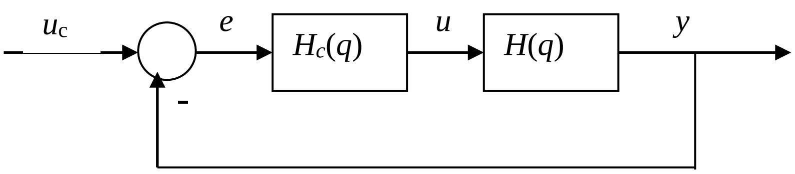
ELEC-E8101 Digital and Optimal Control
Exercise 5,
Solutions

1

System:



a.
The continuous system can be discretized by setting the step responses of the continuous and the discretized system equal. This means that

$$\mathrm{L}^{-1}\left\{\frac{1}{s}G(s)\right\}\Bigg|_{t=kh} = \mathrm{Z}^{-1}\left\{\frac{z}{z-1}H(z)\right\}$$

$$\Rightarrow H(z) = \frac{z-1}{z}\cdot \mathrm{Z}\left\{\mathrm{L}^{-1}\left\{\frac{1}{s}G(s)\right\}\Bigg|_{t=kh}\right\}.$$

Let's first calculate: $\frac{1}{s}G(s) = \frac{1}{s^2} \Rightarrow \mathrm{L}^{-1}\left\{\frac{1}{s}G(s)\right\} = t$,

the corresponding pulse sequence: kh. Its z-transformation: $Z\{kh\} = \frac{zh}{(z-1)^2}$ (from table).

Hence: $H(z) = \frac{z-1}{z}\cdot\frac{zh}{(z-1)^2} = \frac{h}{z-1}$. Note that this result could be obtained directly from the tables of ZOH-equivalent transfer function. (See Tables and Formulas in Materials section: Digital control).

b. $\tau = 0$. Hence $H_c(z) = K$ and the closed loop pulse transfer function is:

$$G_{CL}(z) = \frac{K\frac{h}{z-1}}{1+K\frac{h}{z-1}} = \frac{Kh}{z-1+Kh}.$$

This pulse transfer function has pole in $z_1 = 1 - Kh$. The stability condition is: $|z_1| < 1$
$\Rightarrow |1 - Kh| < 1 \Rightarrow 0 < Kh < 2 \Rightarrow 0\ K < 2 / h$. Note that it was already assumed in the problem statement that $K > 0$. Always $h > 0$ of course.

$\tau = h$ (the computation time is equal to one sample period).

The control law: $: u(kh) = K\left(u_c(kh-\tau) - y(kh-\tau)\right) =$

$$u(kh) = K\left(u_c(kh-h) - y(kh-h)\right) = Kq^{-1}\left(u_c(kh) - y(kh)\right)$$
$$= Kq^{-1}e\left(kh\right)$$
$$\Rightarrow qu\left(kh\right) = Ke\left(kh\right)$$
$$\Rightarrow H_c\left(q\right) = \frac{K}{q}$$

Hence the open loop pulse transfer function is $G_{OL}(z) = \frac{K}{z}\cdot\frac{h}{z-1}$ and the closed loop pulse transfer function is:

$$G_{CL}(z) = \frac{G_{OL}(z)}{1+G_{OL}(z)} = \frac{\frac{K}{z}\cdot\frac{h}{z-1}}{1+\frac{K}{z}\cdot\frac{h}{z-1}} = \frac{Kh}{z(z-1)+Kh}.$$

The characteristic equation becomes: $z^2 - z + Kh = 0$.

Form the Jury's table:

$$\begin{array}{lll} 1 & -1 & Kh \\ Kh & -1 & 1 \\ \hline \end{array}, \quad \alpha_2 = \frac{Kh}{1} = Kh$$

$$\begin{array}{ll} 1-\left(Kh\right)^2 & -1+Kh \\ -1+Kh & 1-\left(Kh\right)^2 \\ \hline \end{array}, \quad \alpha_1 = \frac{-1+Kh}{1-\left(Kh\right)^2} = \frac{-1}{\left(1+Kh\right)}$$

$$1-\left(Kh\right)^2 - \frac{-1\cdot\left(-1+Kh\right)}{\left(1+Kh\right)}$$

Let's examine the last term:

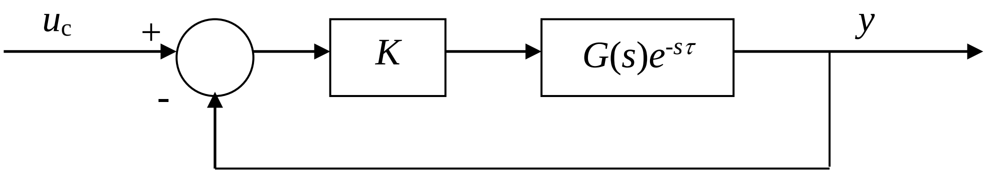
$$1 - (Kh)^2 + \frac{Kh-1}{1+Kh} = \frac{(1+Kh)^2(1-Kh)+Kh-1}{Kh+1} = \frac{(Kh-1)\left[1-(1+Kh)^2\right]}{Kh+1} =$$
$$= \frac{(Kh-1)(-2Kh-K^2h^2)}{Kh+1}$$

Stability conditions:

$1 > 0$
$1 - (Kh)^2 > 0$
$Kh - 1 < 0$ (since $Kh + 1 > 0$ and $-2Kh - K^2h^2 < 0$)
$\Rightarrow 1 - Kh > 0 \Rightarrow K < 1 / h$.

