Lecture 5

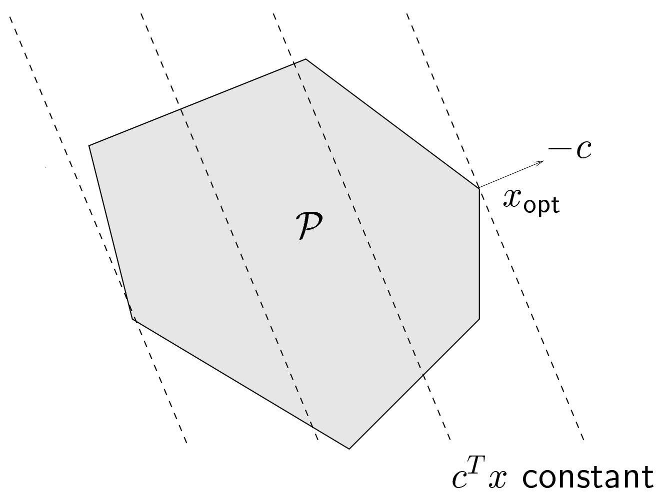
Linear and quadratic problems and Semidefinite programming (SDP)

- linear programming
- examples and applications
- linear fractional programming
- quadratic optimization problems
- (quadratically constrained) quadratic programming
- examples and applications
- Semidefinite programming
- applications

Linear programming (LP)

abstract form: minimize linear obj. over polyhedron $\mathcal{P}$:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P} \end{array}$$



'standard' form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Fx = g \\ & x \succeq 0 \end{array}$$

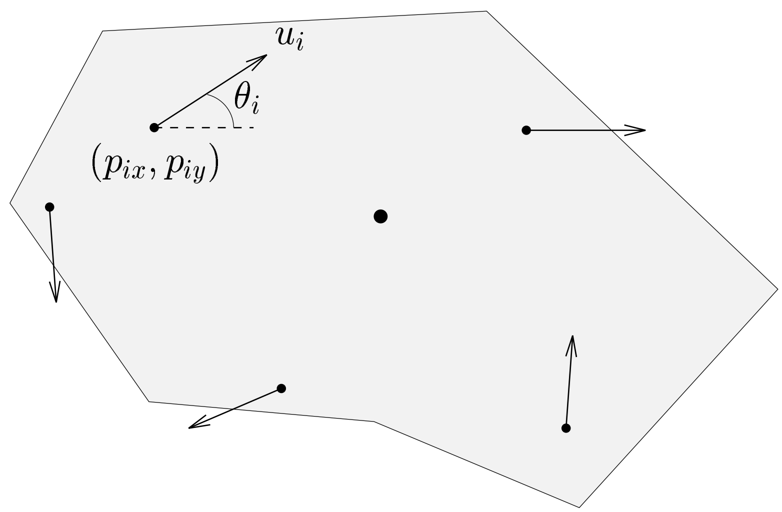
(widely used in LP literature & software)

variations, *e.g.*,

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & Fx = g \end{array}$$

Force/moment generation with thrusters

- rigid body with center of mass at origin $p = 0 \in \mathbf{R}^2$
- n forces with magnitude u_i, acting at $p_i = (p_{ix}, p_{iy})$, in direction θ_i



resulting horizontal force: $F_x = \sum_{i=1}^n u_i \cos\theta_i$

resulting vertical force: $F_y = \sum_{i=1}^n u_i \sin\theta_i$

resulting torque: $T = \sum_{i=1}^n p_{iy} u_i \cos\theta_i - p_{ix} u_i \sin\theta_i$

force limits: $0 \le u_i \le 1$ (thrusters)
fuel usage: $u_1 + \cdots + u_n$

Problem: Find thruster forces u_i that yield given desired forces and torques and minimize fuel usage (if feasible) can be expressed as LP:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & Fu = f^{\mathsf{des}} \\ & 0 \le u_i \le 1, \ i = 1, \ldots, n \end{array}$$

where

$$F = \begin{bmatrix} \cos\theta_1 & \cdots & \cos\theta_n \\ \sin\theta_1 & \cdots & \sin\theta_n \\ p_{1y}\cos\theta_1 - p_{1x}\sin\theta_1 & \cdots & p_{ny}\cos\theta_n - p_{nx}\sin\theta_n \end{bmatrix}$$

