Lecture 3
Convex optimization problems

- abstract form problem
- standard form problem
- convex optimization problem
- standard form with generalized inequalities
- mulitcriterion optimization
- rstriction and relaxation

Optimization problem (abstract form)

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C \end{array}$$

where $f : \mathbf{R}^n \to \mathbf{R}$, $C \subseteq \mathbf{dom}\, f$

- x is *optimization variable*
- f is *objective* or *cost function*
- C is *feasible set* or *constraint set*
- point x is *feasible* if $x \in C$
- problem is *feasible* if $C \neq \emptyset$
- problem is *unconstrained* if $C = \mathbf{R}^n$
- *optimal value* is $f^\star = \inf_{x \in C} f(x)$ (can be $-\infty$)

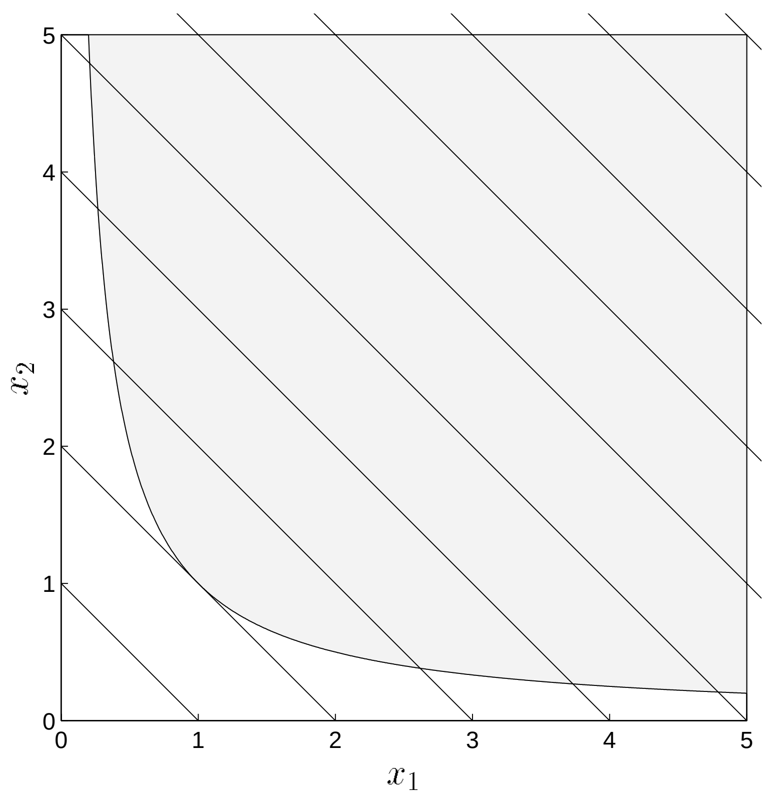
  convention: $f^\star = +\infty$ if infeasible
- *optimal point*: $x \in C$ s.t. $f(x) = f^\star$
- can maximize f by minimizing $-f$

called 'abstract' since we don't say how C is described

Example:

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 \\ \text{subject to} & x_1 \geq 0, \;\; x_2 \geq 0, \;\; x_1 x_2 \geq 1 \end{array}$$

- feasible set C is half-hyperboloid
- optimal value is $f^{\star} = 2$
- only optimal point is $x^{\star} = (1, 1)$



Optimization problem (standard form)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \; i = 1, \ldots, m \\ & g_i(x) = 0, \; i = 1, \ldots, p \end{array}$$

where $f_i, g_i : \mathbf{R}^n \rightarrow \mathbf{R}$

- feasible set is $C = \{x | f_i(x) \leq 0, \; g_i(x) = 0\}$
- f_i are *inequality constraint functions*
- g_i are *equality constraint functions*
- constraint i is *active* at $x \in C$ if $f_i(x) = 0$
- point x is called *strictly feasible* if

  $$f_i(x) < 0, \; i = 1, \ldots, m, \quad g_i(x) = 0, \; i = 1, \ldots, p$$

  $i.e.$, all (inequality) constraints are inactive
- problem is *strictly feasible* if there is a strictly feasible point
- can also have strict inequality constraints