Note: The "Triangle Rule" from the Tables and Formulas could also have been used, because the characteristic polynomial was of the 2nd degree.

c. The system:



$$G_{CL}(s) = \frac{K \cdot \frac{1}{s} e^{-s\tau}}{1 + K \cdot \frac{1}{s} e^{-s\tau}} = \frac{Ke^{-s\tau}}{s + Ke^{-s\tau}}$$

The characteristic equation: $s + Ke^{-s\tau} = 0$.

The open loop transfer function is:

$$G_{OL}(s) = \frac{K}{s} e^{-s\tau} \Rightarrow G_{OL}(i\omega) = \frac{K}{i\omega} e^{-i\omega\tau}$$

The Nyquist diagram corresponding to this system crosses the negative real axis, while the phase is -180° = -π rad.

$$\angle G_{OL}(i\omega) = -\frac{\pi}{2} - \omega\tau = -\pi \Rightarrow \omega = \frac{\pi}{2\tau} = \omega_r$$

$$|G_{OL}(i\omega_r)| = \left|\frac{K}{\omega_r}\right| < 1 \Rightarrow K < \frac{\pi}{2\tau}$$

The result is that the gain can be larger in the continuous time system than in the corresponding ZOH equivalent discrete time system. Although the processes are ZOH equivalent, the closed loop system is not! Also, note that "moving" the delay from the control to the process does not change the stability: the open loop transfer function, which determines stability through the characteristic equation, remains the same.

2. The pulse transfer operator of the process is

$$H(q) = \frac{1}{q(q-0.5)}$$

and for the closed loop system

$$H_T(q) = \frac{H_c(q)H(q)}{1 + H_c(q)H(q)}$$

a. $H_c(q) = K > 0$

$$H_T(q) = \frac{K\dfrac{1}{q(q-0.5)}}{1+K\dfrac{1}{q(q-0.5)}} = \frac{K}{q^2-0.5q+K}$$

The characteristic polynomial is: $p(q) = q^2 - 0.5q + K$. It is of the second order and therefore the triangle rule can be used.

The triangle rule:
$a_2 < 1$, $a_2 > -1 + a_1$ and $a_2 > -1 - a_1$ for a second order equation $z^2 + a_1 z + a_2 = 0$.

The stability conditions in this case are:

$K < 1$
$K > -1-0.5 \Rightarrow K > -1.5$
$K > -1+0.5 \Rightarrow K > -0.5$

The system is stable, if $-0.5 < K < 1$ or actually $0 < K < 1$, because it was assumed that K is positive.

In stationary state:

$$Y(z) = H_T(z)U_c(z) = \frac{K}{z^2-0.5z+K}\cdot\frac{z}{z-1}$$

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(1-z^{-1})Y(z) = \frac{K}{1-0.5+K} = \frac{K}{K+0.5}$$

b. $H_c(q) = \dfrac{Kq}{q-1}$, K > 0, in other words: H_c is integrator.

$$H_T(q) = \frac{\dfrac{Kq}{q-1}\cdot\dfrac{1}{q(q-0.5)}}{1+\dfrac{Kq}{q-1}\cdot\dfrac{1}{q(q-0.5)}} = \frac{Kq}{q(q-1)(q-0.5)+Kq} =$$

$$= \frac{Kq}{q\left(q^2-0.5q-q+0.5+K\right)} = \frac{K}{q^2-1.5q+0.5+K}$$

The characteristic polynomial: $p(q) = q^2 - 1.5q + 0.5 + K$.
As above, the stability conditions are:

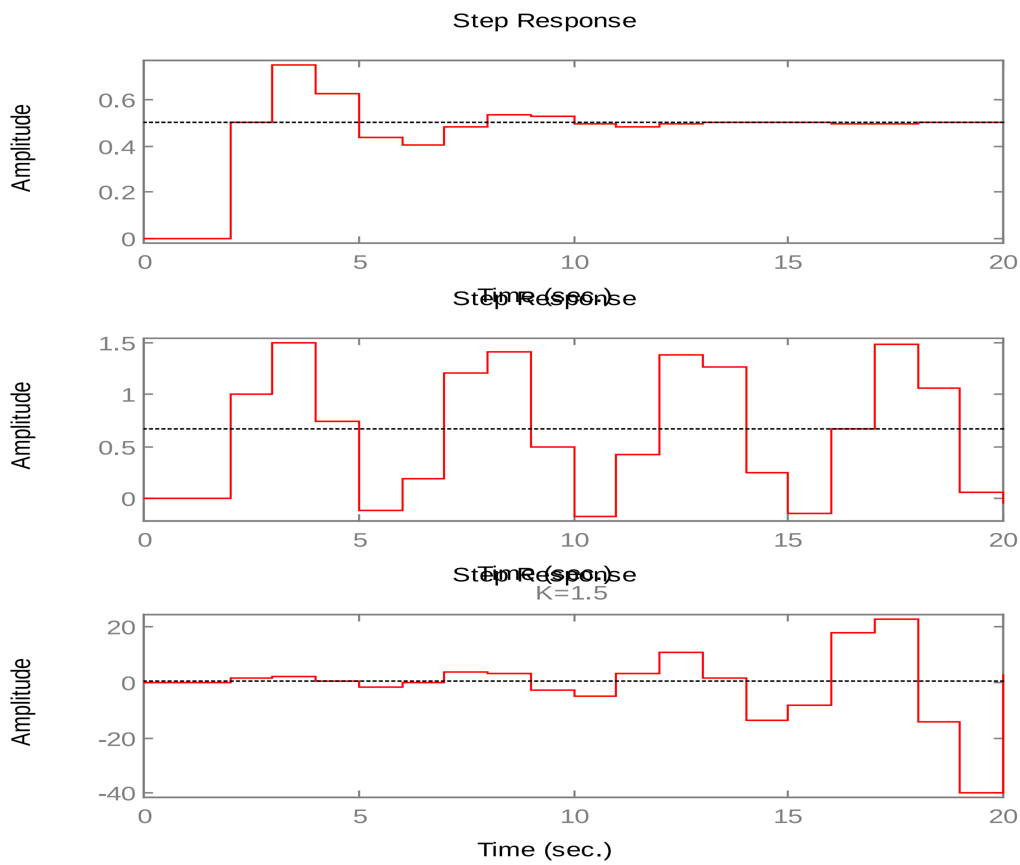
$0.5 + K < 1 \Rightarrow K < 0.5$
$0.5 + K > -1 -1.5 \Rightarrow K > -3$
$0.5 + K > -1 + 1.5 \Rightarrow K > 0$

The system is asymptotically stable, if $0 < K < 0.5$.

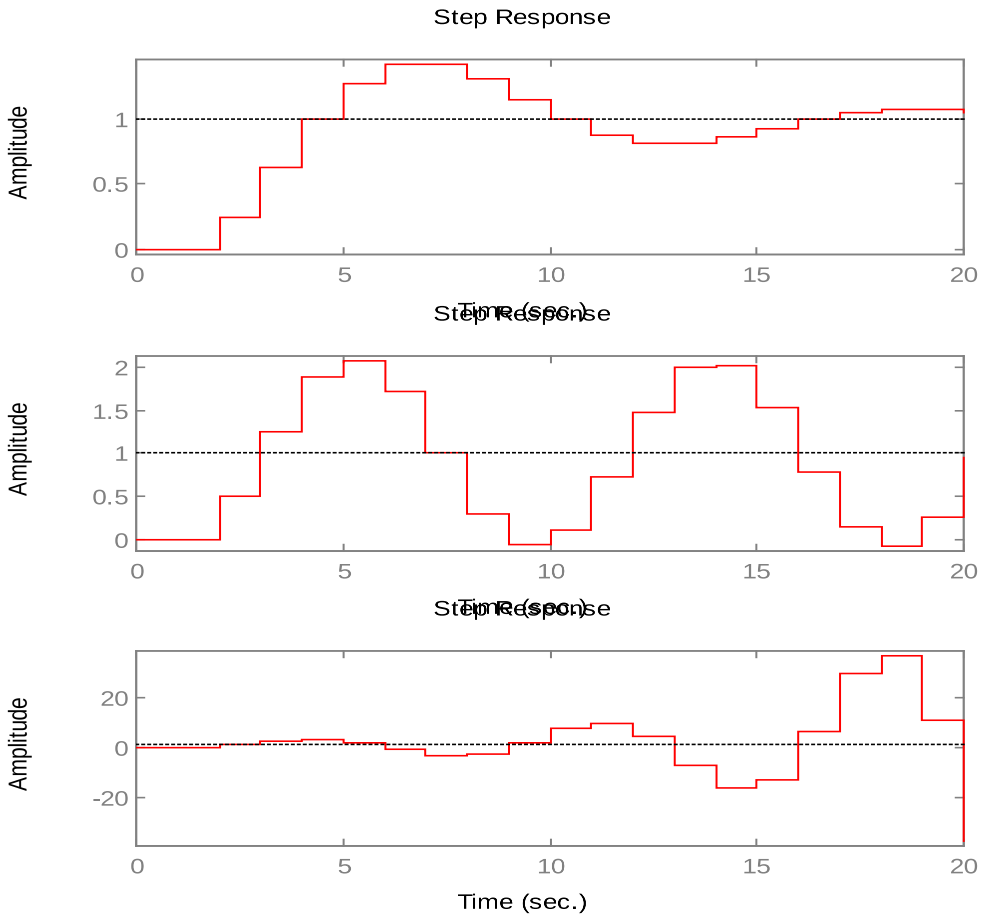
The stationary value:

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(1-z^{-1})Y(z) = \frac{K}{1-1.5+0.5+K} = \frac{K}{K} = 1$$

Step responses of the part **a.** (in order K = 0.5, 1 and 1.5):



Step responses of the part **b.** (in order K = 0.25, 0.5 and 1):



3. The closed loop system is:

$$T(z)=\frac{G(z)}{1+G(z)}=\frac{\dfrac{K}{(z-0,2)(z-0,4)}}{1+\dfrac{K}{(z-0,2)(z-0,4)}}=\frac{K}{(z-0,2)(z-0,4)+K}$$

$$=\frac{K}{z^2-0,6z+0,08+K}=\frac{K}{z^2-0,6z+a},$$

where $a = 0{,}08 + K$.

a)

Jury's stability test:

$$
\begin{array}{lll}
1 & -0,6 & a \\
a & -0,6 & 1 \\
\hline
1-a^2 & -0,6+0,6a & \\
-0,6+0,6a & 1-a^2 & \\
\hline
1-a^2-\frac{\left[-0,6(1-a)\right]^2}{1-a^2} & &
\end{array}
\qquad \alpha_2 = a \qquad \alpha_1 = \frac{-0,6+0,6a}{1-a^2}
$$

Stability conditions:

$$
\begin{cases}
1>0 \\
1-a^2>0 \\
1-a^2-\frac{0,36(1-a)^2}{1-a^2}>0
\end{cases}.
$$

1 > 0 is true, so let's examine the second condition:

$$
\begin{aligned}
&1-a^2>0 \\
&\Leftrightarrow a^2<1 \\
&\Leftrightarrow -1<a<1 \\
&\Leftrightarrow -1<0,08+K<1 \\
&\Leftrightarrow -1,08<K<0,92
\end{aligned}
$$

These are the first limits for *K*. The last condition:

$$
\begin{aligned}
&1-a^2-\frac{0,36(1-a)^2}{1-a^2}>0 \\
&\Leftrightarrow 1-a^2>\frac{0,36(1-a)^2}{1-a^2}
\end{aligned}
$$

We know that 1- $a^2 > 0$, so we can multiply the both sides of the inequality by that without changing the direction of inequality.

$$
\begin{aligned}
&1-a^2>\frac{0,36(1-a)^2}{1-a^2} \\
&\Leftrightarrow \left(1-a^2\right)^2>0,36(1-a)^2
\end{aligned}
$$

Let's take the square root:

$$
\begin{aligned}
&\left(1-a^2\right)^2>0,36(1-a)^2 \\
&\Rightarrow -\left(1-a^2\right)<0,6(1-a)<1-a^2
\end{aligned}.
$$

Let's examine the two inequalities separately:

$$-\left(1-a^2\right) < 0,6\left(1-a\right)$$
$$\Leftrightarrow -1+a^2 < 0,6-0,6a$$
$$\Leftrightarrow \left(0,08+K\right)^2 + 0,6\left(0,08+K\right)-1,6 < 0$$
$$\Leftrightarrow K^2 + 0,16K + 0,0064 + 0,048 + 0,6K - 1,6 < 0$$
$$\Leftrightarrow K^2 + 0,76K - 1,5456 < 0$$
$$\Rightarrow -1,68 < K < 0,92$$

The second inequality:

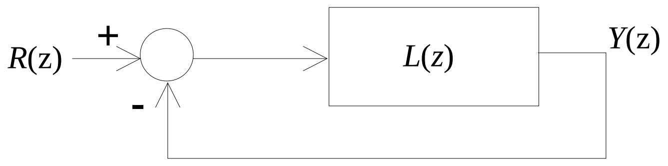
$$0,6\left(1-a\right) < 1-a^2$$
$$\Leftrightarrow 0,6-0,6a < 1-a^2$$
$$\Leftrightarrow \left(0,08+K\right)^2 - 0,6\left(0,08+K\right)-0,4 < 0$$
$$\Leftrightarrow K^2 + 0,16K + 0,0064 - 0,048 - 0,6K - 0,4 < 0$$
$$\Leftrightarrow K^2 - 0,44K - 0,4416 < 0$$
$$\Rightarrow -0,48 < K < 0,92$$

The last condition is the strongest for *K*, so we choose $-0{,}48 < K < 0{,}92$.

The Nyquist stability criterion:

The discrete Nyquist criterion

Let’s examine the stability of the following closed loop system:



The stability of the closed loop system can be examined by the open loop *L(z)* Nyquist curve. $L(e^{i\omega h})$ is the open loop Nyquist curve (in the complex plane) and it encircles the point (-1,0) *N* times clockwise. Then

$$N = Z - P$$

where

N is the number of encirclements of *L(z)* around the point (-1,0) clockwise as *z* encircles the unit circle counterclockwise

Z is the number of zeros of the characteristic equation outside the unit circle

P is the number of poles of the characteristic equation outside the unit circle.

