CS–E4500 Advanced Course in Algorithms

Week 05 – Tutorial

We return to the satisfiability question. For the k-satisfiability (k-SAT) problem, the formula is restricted so that each clause has exactly k literals. Again, we assume that no clause contains both a literal and its negation, as these clauses are trivial. We prove that any k-SAT formula in which no variable appears in too many clauses has a satisfying assignment.

1. If no variable in a k-SAT formula appears in more than $T = 2^k/4k$ clauses, then the formula has a satisfying assignment.

   Solution. Consider the probability space defined by giving a random assignment to the variables. For $i = 1, \ldots, m$, let E_i denote the event that the ith clause is not satisfied by the random assignment. Since each clause has k literals,

   $$\mathrm{P}(E_i) = 2^{-k}\,.$$

   The event E_i is mutually independent of all of the events related to clauses that do not share variables with clause i. Because each of the k variables in clause i can appear in no more than $T = 2^k/4k$ clauses, the degree of the dependency graph is bounded by $d \leq kT \leq 2^{k-2}$. In this case,

   $$4dp \leq 4 \cdot 2^{k-2} \cdot 2^{-k} = 1,$$

   so we can apply the Lovász Local Lemma to conclude that there exists an assignment where none of the E_i's occur.

2. Show that if

   $$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1\,,$$

   then it is possible to 2-color the edges of K_n such that it has no monochromatic K_k as a subgraph.

   Solution. Consider a random 2-coloring of the graph. Let E_i be the event that the ith copy of K_k is a monochromatic clique. Then we have

   $$\mathrm{P}(E_i) = 2^{-\left(\binom{k}{2}-1\right)} = 2^{1-\binom{k}{2}}\,.$$

   Two k-cliques are independent if the two cliques share at most one vertex. For any k-clique, there are at most $\binom{k}{2}\binom{n-2}{k-2} < \binom{k}{2}\binom{n}{k-2}$ other cliques sharing at least two vertices with it. Thus, if we construct the dependency graph for all E_i's, the maximum degree can be bounded by

   $$d \leq \binom{k}{2}\binom{n}{k-2}\,.$$

   Hence, it holds that

   $$4dp = 4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1$$

   and we can apply the Lovász Local Lemma to conclude that there exists a coloring where none of the E_i's occur.