$$f^{\mathsf{des}} = \left[F_x^{\mathsf{des}} \ F_y^{\mathsf{des}} \ T^{\mathsf{des}} \right]^T$$

$$\mathbf{1} = [\, 1 \ 1 \ \cdots \ 1 \,]^T$$

Converting LP to 'standard' form

- inequalities as equality constraints: write $a_i^T x \le b_i$ as

$$\begin{array}{l} a_i^T x + s_i = b_i \\ s_i \ge 0 \end{array}$$

s_i is called *slack variable* associated with $a_i^T x \le b_i$

- unconstrained variables: write $x_i \in \mathbf{R}$ as

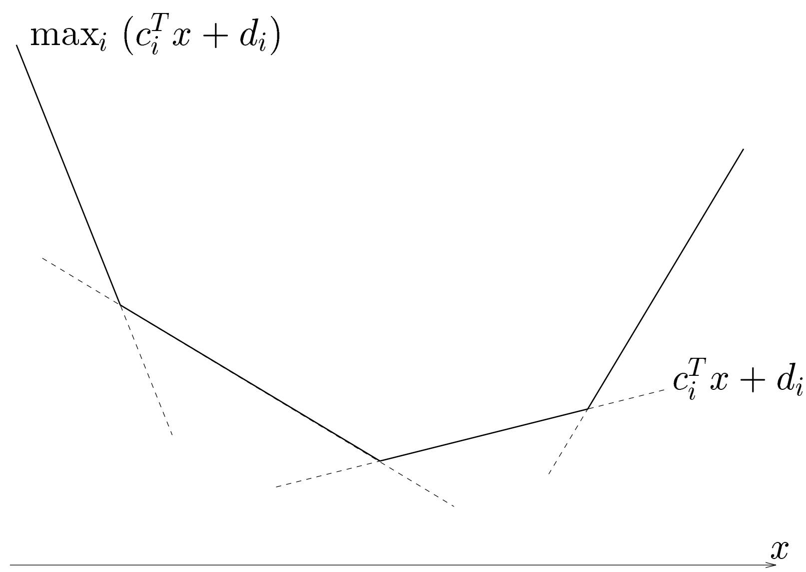
$$\begin{array}{c} x_i = x_i^+ - x_i^- \\ x_i^+, x_i^- \ge 0 \end{array}$$

Example. Thruster problem in 'standard' form

$$\begin{array}{ll} \text{minimize} & \left[\mathbf{1}^T \; 0\,\right] \begin{bmatrix} u \\ s \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} u \\ s \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} F & 0 \\ I & I \end{bmatrix} \begin{bmatrix} u \\ s \end{bmatrix} = \begin{bmatrix} f^{\mathsf{des}} \\ \mathbf{1} \end{bmatrix} \end{array}$$

Piecewise-linear minimization

$$\begin{array}{ll} \text{minimize} & \max_i \, (c_i^T x + d_i) \\ \text{subject to} & Ax \preceq b \end{array}$$



express as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & c_i^T x + d_i \leq t \\ & Ax \preceq b \end{array}$$

an LP in variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$

ℓ_∞- and ℓ_1-norm approximation

Constrained ℓ_∞- (Chebychev) approximation

$$\begin{array}{ll} \text{minimize} & \|Ax-b\|_\infty \\ \text{subject to} & Fx \preceq g \end{array}$$

write as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax-b \preceq t\mathbf{1} \\ & Ax-b \succeq -t\mathbf{1} \\ & Fx \preceq g \end{array}$$

Constrained ℓ_1-approximation

$$\begin{array}{ll} \text{minimize} & \|Ax-b\|_1 \\ \text{subject to} & Fx \preceq g \end{array}$$

write as

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & Ax-b \preceq y \\ & Ax-b \succeq -y \\ & Fx \preceq g \end{array}$$

Extensions of thruster problem

- opposing thruster pairs

$$\begin{array}{ll} \text{minimize} & \sum_i |u_i| \\ \text{subject to} & Fu = f^{\text{des}} \\ & |u_i| \leq 1, \quad i = 1, \ldots, n \end{array}$$

  can express as LP

- given f^{des},

$$\begin{array}{ll} \text{minimize} & \|Fu - f^{\text{des}}\|_\infty \\ \text{subject to} & 0 \leq u_i \leq 1, \quad i = 1, \ldots, n \end{array}$$

  can express as LP

- given f^{des},

$$\begin{array}{ll} \text{minimize} & \#\text{ thrusters on} \\ \text{subject to} & Fu = f^{\text{des}} \\ & 0 \leq u_i \leq 1, \quad i = 1, \ldots, n \end{array}$$