Example:

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 \\ \text{subject to} & x_1 \geq 0, \ \ x_2 \geq 0, \ \ x_1 x_2 \geq 1 \end{array}$$

to put in standard form take $f_0(x) = x_1 + x_2$,

$$f_1(x) = -x_1, \quad f_2(x) = -x_2, \quad f_3(x) = 1 - x_1 x_2$$

note

- third constraint implies first two cannot be active
- first constraint is redundant: second and third imply it

can also put in standard form with $f_0(x) = x_1 + x_2$,

$$f_1(x) = \max\{\ 0, \ \ -x_1, \ \ -x_2, \ 1 - x_1 x_2\ \}$$

- feasible set exactly the same
- one constraint function intead of three
- this standard form problem is not strictly feasible

Feasibility problem

suppose objective $f_0 = 0$, so

$$f^\star = \begin{cases} 0 & \text{if } C \neq \emptyset \\ +\infty & \text{if } C = \emptyset \end{cases}$$

thus, problem is really to

- either find $x \in C$,
- or determine that $C = \emptyset$

i.e., solve the inequality / equality system

$$f_i(x) \leq 0,\ i = 1, \ldots, m, \quad g_i(x) = 0,\ i = 1, \ldots, p$$

or determine that it is inconsistent

Convex optimization problem

abstract form problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C \end{array}$$

is *convex* if C and f are convex (set, fct)

- problem is *quasiconvex* if C is convex and f is quasiconvex
- maximizing concave f over convex C is convex optimization problem

standard form problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \;\; i = 1, \ldots, m \\ & g_i(x) = 0, \;\; i = 1, \ldots, p \end{array}$$

is *convex* if $f_0, \ldots, f_m$ convex, $g_1, \ldots, g_p$ affine

often written as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \;\; i = 1, \ldots, m \\ & Ax = b \end{array}$$

where $A \in \mathbf{R}^{p \times n}$

Example. problem above,

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 \\ \text{subject to} & -x_1 \leq 0, \\ & -x_2 \leq 0, \\ & 1 - x_1 x_2 \leq 0 \end{array}$$

has convex objective and feasible set, hence is convex problem in abstract form

it is **not** a standard form cvx opt problem since

$$f_3(x) = 1 - x_1 x_2$$

is not convex (it is quasiconvex)

problem is easily cast as std form cvx opt problem, *e.g.*,

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 \\ \text{subject to} & -x_1 \leq 0, \\ & -x_2 \leq 0, \\ & 1 - \sqrt{x_1 x_2} \leq 0 \end{array}$$

($1 - \sqrt{x_1 x_2}$ is convex on $\mathbf{R}_+^2$)

many other ways, *e.g.*, replace third constraint with

$$-\log x_1 - \log x_2 \leq 0$$

Example. f_i all affine yields *linear program*

$$\begin{array}{ll} \text{minimize} & c_0^T x + d_0 \\ \text{subject to} & c_i^T x + d_i \leq 0, \ \ i = 1, \ldots, m \\ & Ax = b \end{array}$$

which is a convex optimization problem

Example. minimum norm approximation with limits on variables

$$\begin{array}{ll} \text{minimize} & \|Ax - b\| \\ \text{subject to} & l_i \leq x_i \leq u_i, \ \ i = 1, \ldots, n \end{array}$$

is convex

Example. minimum entropy with lin. equal. constraints

$$\begin{array}{ll} \text{minimize} & \sum_i x_i \log x_i \\ \text{subject to} & x_i \geq 0, \ \ i = 1, \ldots, n \\ & \sum_i x_i = 1 \\ & Ax = b \end{array}$$

is convex

(more on these later)

