Mathematics for Economists

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Quadratic Forms

Quadratic forms

- Quadratic forms are a class of functions that, much like concave or convex functions, have nice properties in optimization problems

- A **quadratic form** on $\mathbb{R}^n$ is a function $Q : \mathbb{R}^n \to \mathbb{R}$ of the form

$$Q(x_1, \ldots, x_n) = \sum_{i \leq j} a_{ij} x_i x_j,$$

  where $j \in \{1, \ldots, n\}$, and a_{ij} are real numbers

- In words, a quadratic form is the sum of monomials of degree two

Quadratic forms

- The general quadratic form on $\mathbb{R}^2$ is

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$$

- The general quadratic form on $\mathbb{R}^3$ is

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3$$

- and so on...

Quadratic forms in matrix form

- The quadratic form $Q(x_1, x_2) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$ can be written in matrix form:

$$Q(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- Equivalently, we can use the following representation in which the 2×2 matrix is symmetric:

$$Q(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Quadratic forms in matrix form

- Similarly, we can represent

$$Q(x_1, x_2, x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3$$

in the following matrix form:

$$Q(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \frac{1}{2}a_{13} \\ \frac{1}{2}a_{12} & a_{22} & \frac{1}{2}a_{23} \\ \frac{1}{2}a_{13} & \frac{1}{2}a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Quadratic forms in matrix form

- The general quadratic form

$$Q(x_1, \ldots, x_n) = \sum_{i \leq j} a_{ij} x_i x_j$$

  can be written as

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \ldots & \frac{1}{2}a_{1n} \\ \frac{1}{2}a_{12} & a_{22} & \ldots & \frac{1}{2}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}a_{1n} & \frac{1}{2}a_{2n} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix},$$

  that is, as

$$\mathbf{x}^T A \mathbf{x},$$

  where A is a unique *symmetric* matrix.

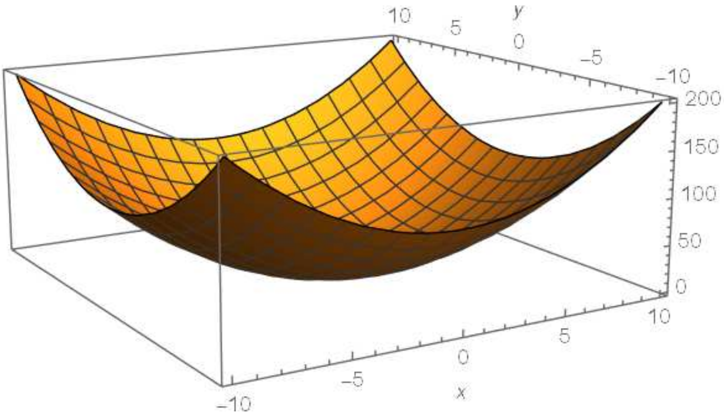
- Conversely, if A is a symmetric $n \times n$ matrix, then the function $Q(x_1, \ldots, x_n) = \mathbf{x}^T A \mathbf{x}$ is a quadratic form

Quadratic forms and unconstrained optimization

- Every quadratic form satisfies $Q(\mathbf{0}) = 0$, where $\mathbf{0} = (0, \ldots, 0)$
- By studying the *definiteness* of the matrix A, we can determine whether $\mathbf{0}$ is a global maximizer or a global minimizer or neither of the quadratic form under consideration
- Recall the following definitions from Lecture 8. An $n \times n$ symmetric matrix A is:
  - **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$
  - **positive semidefinite** if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$
  - **negative definite** if $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$
  - **negative semidefinite** if $\mathbf{x}^T A\mathbf{x} \leq 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$
  - **indefinite** if $\mathbf{x}^T A\mathbf{x} > 0$ for some $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y}^T A\mathbf{y} < 0$ for some $\mathbf{y} \neq \mathbf{x}$ in $\mathbb{R}^n$.