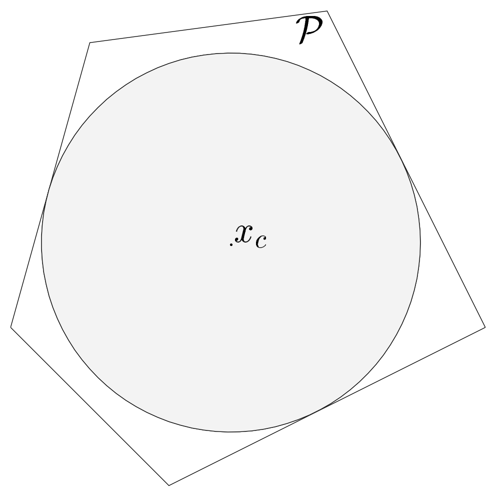
  can not express as LP
  (# thrusters on is quasiconcave!)

Design centering

Find largest ball inside a polyhedron

$$\mathcal{P} = \{x \mid a_i^T x \le b_i,\ i = 1, \ldots, m\}$$

center is called *Chebychev center*



ball $\{x_c + u \mid \|u\| \le r\}$ lies in $\mathcal{P}$ if and only if

$$\sup\{a_i^T x_c + a_i^T u \mid \|u\| \le r\} \le b_i, \quad i = 1, \ldots, m,$$

i.e.,

$$a_i^T x_c + r\|a_i\| \le b_i,\ i = 1, \ldots, m$$

Hence, finding Chebychev center is an LP:

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T x_c + r\|a_i\| \le b_i, \quad i = 1, \ldots, m \end{array}$$

Linear fractional programming

$$
\begin{array}{ll}
\text{minimize} & \dfrac{c^T x + d}{f^T x + g} \\
\text{subject to} & Ax \preceq b \\
& f^T x + g > 0
\end{array}
$$

- objective function is quasiconvex
- sublevel sets are polyhedra
- like LP, can be solved very efficiently

extension:

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\max_{i=1,\ldots,K} \frac{c_i^T x + d_i}{f_i^T x + g_i} \\
\text{subject to} & Ax \preceq b \\
& f_i^T x + g_i > 0, \ i = 1, \ldots, K
\end{array}
$$

- objective function is quasiconvex
- sublevel sets are polyhedra

Nonconvex extensions of LP

Boolean LP or **zero-one LP**:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & Fx = g \\ & x_i \in \{0, 1\} \end{array}$$

integer LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & Fx = g \\ & x_i \in \mathbf{Z} \end{array}$$

these are in general

- not convex problems
- **extremely difficult** to solve

Quadratic functions and forms

definitions:

- *quadratic function*

$$f(x) = x^T P x + 2q^T x + r = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P & q \\ q^T & r \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

  convex if and only if $P \succeq 0$

- *quadratic form* $f(x) = x^T P x$
  convex if and only if $P \succeq 0$

- Euclidean norm $f(x) = \|Ax + b\|$

Minimizing a quadratic function

$$\text{minimize } f(x) = x^T P x + 2q^T x + r$$

nonconvex case ($P \not\succeq 0$): unbounded below ($f^\star = -\infty$)

Proof: take $x = tv$, $t \to \infty$, where $Pv = \lambda v$, $\lambda < 0$

convex case ($P \succeq 0$):
x is optimal iff $\nabla f(x) = 2Px + 2q = 0$

two cases:

- $q \in \mathrm{range}(P)$: $f^\star > -\infty$
- $q \notin \mathrm{range}(P)$: unbounded below ($f^\star = -\infty$)

important special case, $P \succ 0$:
unique optimal point $x_{\mathsf{opt}} = -P^{-1}q$;
optimal value $f^\star = r - q^T P^{-1} q$