Local and global optimality

$x \in C$ is *locally optimal* if it satisfies

$$y \in C, \ \|y - x\| \le R \implies f(y) \ge f(x)$$

for some $R > 0$

c.f. (globally) optimal, which means $x \in C$,

$$y \in C \implies f(y) \ge f(x)$$

for cvx opt problems, any local solution is also global

proof:

- suppose x is locally optimal, but $y \in C$, $f(y) < f(x)$
- take small step from x towards y, $i.e.$, $z = \lambda y + (1 - \lambda)x$ with $\lambda > 0$ small
- z is near x, with $f(z) < f(x)$; contradicts local optimality

An optimality criterion

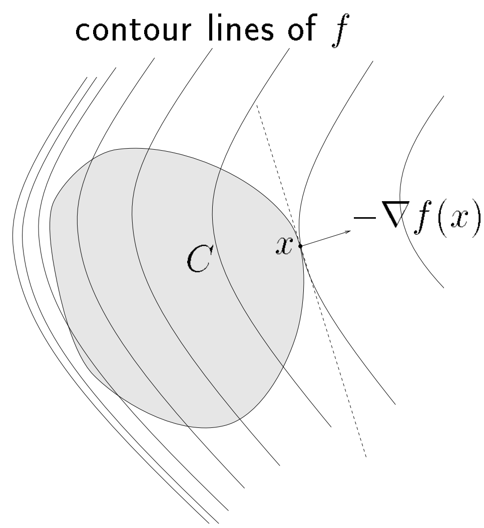
suppose f is differentiable in cvx problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C \end{array}$$

then $x \in C$ is optimal iff

$$y \in C \implies \nabla f(x)^T(y - x) \geq 0$$

- hence $x \in C$, $\nabla f(x) = 0$ implies x optimal
- for unconstrained problems, x is optimal iff $\nabla f(x) = 0$



interpretations:

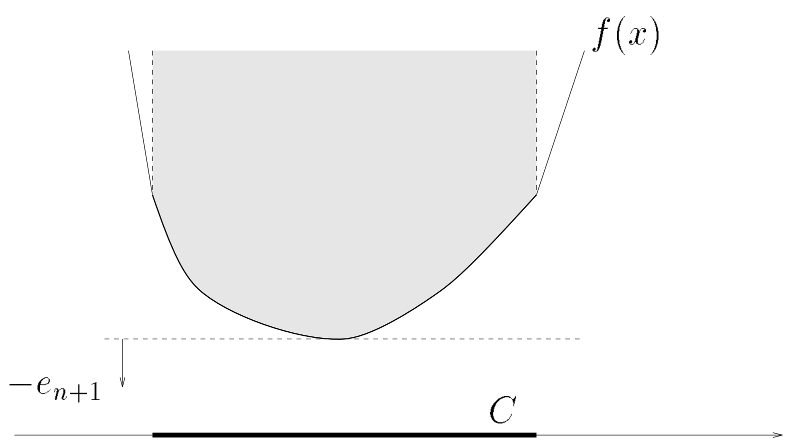
- means $-\nabla f(x)$ defines supporting hyperplane for C at x
- if you move from x towards any feasible y, f does not decrease

Epigraph form

write standard form problem as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & f_0(x) - t \leq 0, \\ & f_i(x) \leq 0, \; i = 1, \ldots, m \\ & g_i(x) = 0, \; i = 1, \ldots, p \end{array}$$

- variables are (x, t)
- $m + 1$ inequality constraints
- objective is *linear*: $t = e_{n+1}^T(x, t)$
- if original problem is cvx, so is epigraph form



linear objective is 'universal' for convex optimization

Std form with generalized inequalities

convex optimization problem in *standard form with generalized inequalities*:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i \preceq_{K_i} 0, \;\; i = 1, \ldots, L \\ & Ax = b \end{array}$$

where:

- $\preceq_{K_i}$ are generalized inequalities on $\mathbf{R}^{m_i}$
- $f_i : \mathbf{R}^n \to \mathbf{R}^{m_i}$ are K_i-convex