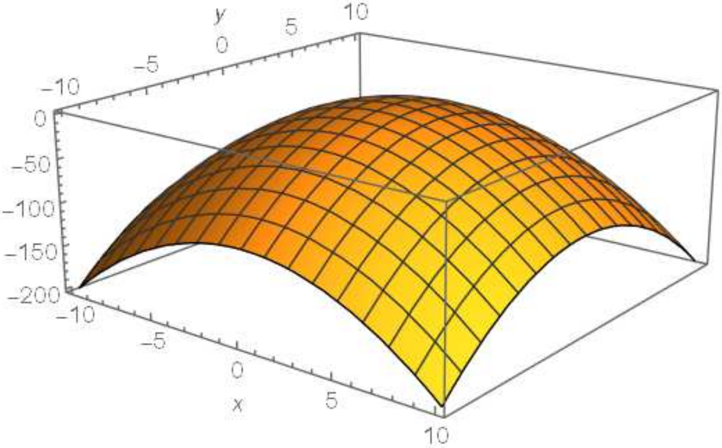
Quadratic forms and unconstrained optimization

- Positive definite quadratic form $Q(x, y) = x^2 + y^2$
- $(0, 0)$ is the unique global minimizer



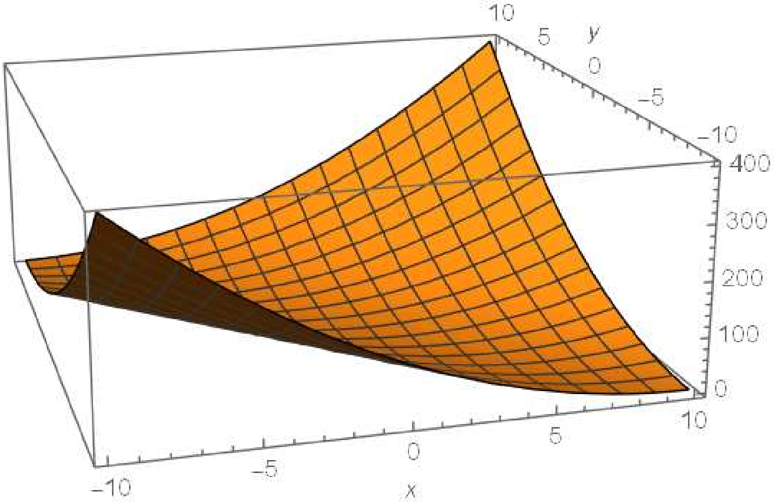
Quadratic forms and unconstrained optimization

- Negative definite quadratic form $Q(x, y) = -(x^2 + y^2)$
- $(0, 0)$ is the unique global maximizer



Quadratic forms and unconstrained optimization

- Positive semidefinite quadratic form $Q(x, y) = (x + y)^2$
- Every point in $\mathbb{R}^2$ such that $x + y = 0$ is a global minimizer



Quadratic forms and unconstrained optimization

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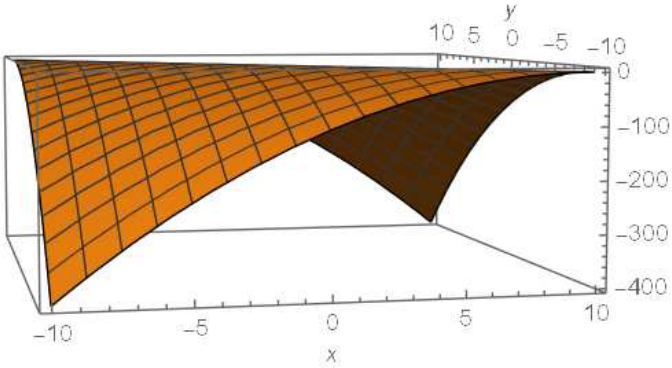
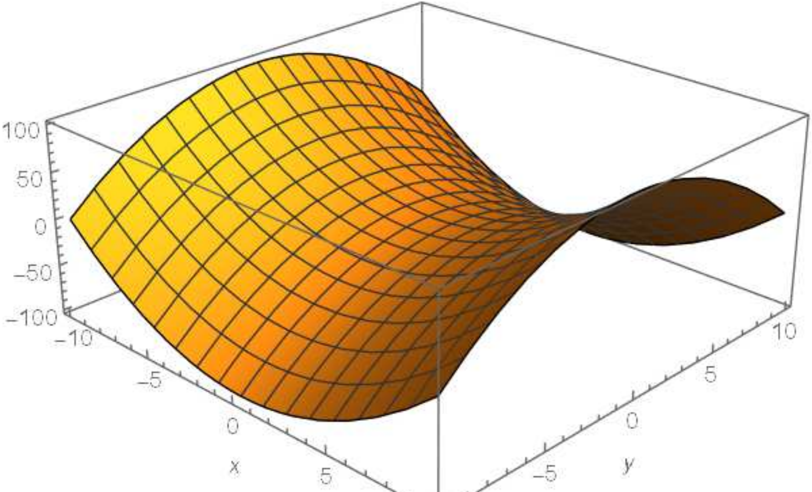