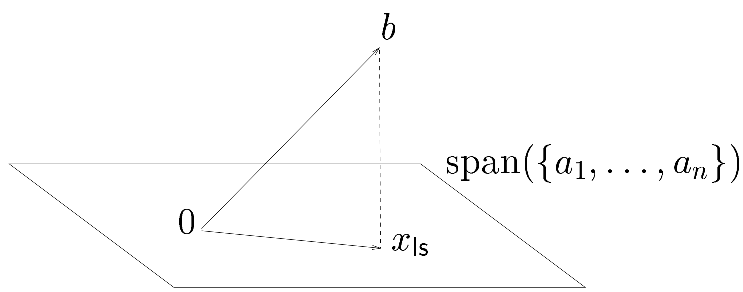
Least-squares problems

Minimize Euclidean norm

$$\text{minimize } \|Ax - b\|$$

$(A = [a_1 \cdots a_n]$ full rank, skinny)

geometrically: project b on $\mathrm{span}(\{a_1, \ldots, a_n\})$

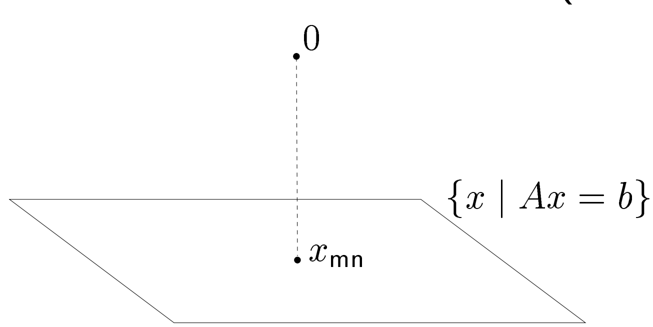


solution: $x_{\mathrm{ls}} = (A^TA)^{-1}A^Tb$

Minimum norm solution

$$\begin{array}{ll} \text{minimize} & \|x\| \\ \text{subject to} & Ax = b \end{array}$$

(A full rank, fat)

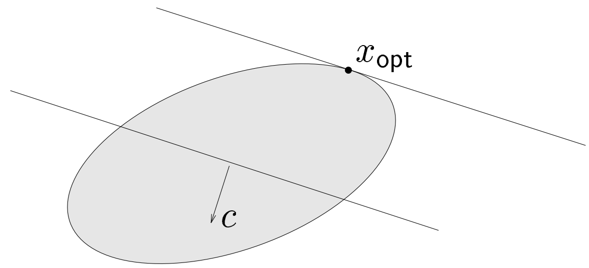


solution: $x_{\mathrm{mn}} = A^T(AA^T)^{-1}b$

Minimizing a linear function with quadratic constraint

minimize $c^T x$
subject to $x^T A x \leq 1$

$(A = A^T \succ 0)$



$$x_{\text{opt}} = -A^{-1}c/\sqrt{c^T A^{-1} c}$$

Proof. Change of variables $y = A^{1/2}x$, $\tilde{c} = A^{-1/2}c$

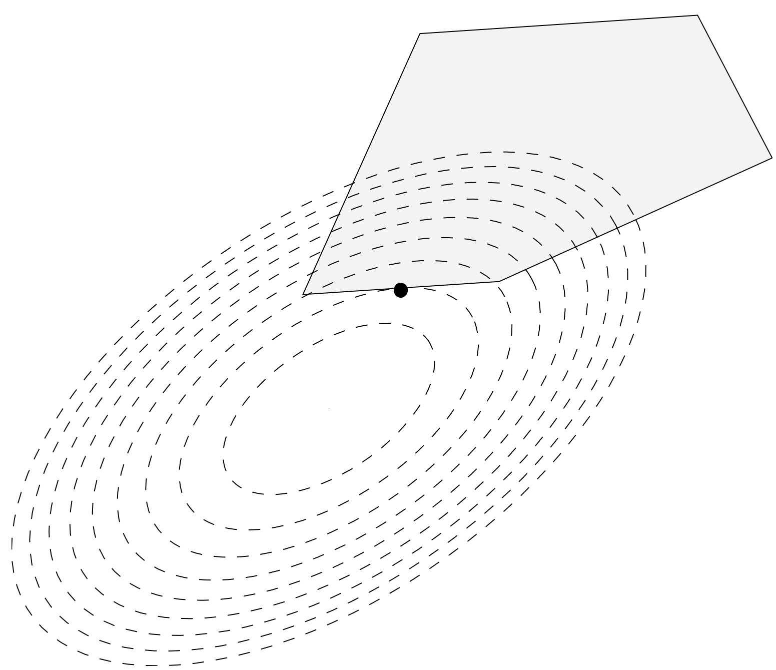
minimize $\tilde{c}^T y$
subject to $y^T y \leq 1$

Optimal solution: $y_{\text{opt}} = -\tilde{c}/\|\tilde{c}\|$.