Example. *semidefinite programming*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & A_0 + x_1 A_1 + \cdots + x_n A_n \preceq 0 \end{array}$$

where $A_i = A_i \in \mathbf{R}^{p \times p}$

- one generalized inequality constraint $(L = 1)$
- K_1 is PSD cone; $\preceq$ is matrix inequality
- f_1 is affine, hence matrix convex

How f_i, g_i are described

analytical form
functions can have analytical form, *e.g.*,

$$f(x) = x^T P x + 2q^T x + r$$

f is specified by giving the problem *data, coefficients, or parameters*, *e.g.*

$$P = P^T \in \mathbf{R}^{n \times n}, \quad q \in \mathbf{R}^n, \quad r \in \mathbf{R}$$

oracle form
functions can be given by *oracle* or *subroutine* that, given x, computes $f(x)$ (and maybe $\nabla f(x)$, $\nabla^2 f(x)$, ...)

- oracle model can be useful even if f has analytic form, *e.g.*, linear but sparse
- how f given affects choice of algorithm, storage required to specify problem, etc.

Some hard problems

'Slight' modification of convex problem can be very hard

- convex maximization, concave minimization, *e.g.*

$$\begin{array}{ll} \text{maximize} & \|x\| \\ \text{subject to} & Ax \preceq b \end{array}$$

- nonlinear equality constraints, *e.g.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x^T P_i x + q_i^T x + r_i = 0, \ i = 1, \ldots, K \end{array}$$

- minimizing over non-convex sets, *e.g.*, integer constraints

$$\begin{array}{ll} \text{find} & x \\ \text{such that} & Ax \preceq b, \quad x_i \in \{0, 1\} \end{array}$$

Multicriterion optimization

Vector objective

$$F(x) = (f_1(x), \ldots, f_N(x))$$

$f_1, \ldots, f_N : \mathbf{R}^n \to \mathbf{R}$
(can include constraint $C \subseteq \mathbf{R}^n \ldots$)

f_i called *objective functions*: roughly speaking, want all f_i small

Family of *specifications* indexed by $t \in \mathbf{R}^N$:

$$F(x) \preceq t$$

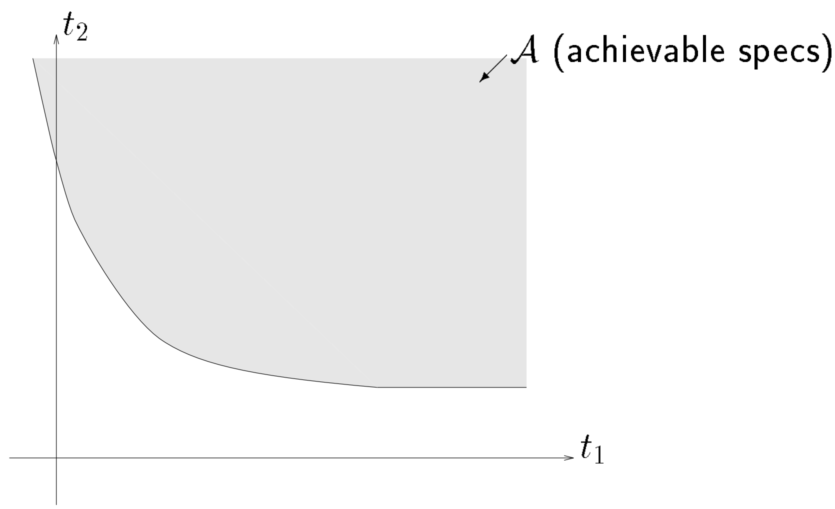
i.e., $f_i(x) \leq t_i$, $i = 1, \ldots, N$.

