Model Solutions 4

1. (a) First, note that the reservation value $v_i \in [-50, 50]$ of household i here indicates the valuation of housing in the core region relative to the periphery. Hence, willingness to pay for i for renting an apartment in the center is given by $v_i - (r_c - r_p)$, where r_c and r_p indicate rents in thousands of euros per year, for center and periphery, respectively. Since i) valuation is uniformly distributed in the given interval, ii) we have 1 million households, and iii) $r_c = 10$, we can express demand (in thousands) for apartments in the center as follows:

$$Q^d(r_c) = (600 - r_c \times 10).$$

Supply, however, is fixed, since no new apartments can be built the core region due to scarcity of land.

Equating supply with demand allows us to solve for the equilibrium rent: $600 - r_c \times 10 = 250 \leftrightarrow r_c^* = 35$. That is, the equilibrium rent for an apartment in the center is 35k/year.

This equilibrium is depicted in figure 1 (where price denotes rent level).

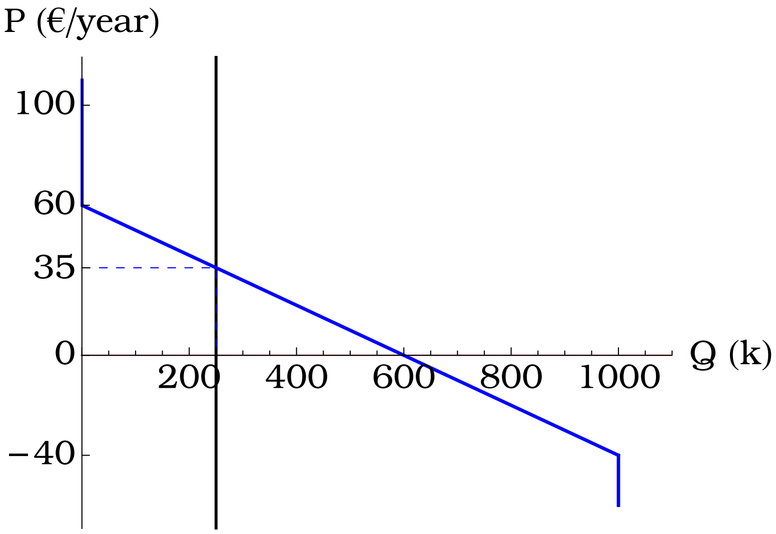

Figure 1: Supply and demand in the rental market of the center region

(b) The price of a house is given by the present value of the infinite stream of rent payments generated by owning one. Since interest rates in the economy are paid at the end of the year, while rents to apartments are paid during the ongoing year, discounting starts already in the first year. Naturally, cost of capital (or the opportunity cost of investing in a house), is given by the 5% interest rate, denoted by r.

We can then express the price by making use of the perpetuity formula

$$p_c^* = \frac{r_c^*}{r} = \frac{35}{0.05} = 700,$$

that is, the equilibrium price of an apartment in center is 700k[1] .

(c) Here the change in demand amounts to a shift in the whole distribution of valuation, that is, from $t = 10$ onward we have $v_i \in [-30, 70]$, and a similar reasoning as in the first subsection means the demand curve in period 10 is given by

$$Q^d_{t=10}(r_p) = (800 - r_c \times 10).$$

Clearly, as the demand shift occurs discontinuously after 10 years, the response in the rental level should also react discontinuously after 10 years. As supply remains fixed at 250k the whole time, we can solve for higher rent level needed to maintain equilibrium after the demand shock: $Q^d_{t=10}(r_c) = (800 - r_c \times 10) = 250 \leftrightarrow r^*_c = 55$. That is, the rent stays constant at 35 k/year from $t = 0$ up to $t = 10$, at which it jumps to 55k/year, in response to the demand shock.

(d) Given the new demand curve at $t \geq 10$, we can analogously solve for the supply that is needed to maintain the initial rent level: $Q^d_{t=10}(35) = (800 - 35 \times 10) = q^* \leftrightarrow q^* = 450$. This amounts to an increase of 200k apartments. This shift in the supply curve is visualized in figure 2.