Quadratic forms and unconstrained optimization

- Indefinite quadratic form $Q(x, y) = x^2 - y^2$
- $(0, 0)$ is a saddle point
- There are no local or global extrema



Quadratic forms and unconstrained optimization

- Consider the following quadratic form in which A is a **diagonal** matrix:

$$Q(x_1, \ldots, x_n) = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix} \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix},$$

which can be written as

$$Q(x_1, \ldots, x_n) = a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2$$

Quadratic forms and unconstrained optimization

- The definiteness of the diagonal matrix A is easy to check:
  - A is *positive definite* if and only if all the a_i's are positive
  - A is *negative definite* if and only if all the a_i's are negative
  - A is *positive semidefinite* if and only if all the a_i's are non-negative
  - A is *negative semidefinite* if and only if all the a_i's are non-positive
  - A is *indefinite* if and only if there are two a_i's of opposite signs

Quadratic forms and unconstrained optimization

- The signs of the terms on the main diagonal are relevant for the definiteness of *any* matrix, not just for diagonal matrices
- For a given symmetric matrix A (not necessarily diagonal), a **necessary condition** for positive definiteness (positive semidefiniteness) is that all the diagonal entries of A be positive (non-negative)
- Similarly, a necessary condition for negative definiteness (negative semidefiniteness) is that all the diagonal entries of A be negative (non-positive)
- Note: the conditions above are necessary but not sufficient. They are necessary *and* sufficient for diagonal matrices (see the previous slide)

Quadratic forms and unconstrained optimization

- Let's prove that a positive definite matrix must have positive diagonal entries
  - Suppose the $n \times n$ symmetric matrix A is positive definite, so that $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$
  - For any $i = 1, \ldots, n$, let $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the vector in $\mathbb{R}^n$ such that its ith entry is 1 and all its other entries are equal to 0
  - For all i, we have $\mathbf{e}_i^T A\mathbf{e}_i = a_{ii} > 0$
  - Thus all the diagonal entries of A are positive
- **Exercise.** Prove the corresponding statement for positive semidefinite, negative definite, and negative semidefinite matrices

Quadratic forms and concavity

- What is the connection between quadratic forms and convexity/concavity? In other words, when is a quadratic form convex? When is it concave?

- Let $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form. The Hessian of Q is

$$D^2 Q(\mathbf{x}) = \begin{pmatrix} 2a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & 2a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & 2a_{nn} \end{pmatrix} = 2A$$

- Therefore, Q is concave if and only if A is negative semidefinite, and Q is convex if and only if A is positive semidefinite

Unconstrained optimization

- **Exercise.** Study the definiteness of the following quadratic forms on $\mathbb{R}^3$:

  1. $Q(x, y, z) = x^2 + 4y^2 + 6z^2 + 4xy + 10yz$
  2. $Q(x, y, z) = -x^2 - y^2 - 2z^2 + 2xy$