Quadratic programming

quadratic objective, linear inequalities

$$\begin{array}{ll} \text{minimize} & x^TPx + 2q^Tx + r \\ \text{subject to} & Ax \preceq b \end{array}$$



convex optimization problem if $P \succeq 0$

very hard problem if $P \not\succeq 0$

QCQP and SOCP

quadratically constrained quadratic programming (QCQP):

$$\begin{array}{ll} \text{minimize} & x^T P_0 x + 2q_0^T x + r_0 \\ \text{subject to} & x^T P_i x + 2q_i^T x + r_i \leq 0, \quad i = 1, \ldots, L \end{array}$$

- convex if $P_i \succeq 0$, $i = 0, \ldots, L$
- nonconvex QCQP **very difficult**

second-order cone programming (SOCP):

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \|A_i x + b_i\| \leq e_i^T x + d_i, \quad i = 1, \ldots, L \end{array}$$

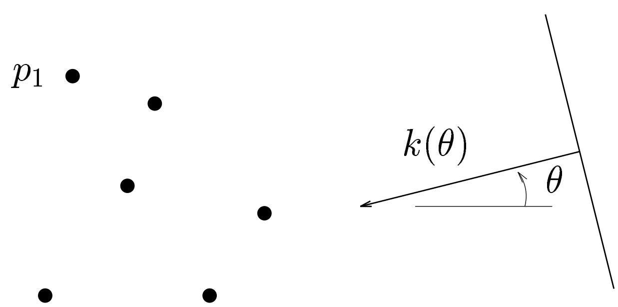
includes QCQP (QP, LP)

Beamforming

- omnidirectional antenna elements at positions $p_1, \ldots, p_n \in \mathbf{R}^2$
- plane wave incident from angle θ:

$$\exp j(k(\theta)^T p - \omega t), \qquad k(\theta) = -[\cos\theta \;\; \sin\theta]^T$$

($j = \sqrt{-1}$)



- output of element i: $y_i(\theta) = \exp(jk(\theta)^T p_i)$
- output of array is weighted sum $y(\theta) = \sum_{i=1}^{n} w_i y_i(\theta)$
- $G(\theta) \triangleq |y(\theta)|$ antenna gain pattern

design variables: $x = [\mathbf{Re}\, w^T \; \mathbf{Im}\, w^T]^T$
(antenna array weights or shading coefficients)

Sidelobe level minimization

make $G(\theta)$ small for $|\theta - \theta_{\text{tar}}| > \alpha$

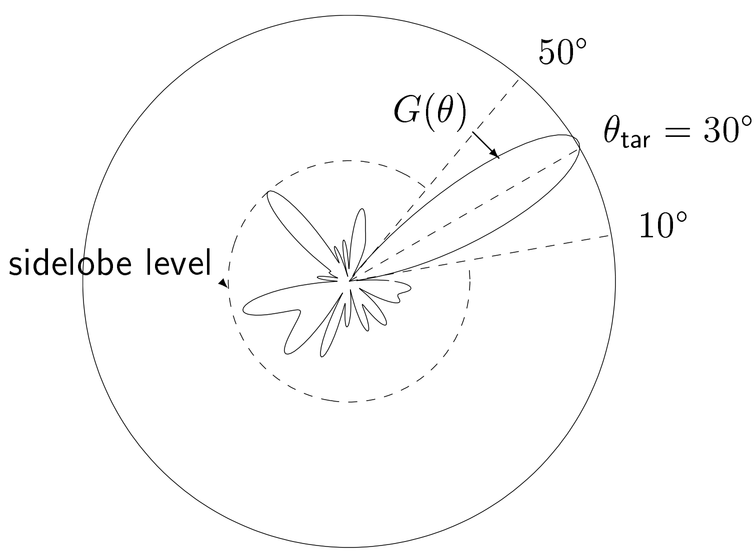
- θ_{tar}: target direction
- 2α: beamwidth

