Mathematics for Economists

Mitri Kitti

Aalto University

Constrained Optimization

Exercise

Consider the following constrained maximization problem:

$$\max_{x_1, x_2} \quad x_1^2 x_2$$
$$\text{s. t.} \quad 2x_1^2 + x_2^2 = 3$$

1. What can you say about the existence of a solution? (Think about Weierstrass's Theorem)

2. Solve this optimization problem

Exercise

- The objective function is continuous. The constraint set is compact. If we restrict the function's domain to the constraint set, we can apply Weierstrass's Theorem and conclude that a solution to this maximization problem exists

- Since $\frac{\partial h}{\partial x_1} = 4x_1$ and $\frac{\partial h}{\partial x_2} = 2x_2$, the only point where both partial derivatives are equal to zero is $(0, 0)$. This point does not belong to the constraint set. Therefore, the constraint qualification is satisfied

- The Lagrangian is

$$L(x_1, x_2, \lambda) = x_1^2 x_2 - \lambda(2x_1^2 + x_2^2 - 3)$$

Exercise

- Critical points of the Lagrangian are found by solving the following system:

$$\frac{\partial L}{\partial x_1} = 2x_1x_2 - 4\lambda x_1 = 0$$
$$\frac{\partial L}{\partial x_2} = x_1^2 - 2\lambda x_2 = 0$$
$$\frac{\partial L}{\partial \lambda} = -(2x_1^2 + x_2^2 - 3) = 0$$

- See p. 419 in the textbook on how to solve the system above

Exercise

- It turns out that the Lagrangian has six critical points:

$$(0, \sqrt{3}, 0), \quad (0, -\sqrt{3}, 0), \quad (1, 1, 0.5)$$
$$(-1, -1, -0.5), \quad (1, -1, -0.5), \quad (-1, 1, 0.5)$$

- Now, we already know that a solution must exist. By the Proposition at p. 14 in the slides from Lecture 12, we also know that the solution must be a critical point of the Lagrangian. Therefore, we can find the solution just by evaluating the objective function at each of the six critical points above

Exercise

- We have:

$$f(1,1) = f(-1,1) = 1$$
$$f(1,-1) = f(-1,-1) = -1$$
$$f(0,\sqrt{3}) = f(0,-\sqrt{3}) = 0$$

- Hence both $(1,1)$ and $(-1,1)$ solve our constrained maximization problem

Constrained Optimization

- The general formulation of a constrained optimization problem with n variables and $m \leq n$ *equality constraints* is to
  - **maximize** or **minimize** the objective function $f(x_1, \ldots, x_n)$
  - subject to the constraints:

$$\begin{aligned} h_1(x_1, \ldots, x_n) &= a_1 \\ h_2(x_1, \ldots, x_n) &= a_2 \\ \cdots \quad \cdots \quad & \cdots \quad \cdots \\ h_m(x_1, \ldots, x_n) &= a_m \end{aligned}$$

- The constraint set is

$$C = \{\mathbf{x} \in \mathbb{R}^n : h_1(\mathbf{x}) = a_1, h_2(\mathbf{x}) = a_2, \ldots, h_m(\mathbf{x}) = a_m\},$$

where $\mathbf{x} = (x_1, \ldots, x_n)$

Constrained Optimization

- In the previous lecture, we introduced a *constraint qualification* condition. To generalize it to n variables and m constraints, we need the **Jacobian derivative** of the constraints. At any given point $\mathbf{x}$, the Jacobian $D\mathbf{h}(\mathbf{x})$ is the $m \times n$ matrix

$$D\mathbf{h}(\mathbf{x}) = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial h_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial h_2}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial h_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial h_m}{\partial x_n}(\mathbf{x}) \end{pmatrix},$$

  where $\mathbf{h} = (h_1, \ldots, h_m)$

- We say that a point $\mathbf{x}$ is a **critical point** of $\mathbf{h}$ if the rank of $D\mathbf{h}(\mathbf{x})$ is strictly less than m

- We say that $\mathbf{h}$ satisfies the **nondegenerate constraint qualification (NDCQ)** at $\mathbf{x}$ if the rank of $D\mathbf{h}(\mathbf{x})$ at $\mathbf{x}$ is m