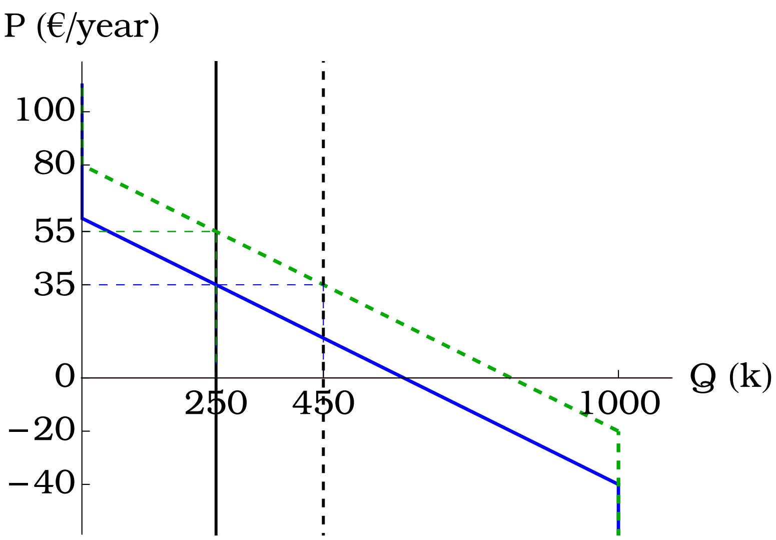

Figure 2

(e) We found that the rent will jump from the initial level of 35, to 55 in $t = 10$, after which it stays constant. Denoting the discount factor $\beta = \frac{1}{1+0.05}$, we can express the price in period 0 as follows:

$$p^*_{t=0} = [\frac{35}{0.05} - \frac{35}{0.05}\beta^{10}] + \beta^{10}\frac{55}{0.05} \approx 946.$$

[1]Note: Here the assumption that rent payments are paid in advance of the ongoing year was also considered as correct. This assumption means the first rent should not be discounted, meaning the price would be given by $\frac{1}{\beta}\frac{35}{0.05} = 735$, where $\beta = \frac{1}{1+0.05}$.

The first brackets captures the payments of 35k up to period 10, we express it here as the difference of two perpetuities. The rightmost term is the perpetuity that starts in period 10, discounted to period zero.

More generally, we can use this logic to express the price for any $t < 10$ as:

$$p_t^* = [\frac{35}{0.05} - \frac{35}{0.05}\beta^{10-t}] + \beta^{10-t}\frac{55}{0.05}, t = 0, ..., 9.$$

For $t >= 10$, we can treat the price as a perpetuity with payments of 55k:

$$p_t^* = \frac{55}{0.05}, t \geq 10.$$

Figure 3 plots the price of housing against time. By dividing the price with the rent level, we quickly see that the price-to-rent ratio increases from $t = 0$ up $t = 9$, and then it drops in $t = 10$. Then it stays constant, as price and rents do not change[2].

This result highlights how the price may increase in anticipation of expected future changes while the rent level does not change. This means that an increase in price need not reflect a price bubble despite the discrepancy between price and rent level.

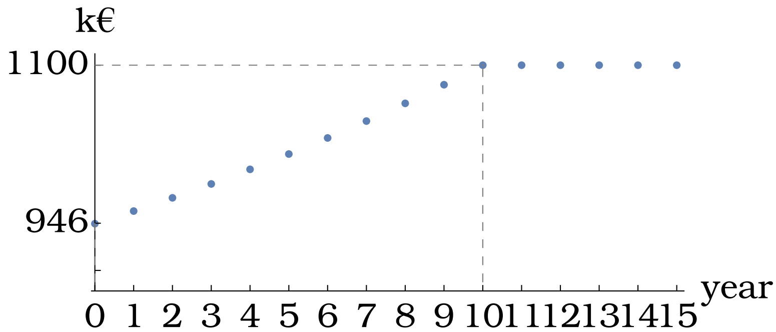

Figure 3

2. First, recall that for CRRA preferences, the Bernoulli utility, indicating utility of risk-free payment x, is given by

$$u(x) = \frac{x^{1-\rho}}{1-\rho}.$$

Second, recall that the certainty equivalent (CE) is the risk-free payment needed for a consumer to be indifferent between this risk-free option, and a lottery (here the risky investment), and that the relevant notion of utility here is expected utility. This indifference

[2]Note: As noted in the second subsection, if one interprets that rents are paid in advance of the ongoing year (which again was also considered as correct), discounting starts in period $t = 1$, meaning that the price will be calculated as explained above, but discounted "once less per period". Then the pries are as shown in the graph, but divided by discount factor β, e.g. $p_{t=0} = \frac{945.56}{\beta} \approx 992.84, ..., p_{t=10} = \frac{1100}{\beta} = 1155$. So the results overall remain qualitatively the same.

