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Differential and Integral Calculus 1

Exercises, Week 6

Work on Warm-up 1–4 during the exercise sessions of Week 6.
Submit on MyCourses your solutions for Homework 1 and 2 by Sunday, October 16th.

Warm-up 1 (Differential equations)**:**

1. Solve the following differential equation and initial-value problems:

$$(a)\quad y''-6y'+10y=0 \qquad (b)\quad \begin{cases} y''-x(y')^2=0 \\ y(0)=0 \\ y'(0)=-2. \end{cases} \qquad (c)\quad \begin{cases} 2y''+5y'-3y=0 \\ y(0)=1 \\ y'(0)=0. \end{cases}$$

   (Hint: For (b), set $v=y'$.)

2. In the lectures we claimed that for a differential equation

$$ay''+by'+cy=0 \tag{†}$$

   with $b^2-4ac>0$, the general solution is of the form $y=Ae^{r_1x}+Be^{r_2x}$, for any real numbers A and B, and where r_1 and r_2 are the two distinct solutions of the equation $ar^2+br+c=0$. Now we prove this fact as follows:

   (a) Assume that $y=e^{r_1x}u$ is a solution to (†), for some function u. Show that u satisfies the equality $u''-(r_2-r_1)u'=0$.

   (b) Let $v=u'$, so that v must satisfy $v'=(r_2-r_1)v$. Recall that the general solution to this (separable) equation is $v=Ce^{(r_2-r_1)x}$, for $C\in\mathbb{R}$. Find u and then y.

Warm-up 2 (Complex numbers)**:**

1. Write the following complex numbers in the form $a+bi$, where a and b are real numbers:

$$(a)\quad (1+2i)(4+5i) \qquad (c)\quad \frac{3+i}{2+3i}$$

$$(b)\quad (4+2i)(4-2i) \qquad (d)\quad \frac{1}{i}+\frac{1}{i-2}.$$

   (Hint: Observe that for any complex number $a+bi$, the product $(a+bi)(a-bi)=a^2+b^2$ is a real number. As long as $a-bi\neq 0$, you can write $1=\frac{a-bi}{a-bi}$ and multiply anything by 1.)

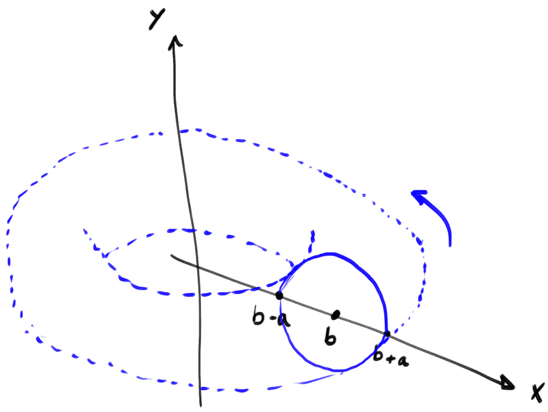
2. Find the multiplicative inverse of $1 + 2i$, that is, find the complex number $a + bi$ such that $(1 + 2i)(a + bi) = 1$.

3. Recall from the lectures that $e^{xi} = \cos x + i \sin x$ (for a real number x). Use this to show that

$$\cos x = \frac{e^{xi} + e^{-xi}}{2} \qquad \text{and} \qquad \sin x = \frac{e^{xi} - e^{-xi}}{2i}.$$

Warm-up 3: 1. Consider the functions $f(x) = x^2 - 2x + 3$ and $g(x) = -x^2 + 4x - 1$. Find their maxima/minima and sketch their graphs. Find the area of the finite region of the plane bounded by their graphs (i.e., simultaneously above f and below g).

2. A circle on the xy-plane is rotated about the y-axis to form a doughnut. The center of the circle is the point $(b, 0)$, and the radius is $a < b$. Find the volume of the doughnut.

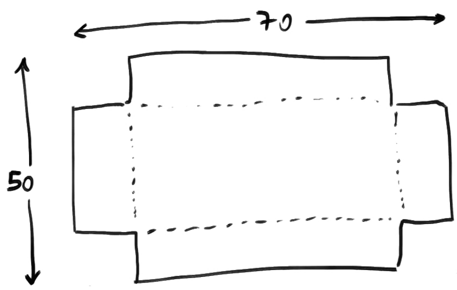


(Hint: The circle is given by the points (x, y) that satisfy $y^2 = a^2 - (x - b)^2$. A circle is not the graph of a function. But the upper half and lower half, taken separately, are. Try to always have a geometric picture in your mind for the integrals you find.)

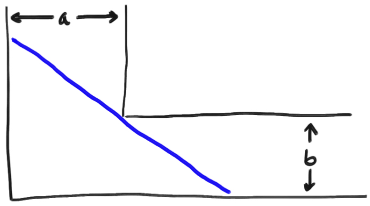
Warm-up 4: Write the following problems in calculus terms and solve them:

1. You are given a rope of fixed length. You can use it to make a rectangle on the ground. Which rectangle has the largest possible area?

2. A box is to be made from a rectangular 70×50 sheet of cardboard by cutting equal squares out of the four corners and bending up the resulting four flaps to make the sides of the box. (The box has no top.) What is the largest possible volume of the box?



3. What is the length of the longest beam that can be carried horizontally around the corner from a hallway of width a to a hallway of width b? (Assume the beam has no width.)



(Hint: The lines going through the point (A, B) have equation $y = s(x - A) + B$, where s is the slope. You might want to minimize a function of the slope s.)

Submit on MyCourses your solutions **only** for the following two problems. Explain the reasoning behind your solutions, do not just return the final result. If you make use of important results from the lectures, state what they are.

Homework 1: Solve the following differential equation and initial-value problem:

$$(a) \quad y'' + 8y' + 16y = 0 \qquad\qquad (b) \quad \begin{cases} y'' + 4y' + 5y = 0 \\ y(0) = 2 \\ y'(0) = 2. \end{cases}$$

[2 points]

Homework 2: The red and blue (non-shaded) parts of the drawing below are the graphs of the functions

$$f(x) = \begin{cases} -x^2 + 2x & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} -\frac{4}{3}x^2 + \frac{8}{3}x - \frac{7}{12} & \text{if } \frac{1}{4} \leq x \leq \frac{7}{4} \\ 0 & \text{otherwise,} \end{cases}$$

respectively. They are rotated about the x-axis to form a lemon. The peel is obtained by rotating the yellow part between the red and blue curves, and the pulp is obtained by rotating the part between the graph of g and the x-axis. Find the volume of the peel. [2 points]