A general characteristic equation is of form:

$$1+\frac{num_{OL}(z)}{den_{OL}(z)}=\frac{den_{OL}(z)+num_{OL}(z)}{den_{OL}(z)}=\frac{num_{CE}(z)}{den_{CE}(z)}=0,$$

where

$num_{OL}(z)$ is the numerator polynomial of the open loop system

$den_{OL}(z)$ is the denominator polynomial of the open loop system.

The open loop poles are the same as the poles of the characteristic equation. The zeros of the characteristic equation determine the stability of the system so that if the characteristic equation has zeros outside the unit circle, then the closed loop system is unstable. The stability criterion is thus obtained by setting *Z* = 0 and by demanding that the Nyquist curve encircles the point (-1,0) *P* times *counterclockwise* (*Z* = *N* + *P* = 0).

We can calculate the stability properties of the system without approximation by using the frequency response. The frequency response of the open loop system:

$$L(z)=\frac{K}{(z-0,2)(z-0,4)}$$

is obtained by setting $z = e^{i\omega h} = e^{i\omega}$, when *h* = 1.

$$L\left(e^{i\omega}\right)=\frac{K}{\left(e^{i\omega}-0,2\right)\left(e^{i\omega}-0,4\right)}=\frac{K}{\left(\cos\omega+i\sin\omega-0,2\right)\left(\cos\omega+i\sin\omega-0,4\right)}=$$

$$=\frac{K}{\left(\cos^2\omega-\sin^2\omega-0,6\cos\omega+0,08\right)+i\left(2\sin\omega\cos\omega-0,6\sin\omega\right)}=$$

$$=\frac{K}{\left(\cos^2\omega-\left(1-\cos^2\omega\right)-0,6\cos\omega+0,08\right)+i\left(2\sin\omega\cos\omega-0,6\sin\omega\right)}=$$

$$=\frac{K}{\left(2\cos^2\omega-0,6\cos\omega-0,92\right)+i\left(2\sin\omega\cos\omega-0,6\sin\omega\right)}.$$

The interesting points when stability is concerned are those where the frequency response crosses the real axis. There the imaginary part of the function is thus zero:

$$\begin{aligned}&2\sin\omega\cos\omega-0,6\sin\omega=0\\&\Leftrightarrow\sin\omega\left(2\cos\omega-0,6\right)=0\\&\Rightarrow\sin\omega=0\vee 2\cos\omega-0,6=0\,.\\&\Leftrightarrow\sin\omega=0\vee\cos\omega=0,3\\&\Rightarrow\omega=0\vee\omega=\arccos\left(0,3\right)\end{aligned}$$

With positive values of K the solution $\omega = 0$ ($z = e^0 = 1$) is the starting point of the frequency response and thus it is more interesting to examine the behaviour of the frequency response at ω = arccos(0,3):

$$L\left(e^{i\omega}\right)=L\left(e^{i\arccos(0,3)}\right)=\frac{K}{2\left(0,3\right)^2-0,6\cdot 0,3-0,92}=\frac{K}{-0,92}\,.$$

To have stable closed loop system, it must hold that $L\left(e^{i\omega}\right)=\frac{K}{-0,92}>-1$. Then the Nyquist curve does not encircle the point (-1,0).

$$\frac{K}{-0,92}>-1\Rightarrow K<0,92\,.$$

If K is negative, the form of the Nyquist curve stays, but it is symmetric with respect to the origin (e.g. a point 1+0.5*j* on the original Nyquist curve becomes now -1-0.5*j*). This means that the stability is now determined by $\omega = 0$.

$$\begin{aligned}&L\left(e^{i\omega}\right)=L\left(e^{i\cdot 0}\right)=\frac{K}{2\left(\cos 0\right)^2-0,6\cos 0-0,92}=\frac{K}{2-0,6-0,92}=\frac{K}{0,48}>-1\\&\Rightarrow K>-0,48\end{aligned}.$$

Altogether we have: $-0,48 < K < 0,92$.

The triangle rule:

If the characteristic equation of a system is:

$$A(z) = z^2 + a_1 z + a_2 ,$$

where a_1 and a_2 are constants, to find out wheather the system is stable or not, it is enough to examine only the following inequalities:

$$\begin{cases} a_2 < 1 \\ a_2 > -a_1 - 1 \\ a_2 > a_1 - 1 \end{cases}.$$

If these hold, then the system is stable. The name is “triangle rule”, because if the area that these inequalities bound is drawn in the a_1,a_2 –plane, the area forms a triangle.