Unconstrained optimization

- **Exercise.** Consider the following function defined over $\mathbb{R}^2$:

$$f(x, y) = -3x^2 - 3y^2 + x^2y + xy^2 - 9xy + 18x + 18y - 27$$

1. Is f a quadratic form?
2. Find all the local and global extrema of f

Linear-quadratic problems

- How to solve $\max \mathbf{c}^T \mathbf{x} + \mathbf{x}^T A \mathbf{x}$?
- First order optimality conditions: $\mathbf{c} + (A^T + A)\mathbf{x} = \mathbf{0}$
- Solution is $\mathbf{x} = -(A^T + A)^{-1}\mathbf{c}$
- Is the objective function concave?
- Yes, if A is negative definite

Linear-quadratic problems: examples

- Solve

$$\min_{\beta} \|\mathbf{y} - X\beta\|^2$$

- Why is this an import problem?
- Portfolio problem $\min -\bar{\mathbf{c}}^T x - a\mathbf{x}^T V\mathbf{x}$ such that $\mathbf{x} \geq \mathbf{0}$ and $\sum_i x_i = I$

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Applications to Decision Making Under Uncertainty

Infinite series and the St. Petersburg paradox

- You learned in Intermediate Microeconomics that preferences over "lotteries" or "gambles" can be represented by *expected utility*
- A lottery is a combination of mutually exclusive outcomes, each with a payoff and a probability
- With n outcomes, a lottery L is a list

$$L = (x_1, p_1; x_2, p_2; \ldots; x_n, p_n),$$

  where x_i is the payoff of outcome i and p_i is the corresponding probability

Infinite series and the St. Petersburg paradox

- When there are finitely many outcomes (and when all the payoffs are bounded), we can always calculate the expected value of the lottery: $E(L) = \sum_{i=1}^{n} p_i x_i$

- However, this is not always true for lotteries having infinitely many outcomes. Consider the following lottery, which gave rise to the so-called **St. Petersburg paradox**:

$$L = (x_1, p_1; x_2, p_2; \dots) = \left(1, \frac{1}{2}; 2, \frac{1}{4}; 4, \frac{1}{8}; \dots\right),$$

  where $x_i = 2^{i-1}$ and $p_i = 2^{-i}$

- The expected value of this lottery is

$$E(L) = 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{8} + \cdots = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty$$

Infinite series and the St. Petersburg paradox

- The key idea behind expected utility theory is to introduce a so-called Bernoulli utility function u that maps payoffs/wealth to utility. In so doing, the expected utility of a lottery L is $EU(L) = \sum_{i=1}^{\infty} p_i u(x_i)$

- The curvature of u reflects the decision maker's attitude toward risk: A decision maker with a *concave* u is risk averse

- Suppose $u(x) = \ln x$ and let's calculate the expected utility of the lottery in the St. Petersburg paradox

Infinite series and the St. Petersburg paradox

- We have:

$$\begin{aligned} EU(L) &= \frac{1}{2}\ln 1 + \frac{1}{4}\ln 2 + \frac{1}{8}\ln 4 + \frac{1}{16}\ln 8 + \frac{1}{32}\ln 16 + \cdots \\ &= 0 + \frac{1}{4}\ln 2 + \frac{2}{8}\ln 2 + \frac{3}{16}\ln 2 + \frac{4}{32}\ln 2 + \cdots \\ &= \frac{1}{2}\ln 2\left(\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots\right) \\ &= \frac{1}{2}\ln 2\left(\sum_{k=1}^{\infty}\frac{k}{2^k}\right) \end{aligned}$$

Infinite series and the St. Petersburg paradox

- The term $\sum_{k=1}^{\infty} \frac{k}{2^k}$ is a **arithmetico-geometric series**
- In general, for $r \in (0, 1)$ we have that $\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2}$
- In our case, $r = \frac{1}{2}$. So we can finally write

$$
\begin{aligned}
EU(L) &= \frac{1}{2} \ln 2 \left(\sum_{k=1}^{\infty} \frac{k}{2^k} \right) \\
&= \frac{1}{2} \ln 2 \, (2) \\
&= \ln 2.
\end{aligned}
$$