Achievable specification: t s.t. $F(x) \preceq t$ feasible

Achievable specifications

set of achievable objectives:

$$\mathcal{A} = \{t \in \mathbf{R}^N \mid \exists x \text{ s.t. } F(x) \preceq t\}$$



if f_i are convex then $\mathcal{A}$ is convex

$\mathcal{A}$ is projection of *vector function epigraph*

$$\mathbf{epi}(F) = \{(x,t) \in \mathbf{R}^n \times \mathbf{R}^N \mid F(x) \preceq t\}$$

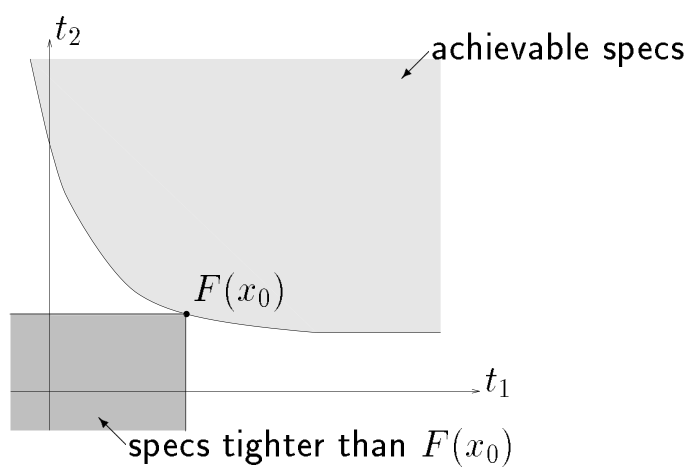
on t-space.

boundary of $\mathcal{A}$ is called (optimal) *tradeoff surface*

Pareto optimality

x *dominates* (is better than) $\tilde{x}$ if $F(x) \preceq F(\tilde{x})$ and $F(x) \neq F(\tilde{x})$
i.e., x is no worse than $\tilde{x}$ in any objective, and better in at least one

x_0 is *Pareto optimal* if no x dominates it

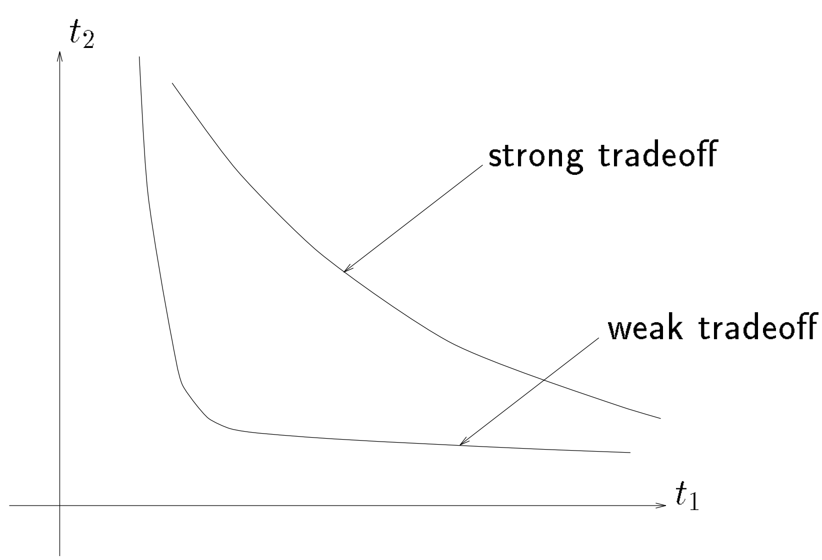


roughly, x_0 Pareto optimal means $F(x_0)$ is on tradeoff surface
x_0 Pareto optimal $\Rightarrow F(x_0) \in \partial\mathcal{A}$
(converse not quite true)

Pareto problem: find Pareto-optimal x

Real (but more vague) engineering problem:
search/explore/characterize tradeoff surface, *e.g.*:

- 'can reduce f_5 below 0.1, but only at huge cost in f_4 and f_2'
- 'can pretty much minimize f_3 independently of other objectives'
- 'f_1 and f_2 tradeoff strongly for $f_1 \leq 1$, $f_2 \leq 2$'