Constrained Optimization

Proposition (First order necessary condition)

Let $f, h_1, h_2, \ldots, h_m$ be C^1 functions defined over $\mathbb{R}^n$. Suppose that:

1. $\mathbf{x}^* = (x_1^*, \ldots, x_n^*) \in C$ *is a local maximizer or a local minimizer of f on the constraint set*

$$C = \{\mathbf{x} \in \mathbb{R}^n : h_1(\mathbf{x}) = a_1, h_2(\mathbf{x}) = a_2, \ldots, h_m(\mathbf{x}) = a_m\};$$

2. $\mathbf{x}^*$ *satisfies the NDCQ.*

Then, there exists real numbers $\lambda_1^*, \ldots, \lambda_m$ such that $(\mathbf{x}, \boldsymbol{\lambda}) := (x_1^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*)$ is a critical point of the following Lagrangian function:

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \left(h_i(\mathbf{x}) - a_i\right).$$

Constrained Optimization

- The proposition in the previous slide does not say that a solution exists. It says that, *if it exists*, it must be a critical point of the Lagrangian

- The NDCQ requires that $D\mathbf{h}(\mathbf{x}^*)$ has full rank m (recall that $m \leq n$)

Constrained Optimization

- The proposition at p. 9 can be applied as follows:

  1. Check the NDCQ by finding all the points (if any) in the constraint set C at which the rank of the Jacobian $D\mathbf{h}(\mathbf{x})$ is strictly less than m
  2. Find the critical points of the Lagrangian function
  3. If there are no points in C at which the NDCQ is violated, the critical points of the Lagrangian are the only candidates for a solution to the original constrained optimization problem
  4. If there points in C at which the NDCQ is violated, then the candidates for a solution to the original optimization problem are both *i)* the critical points of the Lagrangian and *ii)* points in C at which the rank of $D\mathbf{h}(\mathbf{x})$ is strictly less than m

Constrained Optimization

- **Example.** Consider the following constrained maximization problem:

$$\max_{x,y} \quad x^2 + y^2$$
$$\text{subject to} \quad x^2 + xy + y^2 = 3$$

- By Weierstrass's Theorem, we know that a solution exists (why?)

- The Lagrangian is $L(x, y, \lambda) = x^2 + y^2 - \lambda(x^2 + xy + y^2 - 3)$

Constrained Optimization

- **Example (cont'd).** The Jacobian derivative is:

$$D\mathbf{h}(x,y) = \begin{pmatrix} 2x+y & 2y+x \end{pmatrix}$$

- The critical points of L are:
  1. $\left(-\sqrt{3}, \sqrt{3}, 2\right)$
  2. $\left(\sqrt{3}, -\sqrt{3}, 2\right)$
  3. $\left(1, 1, \frac{2}{3}\right)$
  4. $\left(-1, -1, \frac{2}{3}\right)$

- The NDCQ is violated at $(0,0)$, which does not belong to the constraint set C

- Thus we can conclude that $\left(-\sqrt{3}, \sqrt{3}\right)$ and $\left(\sqrt{3}, -\sqrt{3}\right)$ are global constrained maximizers, whereas $(1,1)$ and $(-1,-1)$ are global constrained minimizers (why?)

Constrained Optimization

Proposition (Sufficient condition for the existence of a solution)

Let $f, h_1, h_2, \ldots, h_m$ be C^1 functions defined on an open and convex set $U \subseteq \mathbb{R}^n$. Suppose $\mathbf{x}^* \in U$ and $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ is a stationary point of the Lagrangian function:

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \left(h_i(\mathbf{x}) - a_i \right).$$

- *If L is concave in $\mathbf{x}$ given $\boldsymbol{\lambda}^*$–in particular, if f is concave and $\lambda_j^* h_j$ is convex for all $j = 1, \ldots, m$–then $\boldsymbol{x}^*$ is a solution to the constrained* **maximization** *problem*
- *If L is convex in $\mathbf{x}$ given $\boldsymbol{\lambda}^*$–in particular, if f is convex and $\lambda_j^* h_j$ is concave–then $\boldsymbol{x}^*$ is a solution to the constrained* **minimization** *problem*

