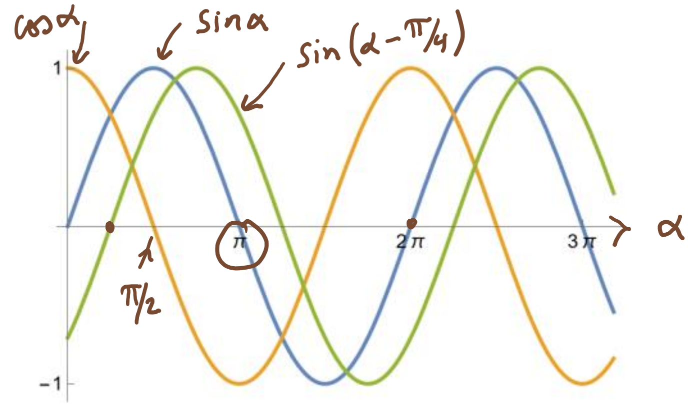
Aikaharmoniset kentät



$$\sin(\omega t)$$

$$\omega = 2\pi f \quad [\text{Hz}]$$

$$\frac{\partial}{\partial t}\sin(\omega t) = \omega\cos(\omega t)$$

$$\frac{\partial}{\partial t}\cos(\omega t) = -\omega\sin(\omega t)$$

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$\frac{\partial}{\partial t}\cancel{e^{j\omega t}} = j\omega\cancel{e^{j\omega t}}$$

REAALISET KENTÄT

$$\nabla\times\bar{E}(\bar{r},t) = -\frac{\partial}{\partial t}\bar{B}(\bar{r},t)$$

AIKAHARMONINEN, KOMPLEKSINEN

$$\nabla\times\bar{E}(\bar{r}) = -j\omega\bar{B}(\bar{r})$$

$$\boxed{\bar{E}(\bar{r},t) = \mathrm{Re}\left\{\bar{E}(\bar{r})\,e^{j\omega t}\right\}}$$

Kompleksiluvut

$$j\cdot j = -1$$

$$a = \underbrace{a_r}_{\mathrm{Re}\{a\}} + j\overset{\mathrm{Im}\{a\}}{a_i}$$

$$a+b = a_r + ja_i + b_r + jb_i$$
$$= (a_r+b_r) + j(a_i+b_i)$$

$$ab = (a_r+ja_i)(b_r+jb_i) = a_rb_r + \overbrace{a_rjb_i}^{ja_rb_i} + ja_ib_r + \overbrace{ja_ijb_i}^{j\cdot j a_ib_i}$$
$$= (a_rb_r - a_ib_i) + j(a_rb_i + a_ib_r)$$

$$a \quad a_r + ja_i \quad (a_r+ja_i)(b_r-jb_i) \quad - \quad \underline{a_rb_r + a_ib_i + j(a_ib_r - a_rb_i)}$$

$$\frac{a}{b} = \frac{a_r + j a_i}{b_r + j b_i} = \frac{(a_r + j a_i)(b_r - j b_i)}{\underbrace{(b_r + j b_i)(b_r - j b_i)}_{b_r^2 - j b_r b_i + j b_r b_i + b_i^2}} = \frac{a_r b_r + a_i b_i + j(a_i b_r - a_r b_i)}{b_r^2 + b_i^2}$$

Kompleksiluvun konjugaatti (liittoluku)

$$a^* = (a_r + j a_i)^* = a_r - j a_i$$

$$(ab)^* = a^* b^*$$

$+ \rightarrow -$

$$(a_r + j a_i)(b_r + j b_i) = a_r b_r - a_i b_i + j(a_r b_i + a_i b_r)$$

$$(a_r - j a_i)(b_r - j b_i) = a_r b_r - a_i b_i - j(a_r b_i + a_i b_r)$$

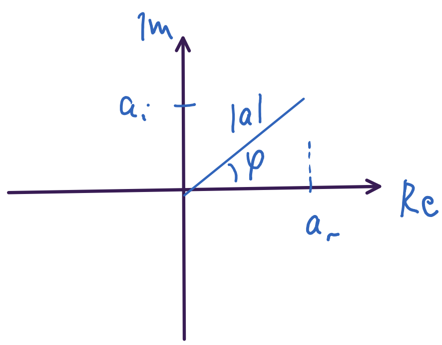
Kompleksiluvun itseisarvo (moduuli)

$$|a|^2 = a_r^2 + a_i^2 \qquad |a|^2 = a a^*$$

$$(a a^*)^* = a^* a^{**} = a^* a$$

$$e^{\alpha} \cdot e^{\beta} = e^{\alpha + \beta}$$

Napakoordinaattiesitys



$$a = |a| e^{j\varphi} = \underbrace{|a| \cos\varphi}_{a_r} + j \underbrace{|a| \sin\varphi}_{a_i}$$

$$ab = |a| e^{j\varphi_a} \cdot |b| e^{j\varphi_b} = |a||b| e^{j(\varphi_a + \varphi_b)}$$