As the general characteristic equation $A(z)$ is compared with the given characteristic equation, we can see that

$$\begin{cases} a_1 = -0,6 \\ a_2 = 0,08 + K \end{cases}.$$

Let’s examine the stability criteria:

$$\begin{cases} 0,08 + K < 1 \\ 0,08 + K > 0,6 - 1 \\ 0,08 + K > -0,6 - 1 \end{cases}$$

$$\begin{cases} K < 0,92 \\ K > -0,48 \\ K > -1,68 \end{cases} .$$

$$\Rightarrow -0,48 < K < 0,92$$

We get the same result as by the Jury test and by the Nyquist criterion (of course).

b)

The closed loop pulse transfer function:

$$T(z) = \frac{G(z)}{1+G(z)} = \frac{\dfrac{K}{z(z-0,2)(z-0,4)}}{1+\dfrac{K}{z(z-0,2)(z-0,4)}} = \frac{K}{z(z-0,2)(z-0,4)+K}$$

$$= \frac{K}{z^3 - 0,6z^2 + 0,08z + K}$$

Let's form the corresponding Jury's table.

$$\begin{array}{cccc} 1 & -0,6 & 0,08 & K \\ K & 0,08 & -0,6 & 1 \end{array} \qquad \alpha_3 = \frac{K}{1} = K$$

$$\begin{array}{ccc} 1-K^2 & -0,6-0,08K & 0,08+0,6K \\ 0,08+0,6K & -0,6-0,08K & 1-K^2 \end{array} \qquad \alpha_2 = \frac{0,08+0,6K}{1-K^2}$$

$$\begin{array}{cc} \dfrac{K^4-2,36K^2-0,096K+0,9936}{1-K^2} & \dfrac{0,08K^3+0,6064K^2+0,016K-0,24}{1-K^2} \\ \dfrac{0,08K^3+0,6064K^2+0,016K-0,24}{1-K^2} & \dfrac{K^4-2,36K^2-0,096K+0,9936}{1-K^2} \end{array}$$

$$\alpha_1 = \frac{0,08K^3+0,6064K^2+0,016K-0,24}{K^4-2,36K^2-0,096K+0,9936}$$

The Jury's stability criterion (as well as the Routh's table) could be useful especially when analytic stability conditions for some parameters are desired (like here). In this case the analytic condition seems to become so difficult that it can not be solved in simple pen and paper calculations.

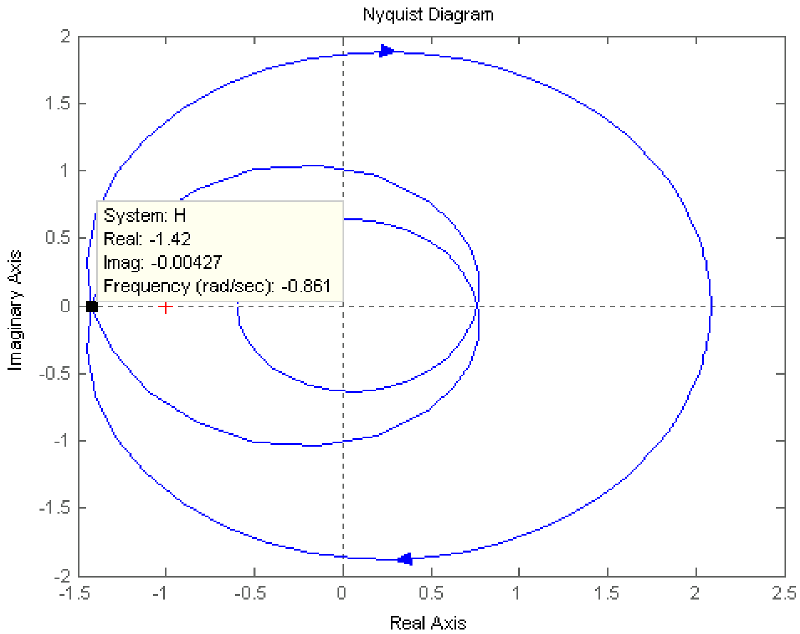
This is why the problem is solved only with using the Nyquist curve in Matlab.

The Nyquist curve for the system under consideration (K = 1)

```
>> K = 1;
>> H = zpk([],[0 0.2 0.4],K,1)

        Zero/pole/gain:
              1
        -----------------
        z (z-0.2) (z-0.4)

>> nyquist(H)
```