Constrained Optimization

- **Example.** Consider the following constrained minimization problem:

$$\begin{aligned} \min_{x,y,z} \quad & x^2 + y^2 + z^2 \\ \text{subject to} \quad & x + 2y + z = 1 \\ & 2x - y - 3z = 4 \end{aligned}$$

- The Lagrangian is

$$L(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda_1(x + 2y + z - 1) - \lambda_2(2x - y - 3z - 4),$$

which is convex for any values of λ_1 and λ_2

- **Example (cont'd).** The Jacobian is

$$D\mathbf{h}(x, y, z) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \end{pmatrix},$$

which has rank 2 for every (x, y, z)

Constrained Optimization

- **Example (cont'd).** The critical points of L can be found by solving the following system:

$$\begin{aligned} 2x - \lambda_1 - 2\lambda_2 &= 0 \\ 2y - 2\lambda_1 + \lambda_2 &= 0 \\ 2z - \lambda_1 + 3\lambda_2 &= 0 \\ x + 2y + z &= 1 \\ 2x - y - 3z &= 4 \end{aligned}$$

- You can verify that the unique solution of the constrained minimization problem is $(x^*, y^*, z^*) = \left(\frac{16}{15}, \frac{1}{3}, -\frac{11}{15}\right)$, and the corresponding multipliers are $\lambda_1 = \frac{52}{75}$ and $\lambda_2 = \frac{54}{75}$

Application: Willingness to Pay and Demand

- Consumer with utility $u(x, y) = v(x) + y$, where y is money and x is the amount of consumption of a goods
- Budget $px + y = w$, where w is the consumer wealth and p is the price of the good
- Lagrange function $v(x) + y - \lambda(px + y - w)$
- First order conditions

$$\begin{aligned} v'(x) - \lambda p &= 0 \\ 1 - \lambda &= 0 \end{aligned}$$

Application: Consumer Surplus and Demand

- Inverse demand (from FOCs) is $p = v'(x)$
- Marginal willingness to pay for amount of good x: $MWTP = v'(x)$, i.e., at price $p = MWTP$ the consumer would be willing to buy an extra unit with the price price
- For a quasilinear utility (linearity in money) MWTP=inverse demand function
- Total willingness to pay is $\int_0^x P(z)dz$, when $P(z)$ is the inverse demand function (marginal WTP)
- The solution of $p = v'(x)$ is the demand function $x(p)$
- Utility can be recovered from the inverse demand $P(z)$, when assuming $v(0) = 0$: because $v(x) - v(0) = \int_0^x P(z)dz$ it holds that

$$u(x, w - px) = v(x) + [w - px] = \int_0^x P(z)dz + [w - px]$$

Application: Consumer Surplus and Demand

- Utility for a consumer i with utility $u_i(x, y) = v_i(x) - y$ from consumption of $x_i(p)$ at price p is $u_i(x_i(p), w_i - px_i(p))$
- Aggregate consumer utility

$$\sum_i [u_i(x_i(p), w_i - px_i(p))] = \sum_i [v_i(x_i(p)) - px_i(p)] - W,$$

  where W is the total wealth and $v_i(x_i(p)) - px_i(p)$ is the surplus of consumer i
- What does the aggregate consumer surplus

$$\sum_i [v_i(x_i(p)) - px_i(p)]$$

  have to do with the are under the demand curve?

Application: Production of Public Goods

- Consumers with utilities $u_i(G, y_i)$, where y_i is private consumption and G is consumption of public good
- Planner's problem with the socially optimal amount of public good production
  - $\max \sum u_i(G, y_i)$ subject to budget constraint: $\sum y_i + c(G) = \sum w_i$, where w_i is the wealth of consumer i
- First order optimality conditions:

$$\partial u_i(G, y_i)/\partial y_i - \lambda = 0$$

$$\sum_i \partial u_i(G, y_i)/\partial G - \lambda c'(G) = 0$$

- Samuelson's condition:

$$\sum_i [\partial u_i(G, y_i)/\partial G]/[\partial u_i(G, y_i)/\partial y_i] = c'(G)$$

Second Order Sufficient Optimality Conditions

- In unconstrained optimization problems, we used **second order conditions** to classify critical points of the objective function as local minimizers or maximizers
- Second order conditions can be established also for constrained optimization. In order to do that, we need to introduce *bordered matrices*