Eulerin kaava

$$e^{j\varphi} = \cos\varphi + j \sin\varphi$$

$$\left(e^{jc}\right)^* = e^{-jc^*}$$

c reaalinen $\Rightarrow e^{-jc^*} = e^{-jc}$

Kompleksilukualgebraa: neliöjuuri

$$\sqrt{a+\mathrm{j}b} = ?$$

$$a+\mathrm{j}b = A\mathrm{e}^{\mathrm{j}\psi} \qquad A = \sqrt{a^2+b^2}$$

$$\psi = \arctan\left(\frac{b}{a}\right)$$

$$\boxed{\sqrt{a+\mathrm{j}b} = \left(A\mathrm{e}^{\mathrm{j}\psi}\right)^{1/2} = \sqrt{A}\mathrm{e}^{\mathrm{j}\psi/2} = \sqrt{A}\cos(\psi/2) + \mathrm{j}\sqrt{A}\sin(\psi/2)}$$

toisaalta:

$$\sqrt{a+\mathrm{j}b} = c+\mathrm{j}d$$

$$a+\mathrm{j}b = c^2-d^2+2\mathrm{j}cd$$

$$d = \frac{b}{2c} \qquad c^2 - \frac{b^2}{4c^2} = a$$

$$4c^4-4ac^2-b^2=0$$

$$\boxed{\begin{aligned} c^2 &= \frac{1}{2}\left(\sqrt{a^2+b^2}+a\right) \\ d^2 &= \frac{1}{2}\left(\sqrt{a^2+b^2}-a\right) \end{aligned}}$$

ETUMERKIT!!

Maxwellin yhtälöt aikaharmonisessa muodossa

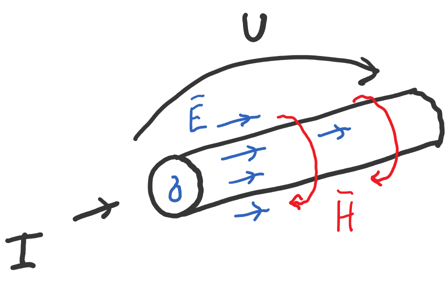
$$\nabla \times \bar{E} = -\mathrm{j}\omega \bar{B} \qquad \nabla \times \bar{H} = \bar{J} + \mathrm{j}\omega\bar{D}$$

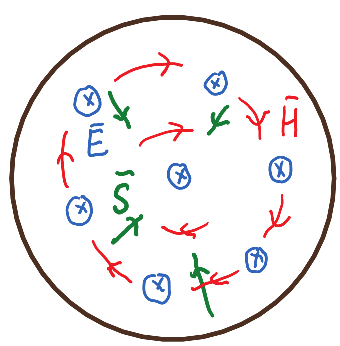
$$\nabla \cdot \bar{D} = \rho \qquad \nabla \cdot \bar{B} = 0$$

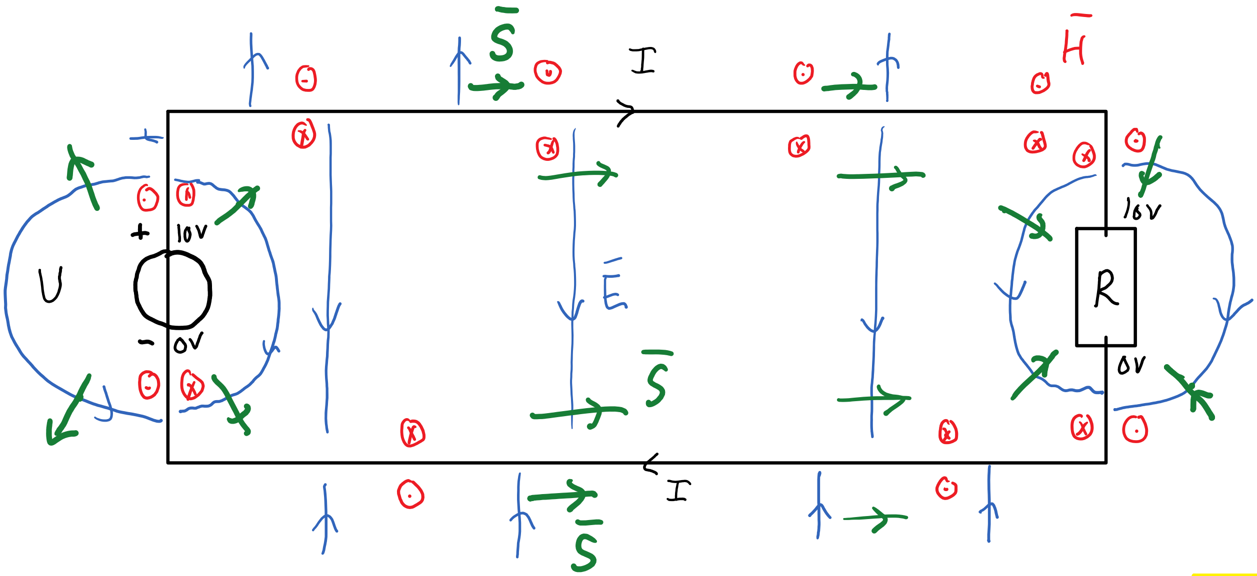
Poyntingin vektori (REAALINEN)