1° K > 0

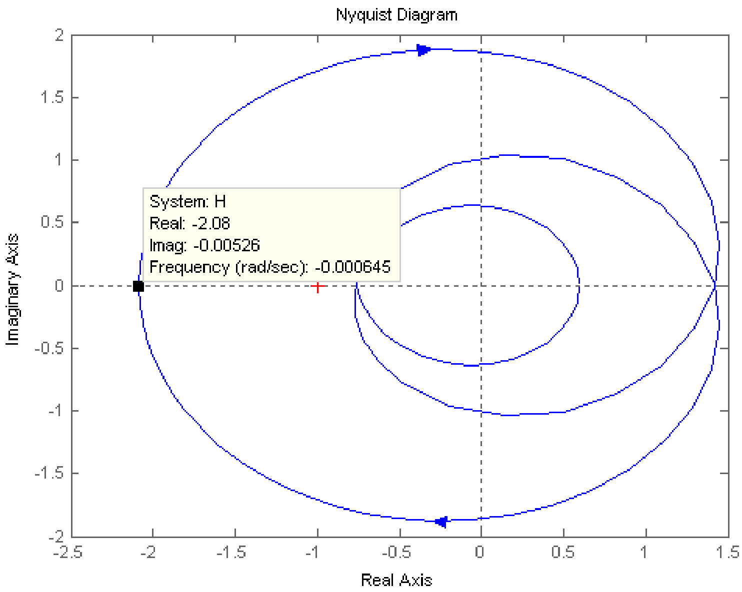
The Nyquist curve encircles the point –1 two times clockwise, so the closed loop system is unstable when $K = 1$. Notice that the system can have 0, 1, 2 or 3 unstable poles depending on the gain ($Z = N + P$)! Now $Z = 0 + 2 = 2$, so the closed loop has two unstable poles. Hint: plot the Nyquist diagram yourself and make it clear to you, how the diagram goes, when ω goes from 0 to pi (first half of the unit circle) and then from pi to 2pi (second half of the unit circle). Note that the Nyquist diagram is always symmetric with respect the real axis.

The Nyquist curve cuts the real axis at point -1,42, so the system is stable when

$$K < \frac{1}{1,42} = 0,70.$$

2° K < 0

If K is negative, the Nyquist plot is rotated 180 degrees.



In this case there are no encirclements of -1 if

$$K > \frac{-1}{2,08} = -0,48\,.$$

Both cases combined, the system is stable, if

$$-0,48 < K < 0,70\,.$$

4. The system:

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(k) \;\; i.e. \;\; \begin{cases} x_1(k+1) = x_2(k) + 2x_3(k) \\ x_2(k+1) = 3x_3(k) + u(k) \\ x_3(k) = 0 \end{cases}$$

a. $\mathbf{x}(0) = [1\;\;1\;\;1]^T$.

Let us calculate the state transition directly from the above equations when the controls are $u(0)$, $u(1)$,… Let's try to get the state to the origin.

1. step:

$x_1(1) = x_2(0) + 2x_3(0) = 1 + 2\cdot1 = 3$

$x_2(1) = 3x_3(0) + u(0) = 3\cdot1 + u(0) = 3 + u(0)$

$x_3(1) = 0$

$\mathbf{x}(1) = [3 \;\; u(0)+3 \;\; 0]^T$, *i.e.* there doesn't exist a control $u(0)$ setting the state to the origin at this point.

2. step:

$x_1(2) = x_2(1) + 2x_3(1) = 3 + u(0) + 2\cdot 0 = 3 + u(0)$

$x_2(2) = 3x_3(1) + u(1) = 3\cdot 0 + u(1) = u(1)$

$x_3(2) = 0$

$\mathbf{x}(2) = [u(0)+3 \;\; u(1) \;\; 0]^T$, by choosing $u(0) = -3$ and $u(1) = 0$ the origin is reached in $k = 2$.

b. See the above result: minimum number of steps is 2.

State $[1 \;\; 1 \;\; 1]^T$ can't be reached from the origin, since the state equations indicate that x_3 is zero all the time.

More formally:

The controllability matrix is:

$$\mathbf{W}_c = [\mathbf{\Gamma} \;\; \mathbf{\Phi\Gamma} \;\; \mathbf{\Phi}^2\mathbf{\Gamma}] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \det\mathbf{W}_c = 0 \Rightarrow \text{not reachable.}$$

However, this is not enough to guarantee that a given state cannot be reached from the origin. The columns $[0 \;\; 1 \;\; 0]^T$ and $[1 \;\; 0 \;\; 0]^T$ of $\mathbf{W}_c$ are linearly independent. According to the theory of state space models, the states that can be reached from the origin are linear combinations of these columns. Hence all of the reachable states are of form:

$$\mathbf{x} = \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}, \alpha, \beta \in \mathbf{R}$$

Since the last component is always zero, the state $[1 \;\; 1 \;\; 1]^T$ can't be reached.

5. In the stationary point

$$x_s(k+1) = x_s(k) = \Phi x_s(k) + \Gamma u_s(k)$$
$$y_s(k) = Cx_s(k) + Du_s(k)$$

Define

$$\tilde{x}(k) = x(k) - x_s$$
$$\tilde{u}(k) = u(k) - u_s$$
$$\tilde{y}(k) = y(k) - y_s$$

and write the new state equations

$$\begin{aligned}\tilde{x}(k+1) &= x(k+1) - x_s = \Phi(\tilde{x}(k) + x_s) + \Gamma(\tilde{u}(k) + u_s) - x_s \\ &= \Phi\tilde{x}(k) + \Gamma\tilde{u}(k) + \underbrace{\Phi x_s + \Gamma u_s - x_s}_{0} \\ &= \Phi\tilde{x}(k) + \Gamma\tilde{u}(k)\end{aligned}$$

$$\begin{aligned}\tilde{y}(k) &= C(\tilde{x}(k) + x_s) + D(\tilde{u}(k) + u_s) - y_s \\ &= C\tilde{x}(k) + D\tilde{u}(k) + \underbrace{Cx_s + Du_s - y_s}_{0} \\ &= C\tilde{x}(k) + D\tilde{u}(k)\end{aligned}$$

The possibility of making this kind of a scaling in the stationary point is the fundamental reason, why we in control engineering can usually assume “zero initial conditions”, even though the real signals in the loop (setpoints etc.) would we non-zero.