Bordered Hessians

- Suppose we want to determine the definiteness of the following quadratic form:

$$Q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$$

subject to the linear constraint $Ax_1 + Bx_2 = 0$, where $A, B \in \mathbb{R}$

- Assuming $A \neq 0$, we get $x_1 = -\frac{B}{A}x_2$ from the linear constraint. Substituting the latter expression in the objective function Q, we obtain

$$Q\left(-\frac{B}{A}x_2, x_2\right) = \frac{aB^2 - 2bAB + cA^2}{A^2}x_2^2$$

- Thus Q is positive definite on the constraint set $Ax_1 + Bx_2 = 0$ if and only if $aB^2 - 2bAB + cA^2 > 0$, and negative definite if and only if $aB^2 - 2bAB + cA^2 < 0$

Bordered Hessians

- The expression $aB^2 - 2bAB + cA^2$ can be written as

$$aB^2 - 2bAB + cA^2 = -\det \begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix}, \tag{1}$$

  where the matrix is obtained by *bordering* the 2×2 coefficient matrix of the quadratic form on the top and left by the coefficients A and B of the linear constraint

- Thus the definiteness of Q can be studied by looking at the determinant of the bordered matrix in (1)

Bordered Hessians

- More generally, suppose we want to study the definiteness of the quadratic form $Q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$, where A is an $n \times n$ coefficient matrix, subject to the linear constraint set:

$$\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Bordered Hessians

- The corresponding bordered matrix is

$$H = \begin{pmatrix} 0 & \dots & 0 & B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & B_{m1} & \cdots & B_{mn} \\ B_{11} & \cdots & B_{m1} & a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{1n} & \cdots & B_{mn} & a_{1n} & \cdots & a_{nn} \end{pmatrix}$$

- In more compact form,

$$H = \begin{pmatrix} \mathbf{0} & B \\ B^T & A \end{pmatrix}$$

Bordered Hessians

- The definiteness of $Q(\boldsymbol{x})$ when restricted to the linear constraint $B\mathbf{x} = \mathbf{0}$ can be determined by checking the last $n - m$ leading principal minors of H, starting with the determinant of H itself.

  1. If $\det(H)$ has the same sign as $(-1)^n$, and if the last $n - m$ leading principal minors alternate in sign, then $Q(\boldsymbol{x})$ is negative definite on the constraint set $B\mathbf{x} = \mathbf{0}$, and $\mathbf{x} = \mathbf{0}$ is a strict global constrained maximizer

  2. If $\det(H)$ and the last $n - m$ leading principal minors all have the same sign as $(-1)^m$, then $Q(\boldsymbol{x})$ is positive definite on the constraint set $B\mathbf{x} = \mathbf{0}$, and $\mathbf{x} = \mathbf{0}$ is a strict global constrained minimizer

  3. If both conditions 1. and 2. are violated by some non-zero leading principal minor, then $Q(\boldsymbol{x})$ is indefinite on the constraint set $B\mathbf{x} = \mathbf{0}$, and $\mathbf{x} = \mathbf{0}$ is neither a constrained maximizer nor a minimizer

Second Order Sufficient Optimality Conditions

Proposition (Second order sufficient condition)

Let $f, h_1, h_2, \dots, h_m$ be C^2 functions defined over $\mathbb{R}^n$. Consider the problem of **maximizing** *f on the constraint set*

$$C = \{\mathbf{x} \in \mathbb{R}^n : h_1(\mathbf{x}) = a_1, h_2(\mathbf{x}) = a_2, \dots, h_m(\mathbf{x}) = a_m\}.$$

Suppose that:

- $\boldsymbol{x}^* \in C$
- $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ *is a critical point of the Lagrangian L for the maximization problem under consideration*
- *the Hessian of L with respect to $\mathbf{x}$ at $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ is negative definite on the linear constraint set $D\mathbf{h}(\boldsymbol{x}^*)\boldsymbol{v} = 0$. That is,*

$$\boldsymbol{v} \neq 0 \text{ and } D\mathbf{h}(\boldsymbol{x}^*)\boldsymbol{v} = 0 \implies \boldsymbol{v}^T(D^2_{\boldsymbol{x}} L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*))\boldsymbol{v} < 0.$$