Scalarization

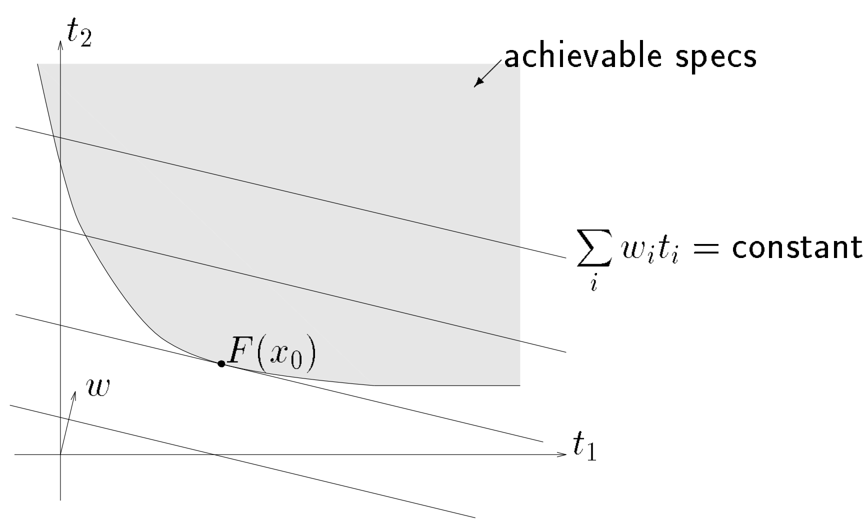
multicriterion problem with $f_1, \ldots, f_N$

weighted sum of objectives: choose weights $w_i > 0$, solve

$$\text{minimize} \ \sum_i w_i f_i(x)$$

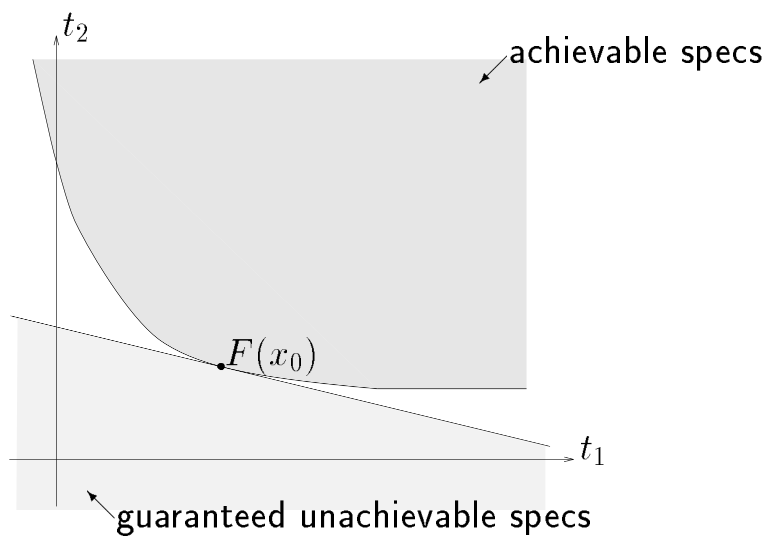
which is the same as

$$\begin{array}{ll} \text{minimize} & w^T t \\ \text{subject to} & t \in \mathcal{A} \end{array}$$



- solution x_0 is Pareto optimal
- for many cvx problems, all Pareto optimal points can be found this way, as weights vary over $\mathbf{R}_+^N$

halfspace of specifications $\{t \mid w^T t < w^T F(x)\}$ are unachievable ($i.e.$, supports $\mathcal{A}$ at x)



Restriction and relaxation

original problem, with optimal value $f^\star$:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C \end{array}$$

new problem, with optimal value $\tilde{f}^\star$:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \tilde{C} \end{array}$$

new problem is

- *relaxation* (of original) if $\tilde{C} \supseteq C$
  (in which case $\tilde{f}^\star \leq f^\star$)
- *restriction* if $\tilde{C} \subseteq C$
  (in which case $\tilde{f}^\star \geq f^\star$)

Example. f is convex, C is nonconvex; $\tilde{C} = \mathbf{Co}C$
relaxation is convex problem that gives lower bound for original, nonconvex problem