6.

A state is reachable, if it can be controlled from any state to any other state in finite time. The system is reachable, if all states are.

Starting from origin:

$$\begin{aligned}&\mathbf{x}(1) = \Gamma u(0) \\ &\mathbf{x}(2) = \Phi\,\mathbf{x}(1) + \Gamma u(1) = \Phi\Gamma u(0) + \Gamma u(1) \\ &\mathbf{x}(3) = \Phi\,\mathbf{x}(2) + \Gamma u(2) = \Phi^2\Gamma u(0) + \Phi\Gamma u(1) + \Gamma u(2) \\ &\vdots \\ &\mathbf{x}(N) = \Phi^{N-1}\Gamma u(0) + \Phi^{N-2}\Gamma u(1) + \cdots + \Gamma u(N-1)\end{aligned}$$

It turns out that if the system is of order N, the final state can be reached in N time steps, if it is possible at all.

Let’s write the previous into matrix form:

$$\mathbf{x}(N) = \begin{bmatrix}\Gamma & \Phi\Gamma & \Phi^2\Gamma & \cdots & \Phi^{N-1}\Gamma\end{bmatrix}\begin{bmatrix}u(N-1) \\ \vdots \\ u(1) \\ u(0)\end{bmatrix}$$

where

$$\begin{bmatrix}\Gamma & \Phi\Gamma & \Phi^2\Gamma & \cdots & \Phi^{N-1}\Gamma\end{bmatrix} = \mathbf{W}_C$$

is called a controllability matrix.

Arbitrary final state $\boldsymbol{x}(N)$ can be reached only if the rank of the controllability (reachability) matrix is full, *i.e.* in the case of square matrices, the determinant is not 0.

In the case of this problem the system is:

$$\mathbf{x}(k+1) = 0,2\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbf{x}(k)$$

The corresponding controllability matrix:

$$\mathbf{W}_C = \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad 0,2\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad 0,2^2\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 0 & 0,2 & 0,08 \\ 0 & 0,2 & 0,04 \\ 1 & 0 & 0,04 \end{bmatrix}.$$

Since det($\mathbf{W}_c$) = -0,008, the system is reachable. The control values taking the state of the system to the final state $\boldsymbol{x}(3)$ can be obtained in the following way:

$$\mathbf{x}(3) = \mathbf{W}_C\begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} = \mathbf{W}_C^{-1}\mathbf{x}(3)$$

7.

Observability: The system is observable, if the initial state can be determined by the knowledge of a finite number of system inputs and outputs.

$$\begin{aligned} \mathbf{x}(0) &= \mathbf{x}(0) & y(0) &= Cx(0) \\ \mathbf{x}(1) &= \Phi\,\mathbf{x}(0) & y(1) &= Cx(1) = C\Phi x(0) \\ \mathbf{x}(2) &= \Phi\,\mathbf{x}(1) = \Phi^2\,\mathbf{x}(0) & y(2) &= Cx(2) = C\Phi^2 x(0) \\ &\vdots & &\vdots \\ \mathbf{x}(N-1) &= \Phi\,\mathbf{x}(N-2) = \Phi^{N-1}\,\mathbf{x}(0) & y(N-1) &= Cx(N-1) = C\Phi^{N-1}x(0) \end{aligned}$$

In the matrix form:

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C\Phi} \\ \mathbf{C\Phi}^2 \\ \vdots \\ \mathbf{C\Phi}^{N-1} \end{bmatrix} \mathbf{x}(0) = \mathbf{W}_O \mathbf{x}(0)$$

$\boldsymbol{x}(0)$ can be solved uniquely if the rank of $\boldsymbol{W}_O$ (observability matrix) is full, meaning that the inverse matrix exists.

⇒ A system is observable, if rank($\mathbf{W}_o$) is full.

In the case of this problem:

$$\mathbf{W}_O = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} 0,2 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} 0,2^2 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0,2 & 0 & 0 \\ 0,04 & 0,04 & 0,04 \end{bmatrix}$$

The determinant is 0.008, *i.e.* the rank is full, and the system is observable. The initial state can be solved from three samples:

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \end{bmatrix} = \mathbf{W}_O \mathbf{x}(0) \quad \Rightarrow \quad \mathbf{x}(0) = \mathbf{W}_O^{-1} \begin{bmatrix} y(0) \\ y(1) \\ y(2) \end{bmatrix}$$