Via least-squares (discretize angles)

$$\begin{array}{ll} \text{minimize} & \sum_i G(\theta_i)^2 \\ \text{subject to} & y(\theta_{\text{tar}}) = 1 \end{array}$$

(sum over angles outside beam)

least-squares problem with two linear equality constraints



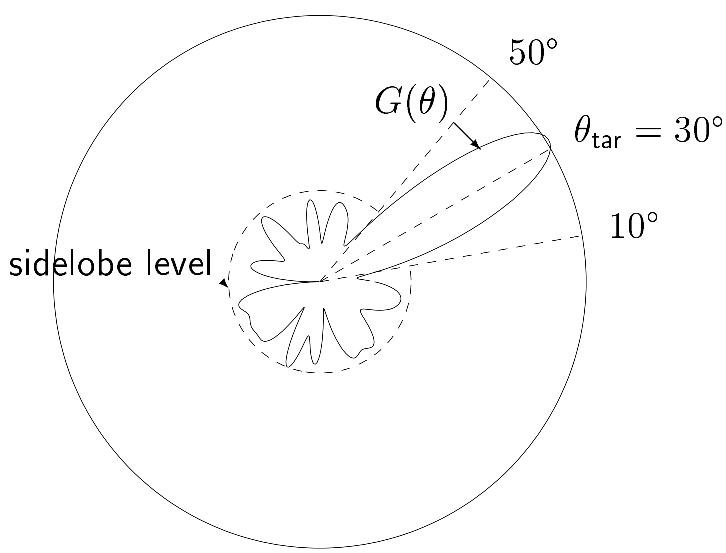
Via QCQP

$$\begin{array}{ll} \text{minimize} & \max_i G(\theta_i) \\ \text{subject to} & y(\theta_{\text{tar}}) = 1 \end{array}$$

(max over angles outside beam)

Quadratically constrained quadratic program

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & G(\theta_i) \le t \\ & y(\theta_{\text{tar}}) = 1 \end{array}$$



Extensions

- $G(\theta_0) = 0$ (null in direction θ_0)
- w is real (amplitude only shading)
- $|w_i| \leq 1$ (attenuation only shading)
- minimize $\sigma^2 \sum_i |w_i|^2$ (thermal noise power in y)
- minimize beamwidth given a maximum sidelobe level
- maximize number of zero weights

Semidefinite programming (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F(x) \preceq 0 \end{array}$$

where

$$F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n, \quad F_i = F_i^T \in \mathbf{R}^{p \times p}$$

- SDP is cvx opt problem in generalized standard form ($\preceq$ is matrix inequality)
- LMI $F(x) \preceq 0$ is equivalent to a set of polynomial inequalities in x (nonnegative diagonal minors of $-F$)
- multiple LMIs can be combined into one (block diagonal) LMI

cf. LP, written as

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & G(x) \preceq 0 \end{array}$$

where

$$G(x) = g_0 + x_1 g_1 + \cdots + x_n g_n$$

(and $\preceq$ is componentwise inequality)

LP as SDP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

can be expressed as SDP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{diag}(Ax - b) \preceq 0 \end{array}$$

since $Ax - b \preceq 0 \;\Leftrightarrow\; \mathbf{diag}(Ax - b) \preceq 0$
(that's tricky notation!)

Maximum eigenvalue minimization

$$\text{minimize}_x \; \lambda_{\mathsf{max}}(A(x))$$

$A(x) = A_0 + x_1 A_1 + \cdots + x_m A_m$, $A_i = A_i^T$
SDP with variables $x \in \mathbf{R}^m$ and $t \in \mathbf{R}$:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) - tI \preceq 0 \end{array}$$

Schur complements

$$X = X^T = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

$S = C - B^T A^{-1} B$ is the *Schur complement* of A in X (provided $\det A \neq 0$)

- arises in many contexts
- useful to represent nonlinear convex constraints as LMIs

Facts: (homework)

- $X \succ 0$ if and only if $A \succ 0$ and $S \succ 0$
- if $A \succ 0$, then $X \succeq 0$ if and only if $S \succeq 0$

