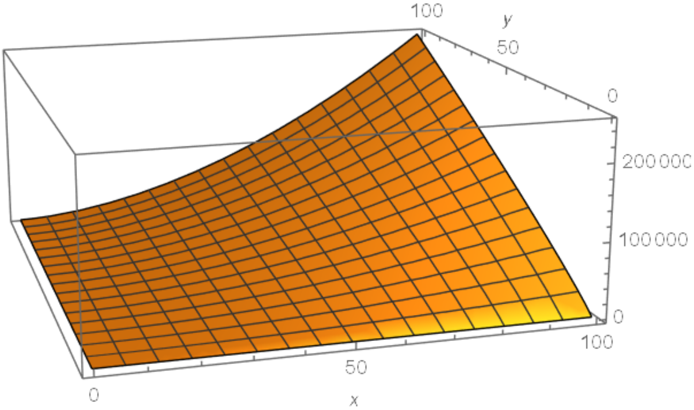
Second order test of concavity

- **Example (cont'd).** The concave production function $f(x, y) = x^{0.3}y^{0.5}$



Second order test of concavity

- **Example (cont'd).** The production function $f(x, y) = x^{1.8}y^{0.9}$, which is neither convex nor concave



Unconstrained optimization

- **Exercise.** Which of the following functions defined on $\mathbb{R}^n$ are concave or convex?

1. $f(x) = 3e^x + 5x^4 - \ln x$
2. $f(x, y) = -3x^2 + 2xy - y^2 + 3x - 4y + 1$
3. $f(x, y, z) = 3e^x + 5y^4 - \ln z$

Constrained optimization with concave functions

- The result stated at the beginning of this lecture that critical points of concave or convex functions are global extrema holds for functions having an *open* domain
- If the domain of a concave or convex function is not an open set, it could be the case that global extrema are on the *boundary* of the domain, e.g. $\max_{x\in[0,1]} x^2$. In those cases, critical points need not be global extrema
- The following result will give us conditions to identify global extrema even when they are possibly located on the boundary of the domain

Constrained optimization with concave functions

Proposition

Let $f : U \to \mathbb{R}$ be a C^1 function, where $U \subseteq \mathbb{R}^n$ is a convex (and not necessarily open) set.

1. *If f is a* **concave function**, $\mathbf{x}^* \in U$ *is a global maximizer of f if and only if* $\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \leq 0$ *for all* $\mathbf{x} \in U$.

2. *If f is a* **convex function**, $\mathbf{x}^* \in U$ *is a global minimizer of f if and only if* $\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0$ *for all* $\mathbf{x} \in U$.

Note: $\nabla f(x)^T$ is the $1 \times n$ matrix whose columns are the partial derivatives of f with respect to x_i.

$$\nabla f(\mathbf{x})^T = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) & \frac{\partial f}{\partial x_2}(x) & \dots & \frac{\partial f}{\partial x_n}(x) \end{pmatrix}$$

Constrained optimization with concave functions

- **Example.** Let the concave function $f(x,y) = x^{\frac{1}{4}} y^{\frac{1}{4}}$ be defined on the following **compact** and convex domain

$$U = \left\{(x,y) \in \mathbb{R}^2 : x + y \leq 2\right\}.$$

- The global maximizer of f on U is the point $(1,1)$. Indeed, for any $(x,y) \in U$ we have

$$\begin{aligned}
\begin{pmatrix} \frac{\partial f}{\partial x}(1,1) & \frac{\partial f}{\partial y}(1,1) \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \end{pmatrix} &= \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1) \\
&= \frac{1}{4}(x-1) + \frac{1}{4}(y-1) \\
&= \frac{1}{4}(x+y-2) \\
&\leq 0,
\end{aligned}$$

where the inequality follows from $x + y \leq 2$ in the definition of U.

Constrained optimization with concave functions

- Assume that f is a concave function defined on a convex set U
- Does $\max_{\mathbf{x} \in U} f(\mathbf{x})$ have a solution?

  concavity of f and convexity of U do not guarantee the existence of a solution (continuity and compactness are needed)
  note: a concave/convex function is continuous on the interior of its domain (discontinuities are possible on the boundary)

- If there is a solution, is it unique?

  no, but the set of maximizers is a convex set
  for uniqueness strict concavity is needed

Concavity and transformations

- Assume that $f : U \mapsto \mathbb{R}$ is a concave function, and $U \subseteq \mathbb{R}^n$ is a convex set
- Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a strictly increasing function
- Reminder: $\mathbf{x}^*$ is a solution of $\max_{\mathbf{x} \in U} f(\mathbf{x})$ is and only if it is a solution of $\max_{\mathbf{x} \in U} g(f(\mathbf{x}))$
- Assume that $\mathbf{x}^*$ satisfies first order optimality conditions of $\max_{\mathbf{x} \in U} g(f(\mathbf{x}))$, and $g(f(\mathbf{x}))$ is a concave function
- What can you say about the optimality of $\mathbf{x}^*$?

Log-concavity

- $f : U \mapsto \mathbb{R}_{++}$ is said to be log-concave if $\ln(f(x)$ is concave

  note that $\ln(x)$ is strictly increasing function
- Example: $f(x, y) = Ax^a y^b$, where $A, a, b > 0$
  - this function is log-concave

  → even if a Cobb-Douglas function does not satisfy constant or decreasing returns to scale, we may be able to say something about the optimality of points that satisfy the first order optimality conditions