

# MEC-E1050 Finite Element Method in Solids; Formulae

## LINEAR ELASTICITY

**Coordinate systems:** 
$$\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{bmatrix} i_X & i_Y & i_Z \\ j_X & j_Y & j_Z \\ k_X & k_Y & k_Z \end{bmatrix} \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix} = \{\mathbf{i} \ \mathbf{j} \ \mathbf{k}\}^T \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix}$$

**Strain-stress:** 
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{G} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}, \quad G = \frac{E}{2(1+\nu)} \quad \text{or}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} \equiv [E] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = G \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}, \quad [E]_{\varepsilon} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

**Strain-displacement:** 
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_z / \partial z \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial y + \partial u_y / \partial x \\ \partial u_y / \partial z + \partial u_z / \partial y \\ \partial u_z / \partial x + \partial u_x / \partial z \end{Bmatrix}$$

## ELEMENT CONTRIBUTION (constant load)

**Bar (axial):** 
$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} \mathbf{ii}^T & -\mathbf{ii}^T \\ -\mathbf{ii}^T & \mathbf{ii}^T \end{bmatrix} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} \mathbf{i} \\ \mathbf{i} \end{Bmatrix}, \quad \text{in which } \mathbf{i} = \frac{1}{h} \begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{Bmatrix}$$

**Bar (torsion):** 
$$\begin{Bmatrix} M_{x1} \\ M_{x2} \end{Bmatrix} = \frac{GI_{rr}}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

**Beam (xz):** 
$$\begin{Bmatrix} F_{z1} \\ M_{y1} \\ F_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$$

$$\text{Point loads: } \begin{cases} F_{X1} \\ F_{Y1} \\ F_{Z1} \end{cases} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{cases} u_{X1} \\ u_{Y1} \\ u_{Z1} \end{cases} - \begin{cases} F_X \\ F_Y \\ F_Z \end{cases}, \quad \begin{cases} M_{X1} \\ M_{Y1} \\ M_{Z1} \end{cases} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{cases} \theta_{X1} \\ \theta_{Y1} \\ \theta_{Z1} \end{cases} - \begin{cases} M_X \\ M_Y \\ M_Z \end{cases}$$

## PRINCIPLE OF VIRTUAL WORK

$$\delta W = \delta W^{\text{ext}} + \delta W^{\text{int}}, \quad \delta W = \sum_{e \in E} \delta W^e = 0 \quad \forall \delta \mathbf{a}, \quad \delta W = \int_{\Omega} \delta w d\Omega$$

$$\text{Bar: } \delta w^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx}, \quad \delta w^{\text{ext}} = \delta u f_x$$

$$\text{Torsion: } \delta w^{\text{int}} = -\frac{d\delta\phi}{dx} GI_{rr} \frac{d\phi}{dx}, \quad \delta w^{\text{ext}} = \delta\phi m_x$$

$$\text{Beam bending (xz-plane): } \delta w^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2}, \quad \delta w^{\text{ext}} = \delta w f_z$$

$$\text{Beam bending (xy-plane): } \delta w^{\text{int}} = -\frac{d^2\delta v}{dx^2} EI_{zz} \frac{d^2 v}{dx^2}, \quad \delta w^{\text{ext}} = \delta v f_y$$

### Beam (Bernoulli):

$$\delta w^{\text{int}} = -\begin{Bmatrix} d\delta u/dx \\ d^2\delta v/dx^2 \\ d^2\delta w/dx^2 \end{Bmatrix}^T E \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{zy} \\ -S_y & I_{yz} & I_{yy} \end{bmatrix} \begin{Bmatrix} du/dx \\ d^2v/dx^2 \\ d^2w/dx^2 \end{Bmatrix} - \frac{d\delta\phi}{dx} GI_{rr} \frac{d\phi}{dx},$$

$$\delta w^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^T \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ -d\delta w/dx \\ d\delta v/dx \end{Bmatrix}^T \begin{Bmatrix} -S_y f_y + S_z f_z \\ S_y f_x \\ -S_z f_z \end{Bmatrix}$$

### Thin slab (plane-stress):

$$\delta w^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^T t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix}, \quad \delta w^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^T \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$$

### Thin slab (plane-strain):

$$\delta w^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^T t[E]_{\varepsilon} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix}, \quad \delta w^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^T \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$$

### Kirchhoff plate:

$$\delta w^{\text{int}} = - \begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / (\partial x \partial y) \end{Bmatrix}^T \frac{t^3}{12} [E]_{\sigma} \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / (\partial x \partial y) \end{Bmatrix}, \quad \delta w^{\text{ext}} = \delta w f_z$$