condition will directly allow us to solve for R_i, which is the only unknown for a given individual i. Denoting investment by I, we have:

$$\frac{(I(1+R_i))^{1-\rho_i}}{1-\rho_i} = 0.5\frac{(0.9I)^{1-\rho_i}}{1-\rho_i} + 0.5\frac{(1.2I)^{1-\rho_i}}{1-\rho_i}$$
$$\leftrightarrow R_i = [0.5(1.2^{1-\rho_i} + 0.9^{1-\rho_i})]^{\frac{1}{1-\rho_i}}, i = A, B, C.$$

The last equality shows that R is independent of wealth level I. Note that while CRRA preferences does not imply that the CE is independent of the investment into the lottery, it does imply the CE as a share of investment remains unchanged. That is, $\frac{CE}{I} = (1+R)$, the return on investment required remains unchanged. This is indeed the key property of CRRA preferences.

(a) We can now use the above observations to solve for R, given $I = 1M$. Here, note that for Bob, we have $\rho_C = 1$, in which case the Bernoulli utility can be expressed as $ln(x)$, which means that similarly solving for R from the indifference condition gives us

$$R_B = \sqrt{1.2 \times 0.9} - 1 \approx 0.039.$$

For Ann and Cecilia, we use the derived result to calculate R, and we get $R_A \approx 0.045$, and $R_C = 0.05$.

(b) As we observed in the previous subsection, R is independent of initial wealth in case of CRRA preferences. This means that for $I = 10M$, we have an identical result: $R_A \approx 0.045, R_B \approx 0.039, R_C = 0.05$.

3. (a) Here, we simply compare the expected values for the two options. Denoting C for Cumin, and F for Fava beans, we have

$$\begin{aligned} E(\pi^C) &= 0.5 \times 300 + 0.5 \times 100 = 200, \\ E(\pi^F) &= 180. \end{aligned}$$

So 200k is the highest expected value that can be achieved.

(b) Here, sensitivity refers to possible values that $p \in [0, 1]$ can take, that do not change the optimal decision found above. Since expected profits is what matters for the decision, we can simply solve for p as follows:

$$\begin{aligned} E(\pi^C) \geq E(\pi^F) &\leftrightarrow (1-p)300 + p100 \geq 180 \\ &\leftrightarrow p \leq 0.6. \end{aligned}$$

That is, the optimal decision remains unchanged for any $p \in [0, 0.6]$.

(c) Here we should compare the expected profits for the different decisions, which can be considered as sequences of actions. The decision tree for this exercise is illustrated in figure 4.

Consider first the option of waiting to find out whether $p = 0.2$ or $p = 0.8$. As the expected value of investing in cumin now should be independent of the possibility of waiting, we know either event $p = 0.2$ or $p = 0.8$ happens with probability 0.5. Now consider the event that after waiting, the agent learns that $p = 0.2$. The expected profits of investing in cumin is then $0.9(0.2 \times 300 + 0.8 \times 100) = 126$. Since the agent gets $0.9. \times 180 = 162$ from then investing in Fava beans, since $162 > 126$, the payoff after learning $p = 0.2$ is 162. Now consider the case where the agent learns that $p = 0.8$. Analogous calculations gives an expected profit of $0.9 \times (0.8 \times 300 + 0.2 \times 100) = 234$ for investing in cumin, and this is higher than 162, the profit from Fava beans.

Given these observations, we can determine the optimal initial decision. In expectation, waiting give profits of $0.5 \times 234 + 0.5 \times 162 = 198$. The remaining decisions are to invest now in Cumin, yielding the original $0.5 \times 300 + 0.5 \times 100 = 200$, and investing in Fava beans, which gives 180. So we conclude that it is optimal to invest immediately in Cumin.