Then, $\mathbf{x}^*$ is a strict local constrained maximizer of f on C

Second Order Sufficient Optimality Conditions

Proposition (Second order sufficient condition)

Let $f, h_1, h_2, \ldots, h_m$ be C^2 functions defined over $\mathbb{R}^n$. Consider the problem of **minimizing** *f on the constraint set*

$$C = \{\mathbf{x} \in \mathbb{R}^n : h_1(\mathbf{x}) = a_1, h_2(\mathbf{x}) = a_2, \ldots, h_m(\mathbf{x}) = a_m\}.$$

Suppose that:

- $\boldsymbol{x}^* \in C$
- $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ *is a critical point of the Lagrangian L for the minimization problem under consideration*
- *the Hessian of L with respect to $\mathbf{x}$ at $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ is positive definite on the linear constraint set $D\mathbf{h}(\boldsymbol{x}^*)\boldsymbol{v} = 0$. That is,*

$$\boldsymbol{v} \neq 0 \text{ and } D\mathbf{h}(\boldsymbol{x}^*)\boldsymbol{v} = 0 \implies \boldsymbol{v}^T(D^2_{\boldsymbol{x}} L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*))\boldsymbol{v} > 0.$$

Then, $\mathbf{x}^*$ is a strict local constrained minimizer of f on C

Second Order Sufficient Optimality Conditions

- **Example.** Consider the following constrained maximization problem:

$$\max_{(x,y,z)\in\mathbb{R}^3_+} \quad x^2y^2z^2$$

$$\text{subject to} \quad x^2+y^2+z^2=3$$

- The Lagrangian is

$$L(x,y,z,\lambda)=x^2y^2z^2-\lambda\left(x^2+y^2+z^2-3\right)$$

Second Order Sufficient Optimality Conditions

- **Example (cont'd).** The first order conditions are:

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 2xy^2z^2 - 2\lambda x = 0 \\
\frac{\partial L}{\partial y} &= 2x^2yz^2 - 2\lambda y = 0 \\
\frac{\partial L}{\partial z} &= 2x^2y^2z - 2\lambda z = 0 \\
\frac{\partial L}{\partial \lambda} &= -(x^2 + y^2 + z^2 - 3) = 0,
\end{aligned}$$

which solve for $x = y = z = \lambda = 1$

Second Order Sufficient Optimality Conditions

- **Example (cont'd).** The bordered Hessian is

$$H = \begin{pmatrix} 0 & 2x & 2y & 2z \\ 2x & 2y^2z^2 - 2\lambda & 4xyz^2 & 4xy^2z \\ 2y & 4xyz^2 & 2x^2z^2 - 2\lambda & 4x^2yz \\ 2z & 4xy^2z & 4x^2yz & 2x^2y^2 - 2\lambda \end{pmatrix}$$

Second Order Sufficient Optimality Conditions

- **Example (cont'd).** At the critical point $(x, y, z, \lambda) = (1, 1, 1, 1)$, the bordered Hessian becomes:

$$H = \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ 2 & 4 & 4 & 0 \end{pmatrix}$$

- The definiteness of H depends on the signs of the last $n - m = 3 - 1$ leading principal minors

Constrained Optimization

- **Example (cont'd).** The last leading principal minor is the determinant of H itself. The second to last leading principal minor is the submatrix H_3:

$$H_3 = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 4 \\ 2 & 4 & 0 \end{pmatrix}$$

- We have that $\det(H) = -192$ and $\det(H_3) = 32$. Consequently, H is negative definite (on the constrained set)

- Thus we can conclude that $(x, y, z) = (1, 1, 1)$ is a local constrained maximizer

Exercise

Study the definiteness of the quadratic form

$$Q(x_1, x_2, x_3, x_4) = x_1^2 - x_2^2 + x_3^2 + x_4^2 + 4x_2x_3 - 2x_1x_4$$

on the following constraint set:

$$\begin{aligned} x_2 + x_3 + x_4 &= 0 \\ x_1 - 9x_2 + x_4 &= 0. \end{aligned}$$