**Reissner-Mindlin plate:**

$$\delta w^{\text{int}} = - \begin{Bmatrix} -\partial \delta \theta / \partial x \\ \partial \delta \phi / \partial y \\ \partial \delta \phi / \partial x - \partial \delta \theta / \partial y \end{Bmatrix}^T \frac{t^3}{12} [E]_{\sigma} \begin{Bmatrix} -\partial \theta / \partial x \\ \partial \phi / \partial y \\ \partial \phi / \partial x - \partial \theta / \partial y \end{Bmatrix} - \begin{Bmatrix} \partial \delta w / \partial y - \delta \phi \\ \partial \delta w / \partial x + \delta \theta \end{Bmatrix}^T tG \begin{Bmatrix} \partial w / \partial y - \phi \\ \partial w / \partial x + \theta \end{Bmatrix},$$

$$\delta w^{\text{ext}} = \delta w f_z$$

$$\text{Body: } \delta w^{\text{int}} = - \begin{Bmatrix} \delta \varepsilon_{xx} \\ \delta \varepsilon_{yy} \\ \delta \varepsilon_{zz} \end{Bmatrix}^T \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta \gamma_{xy} \\ \delta \gamma_{yz} \\ \delta \gamma_{zx} \end{Bmatrix}^T \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}, \quad \delta w^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^T \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} \quad \text{or}$$

$$\delta w^{\text{int}} = - \begin{Bmatrix} \partial \delta u / \partial x \\ \partial \delta v / \partial y \\ \partial \delta w / \partial z \end{Bmatrix}^T [E] \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial w / \partial z \end{Bmatrix} - \begin{Bmatrix} \partial \delta u / \partial y + \partial \delta v / \partial x \\ \partial \delta v / \partial z + \partial \delta w / \partial y \\ \partial \delta w / \partial x + \partial \delta u / \partial z \end{Bmatrix}^T G \begin{Bmatrix} \partial u / \partial y + \partial v / \partial x \\ \partial v / \partial z + \partial w / \partial y \\ \partial w / \partial x + \partial u / \partial z \end{Bmatrix}$$

**APPROXIMATIONS (some)**  $u = \mathbf{N}^T \mathbf{a}$ ,  $\xi = \frac{x}{h}$

$$\text{Quadratic line: } \mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 1 - 3\xi + 2\xi^2 \\ 4\xi(1 - \xi) \\ \xi(2\xi - 1) \end{Bmatrix}, \quad \mathbf{a} = \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix} \quad (\text{bar})$$

$$\text{Cubic line: } \mathbf{N} = \begin{Bmatrix} N_{10} \\ N_{11} \\ N_{20} \\ N_{21} \end{Bmatrix} = \begin{Bmatrix} (1 - \xi)^2(1 + 2\xi) \\ h(1 - \xi)^2 \xi \\ (3 - 2\xi)\xi^2 \\ h\xi^2(\xi - 1) \end{Bmatrix}, \quad \mathbf{a} = \begin{Bmatrix} u_{10} \\ u_{11} \\ u_{20} \\ u_{21} \end{Bmatrix} (= \begin{Bmatrix} u_{z1} \\ -\theta_{y1} \\ u_{z2} \\ -\theta_{y2} \end{Bmatrix}) \quad (\text{beam bending})$$

$$\text{Linear: } \mathbf{N} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$$

**VIRTUAL WORK EXPRESSIONS**

$$\mathbf{Force: } \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{Xi} \\ \delta u_{Yi} \\ \delta u_{Zi} \end{Bmatrix}^T \begin{Bmatrix} F_{Xi} \\ F_{Yi} \\ F_{Zi} \end{Bmatrix} + \begin{Bmatrix} \delta \theta_{Xi} \\ \delta \theta_{Yi} \\ \delta \theta_{Zi} \end{Bmatrix}^T \begin{Bmatrix} M_{Xi} \\ M_{Yi} \\ M_{Zi} \end{Bmatrix}$$

$$\mathbf{Bar: } \delta W^{\text{int}} = - \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^T \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^T \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\mathbf{Torsion: } \delta W^{\text{int}} = - \begin{Bmatrix} \delta \theta_{x1} \\ \delta \theta_{x2} \end{Bmatrix}^T \frac{GI_{rr}}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta \theta_{x1} \\ \delta \theta_{x2} \end{Bmatrix}^T \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

**Beam bending (xz-plane):**

$$\delta W^{\text{int}} = - \begin{Bmatrix} \delta u_{z1} \\ \delta \theta_{y1} \\ \delta u_{z2} \\ \delta \theta_{y2} \end{Bmatrix}^T \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta \theta_{y1} \\ \delta u_{z2} \\ \delta \theta_{y2} \end{Bmatrix}^T \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$$

**Beam bending (xy-plane):**

$$\delta W^{\text{int}} = - \begin{Bmatrix} \delta u_{y1} \\ \delta \theta_{z1} \\ \delta u_{y2} \\ \delta \theta_{z2} \end{Bmatrix}^T \frac{EI_{zz}}{h^3} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{y1} \\ \delta \theta_{z1} \\ \delta u_{y2} \\ \delta \theta_{z2} \end{Bmatrix}^T \frac{f_y h}{12} \begin{Bmatrix} 6 \\ h \\ 6 \\ -h \end{Bmatrix}$$

**CONSTRAINTS**

**Frictionless contact:**  $\vec{n} \cdot \vec{u}_A = 0$

**Joint:**  $\vec{u}_B = \vec{u}_A$

**Rigid body (link):**  $\vec{u}_B = \vec{u}_A + \vec{\theta}_A \times \vec{\rho}_{AB}$ ,  $\vec{\theta}_B = \vec{\theta}_A$ .