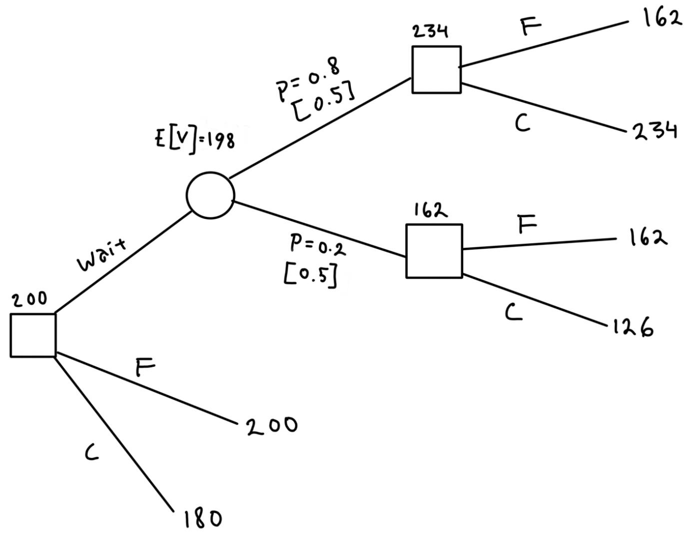

Figure 4: Decision tree of old MacDonald

(d) We found that when risk neutral, it was optimal to invest in Cumin immediately. We will now consider how a gradual increase in risk aversion affects the optimal decision, which should be considered as a sequence of actions (in the case of waiting, in particular).

First, observe that the agent will never invest in Cumin after bad news ($p = 0.2$), since this is both riskier and gives a lower expected payoff than investing in Fava beans. Hence, action after bad news remains the same regardless of the level of risk aversion. This means we can reduce the set of conceivable sequences of actions to the following set with corresponding expected profits, where we treat them as lotteries for conciseness:

$$L^C = (\{0.5, 0.5\}, \{300, 100\}), E(L^C) = 200$$
$$L^F = (\{1\}, \{180\}), E(L^F) = 180$$
$$L^{W,C} = (\{0.5, 0.4, 0.1\}, \{162, 270, 90\}), E(L^{W,C}) = 198$$
$$L^{W,F} = (\{0.5, 0.5\}, \{162, 162\}), E(L^{W,F}) = 162.$$

Here L^C and L^F denote the lotteries of directly investing in Cumin and Fava beans, respectively. Similarly, $L^{W,C}$ and $L^{W,F}$ denote the lotteries of first waiting, and after good news investing in Cumin and Fava beans, respectively.

Now observe that the expected payoff for $L^{W,C}$ is only slightly lower than the risk neutral optimum L^C, and that the variability of outcomes is relatively small. This means that as risk aversion increases, the farmer will first switch his decision to waiting, after which he invests in Cumin after good news (and trivially in Fava beans after bad news). If risk aversion still increases, at some point the farmer will switch to L^F, meaning the optimal decision will never be to wait and invest in Cumin after bad news. This is the case because this lottery has a certain payoff of 162, which is worse than 180.

To conclude, as risk aversion increases, Old MacDonald first switches to waiting and investing in Cumin after good news and Fava beans after bad news. As risk aversion still increases, the MacDonald eventually chooses to invest in Fava beans immediately.

4. (a) Since only one event brings any value, namely the one where each programmer succeeds in their respective task, the expected value of hiring programmers of type $k \in \{above, average\}$, is given by:

$$\begin{aligned} E_k(V) &= Prob(succes_k) \times V \\ &= p_k^n \times V, \end{aligned}$$

where we used that programmers successes are independent, and hence the probability is given by the product of the probability that an individual programmer succeeds. So we have:

$$\begin{aligned} E_{avg}(V) &= 0.9^5 \times 100 \approx 59.05 \\ E_{above}(V) &= 0.91^5 \times 100 \approx 62.4, \end{aligned}$$

i.e. the expected value of hiring average and above average programmers is 59,05 and 62,4 million euros, respectively.

(b) Here, we simply equate the expected value net of wages for the two scenarios, and solve for the only unknown, i.e. the pay level of an individual above-average programmer:

$$\begin{aligned} E_{avg}(V) - TC_{avg} &= E(V_{above}) - TC_{above} \\ \leftrightarrow 0.9^5 \times 100 - 5 \times 0.1 &= 0.91^5 \times 100 - 5 \times w_{above} \\ &\leftrightarrow w^*_{above} \approx 0.771. \end{aligned}$$

That is, the pay level of above-average programmers is 771k euros.

(c) Now $V = 200M$ and $n = 10$, and we can proceed as above to solve for w_{above}:

$$\begin{aligned} E_{avg}(V) - TC_{avg} &= E(V_{above}) - TC_{above} \\ \leftrightarrow 0.9^{10} \times 200 - 10 \times 0.1 &= 0.91^{10} \times 200 - 10 \times w_{above} \\ &\leftrightarrow w^*_{above} \approx 0.915. \end{aligned}$$

That is, the pay level of above-average programmers increases to 915k euros.