Infinite series and the St. Petersburg paradox

- Now suppose that the Bernoulli utility function is $u(x) = \sqrt{x}$
- The expected value of our lottery becomes:

$$
\begin{aligned}
EU(L) &= \frac{1}{2}\sqrt{1} + \frac{1}{4}\sqrt{2} + \frac{1}{8}\sqrt{4} + \frac{1}{16}\sqrt{8} + \frac{1}{32}\sqrt{16} + \cdots \\
&= \frac{1}{2} + \frac{1}{4}\sqrt{2} + \frac{1}{4} + \frac{1}{8}\sqrt{2} + \frac{1}{8} + \cdots \\
&= \frac{1}{2}\left(1 + \frac{1}{2}\sqrt{2} + \frac{1}{2} + \frac{1}{4}\sqrt{2} + \frac{1}{4} + \cdots\right) \\
&= \frac{1}{2}\left[1 + \sqrt{2}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)\cdots\right]
\end{aligned}
$$

Infinite series and the St. Petersburg paradox

- The term $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is a **geometric series**
- In general, for $r \in (0, 1)$ we have that $\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$
- In our case, $r = \frac{1}{2}$. So we can finally write:

$$
\begin{aligned}
EU(L) &= \frac{1}{2}\left[1 + \sqrt{2}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)\cdots\right] \\
&= \frac{1}{2}\left(1 + \sqrt{2} + 1\right) \\
&= 1 + \frac{\sqrt{2}}{2}.
\end{aligned}
$$

CRRA utility

- Consider the following version of the CRRA utility function:

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}, \tag{1}$$

with $\gamma \geq 0$ and $\gamma \neq 1$

- A limiting case when γ goes to one is $\ln(c)$ (log utility)
- BUT how to come up with the limit result?

L'Hôpital's rule

- When calculating limits of functions, you may encounter (among others) the indeterminate form $\frac{0}{0}$
- For example,

$$\lim_{x \longrightarrow 0} \frac{e^x - 1}{x}$$

  is an indeterminate form $\frac{0}{0}$

L'Hôpital's rule

- In cases like this, we can apply the **l'Hôpital's rule**, which says the following
- If $\lim_{x \longrightarrow c} f(x) = \lim_{x \longrightarrow c} g(x) = 0$, if $g'(x) \neq 0$ for $x \neq c$, and if $\lim_{x \longrightarrow c} f'(x)/g'(x) = L$, then

$$\lim_{x \longrightarrow c} \frac{f(x)}{g(x)} = \lim_{x \longrightarrow c} \frac{f'(x)}{g'(x)} = L,$$

  where L can be either a finite number or $\pm\infty$
- In the previous example, we have

$$\lim_{x \longrightarrow 0} \frac{e^x - 1}{x} = \lim_{x \longrightarrow 0} \frac{e^x}{1} = 1$$

Arrow-Pratt measure of absolute risk aversion

- Arrow-Pratt measure of risk aversion

$$ARA(c) = -u''(c)/u'(c)$$

- Note: the measure of relative risk aversion is

$$RRA(c) = -u''(c)c/u'(c)$$

- How to derive ARA? Start from the definition of certainty equivalent:

$$\mathbb{E}[u(c+z)] = u(c - CE),$$

where z is a random variable with mean zero and variance σ (tip: try second order approximation on the left hand side and first order on the right hand side, then derive a formula for CE)

CARA utility and portfolio choice

- CARA (constant absolute risk aversion) utility $u(c) = 1 - e^{-ac}$
- Multivariate normal distribution of returns $\mathbf{c} \sim N(\bar{\mathbf{c}}, V)$
- Assume that $\mathbf{x}$ is the vector of amount of money invested in different assets, total return is $\mathbf{c}^T\mathbf{x}$, the expected value is $\bar{\mathbf{c}}^T\mathbf{x}$ and the variance $\mathbf{x}^T V\mathbf{x}$
- Expected utility of z:

$$\mathbb{E}\left[u(\mathbf{c}^T\mathbf{x})\right] = 1 - e^{-a(\mathbf{c}^T\mathbf{x})+(1/2)a^2(\mathbf{x}^T V\mathbf{x})}$$

- What kind of monotone transformations can you apply to $\mathbb{E}[u(\mathbf{c}^T\mathbf{x})$ to get a more tractable objective function for an optimization problem?