Example. (convex) quadratic inequality

$$(Ax+b)^T(Ax+b) - c^T x - d \leq 0$$

is equivalent to the LMI

$$\begin{bmatrix} I & Ax+b \\ (Ax+b)^T & c^T x + d \end{bmatrix} \succeq 0$$

QCQP as SDP

The quadratically constrained quadratic program

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, L \end{array}$$

where $f_i(x) \triangleq (A_i x + b)^T (A_i x + b) - c_i^T x - d_i$

can be expressed as SDP (in x and t)

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} I & A_0 x + b_0 \\ (A_0 x + b_0)^T & c_0^T x + d_0 + t \end{bmatrix} \succeq 0, \\ & \begin{bmatrix} I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \ldots, L \end{array}$$

extends to problems over second-order cone:

$$\|Ax + b\| \leq e^T x + d$$

is equivalent to LMI

$$\begin{bmatrix} (e^T x + d) I & Ax + b \\ (Ax + b)^T & e^T x + d \end{bmatrix} \succeq 0$$

Simple nonlinear example

$$\begin{array}{ll} \text{minimize} & \dfrac{(c^T x)^2}{d^T x} \\ \text{subject to} & Ax \preceq b \end{array}$$

(assume $d^T x > 0$ whenever $Ax \preceq b$)

1. equivalent problem with linear objective:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax \preceq b \\ & t - \dfrac{(c^T x)^2}{d^T x} \geq 0 \end{array}$$

2. SDP (in x, t) using Schur complement:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} \mathbf{diag}(b - Ax) & 0 & 0 \\ 0 & t & c^T x \\ 0 & c^T x & d^T x \end{bmatrix} \succeq 0 \end{array}$$

Matrix norm minimization

$$\text{minimize } \|A(x)\|$$

where

$$A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n, \quad A_i \in \mathbf{R}^{p \times q}$$

and $\|A\| = \left(\lambda_{\mathsf{max}}(A^T A)\right)^{1/2}$

can cast as SDP:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0 \end{array}$$

Measurements with unknown sensor noise variance

Random vectors $y = x + v \in \mathbf{R}^k$

- x: random vector of interest,
  $\mathbf{E}\, x = \bar{x}$, $\mathbf{E}(x - \bar{x})(x - \bar{x})^T = \Sigma$
- v: measurement noise, independent of x,
  $\mathbf{E}\, v = 0$, $\mathbf{E}\, vv^T = F$, diagonal but otherwise unknown
- y: measured data, $\mathbf{E}\, y = \bar{x}$,
  $\mathbf{E}(y - \bar{x})(y - \bar{x})^T = \widehat{\Sigma} = \Sigma + F$

take **many** samples of $y \Rightarrow \bar{x}$, $\widehat{\Sigma}$ known

covariance Σ is unknown, but lies in (convex) set

$$\mathsf{S} = \{\widehat{\Sigma} - D \mid D \succeq 0 \text{ diagonal}, \widehat{\Sigma} - D \succeq 0\}$$

can bound linear function of Σ by solving SDP over **S**

Example. can bound variance of $c^T x$ by solving SDP:

$$\begin{aligned} c^T \widehat{\Sigma} c \;&\geq\; \mathbf{E}(c^T x - c^T \bar{x})^2 \\ &\geq\; \inf\{c^T \widehat{\Sigma} c - c^T D c \mid D \text{ diag.},\; D \succeq 0,\; \widehat{\Sigma} - D \succeq 0\} \end{aligned}$$

Special case. 'educational testing problem' ($c = \mathbf{1}$)

- x: 'ability' of a random student on k tests
- y: score of a random student on k tests
- v: testing error of k tests
- $\mathbf{1}^T x$: total ability on tests
- $\mathbf{1}^T y$: total test score
- $\mathbf{1}^T \Sigma \mathbf{1}$: variance in total ability
- $\mathbf{1}^T \widehat{\Sigma} \mathbf{1}$: variance in total score
- reliability of the test:

$$\frac{\mathbf{1}^T \Sigma \mathbf{1}}{\mathbf{1}^T \widehat{\Sigma} \mathbf{1}} = 1 - \frac{\mathbf{Tr}\, F}{\mathbf{1}^T \widehat{\Sigma} \mathbf{1}}$$

can bound reliability by solving SDP:

$$\begin{array}{ll} \text{maximize} & \mathbf{Tr}\, D \\ \text{subject to} & D \text{ diagonal}, \;\; D \succeq 0 \\ & \widehat{\Sigma} - D \succeq 0 \end{array}$$