$$\bar{S}(\bar{r},t) = \bar{E}(\bar{r},t) \times \bar{H}(\bar{r},t)$$

$$[\bar{S}] = \frac{\mathrm{V}}{\mathrm{m}}\,\frac{\mathrm{A}}{\mathrm{m}} = \frac{\mathrm{W}}{\mathrm{m}^2}$$







$a + a^* = 2\,Re\{a\}$

$a^{**} = a$

$e^{j\omega t} \cdot e^{-j\omega t} = 1$

$$\bar{S}(\bar{r},t) = \bar{E}(\bar{r},t) \times \bar{H}(\bar{r},t)$$

$$= Re\{\bar{E}(\bar{r})e^{j\omega t}\} \times Re\{\bar{H}(\bar{r})e^{j\omega t}\}$$

$$= \frac{1}{2}(\bar{E}e^{j\omega t} + \bar{E}^* e^{-j\omega t}) \times \frac{1}{2}(\bar{H}e^{j\omega t} + \bar{H}^* e^{-j\omega t})$$

$$= \frac{1}{4}\Big(\bar{E}\times\bar{H}\,e^{2j\omega t} + \underbrace{\bar{E}\times\bar{H}^* + \bar{E}^*\times\bar{H}}_{\bar{E}\times\bar{H}^* + (\bar{E}\times\bar{H}^*)^* = 2\,Re\{\bar{E}\times\bar{H}^*\}} + \bar{E}^*\times\bar{H}^*\,e^{-2j\omega t}\Big)$$

$$= \frac{1}{2}Re\{\bar{E}\times\bar{H}^*\} + \frac{1}{2}Re\{\bar{E}\times\bar{H}\,e^{2j\omega t}\}$$

aikakeskiarvo on nolla

Poyntingin vektorin aikakeskiarvo

$$\langle \bar{S}(\bar{r},t)\rangle = \frac{1}{2}Re\{\bar{E}\times\bar{H}^*\}$$

Kompleksinen Poyntingin vektori

$$\bar{S}(\bar{r}) = \frac{1}{2}\bar{E}(\bar{r})\times\bar{H}^*(\bar{r})$$

Aikaharmonisuus piiriteorian terminologiassa: kompleksiluku — osoitin

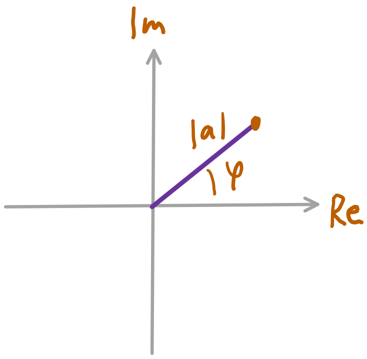
AIKA-ALUE

$$U(t) = U_0 \cos(\omega t + \varphi)$$

TAAJUUSALUE

$$U_0 e^{j\varphi} = U_0 \angle\varphi = \tilde{U}$$

KOMPLEKSITASO



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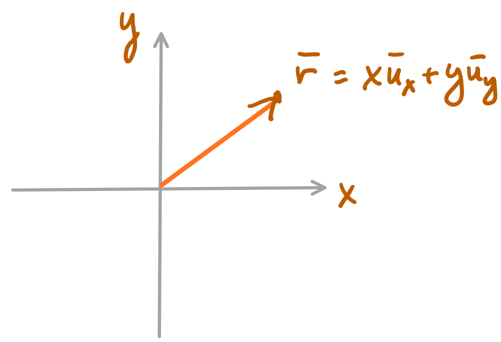
REAALINEN VEKTORITASO



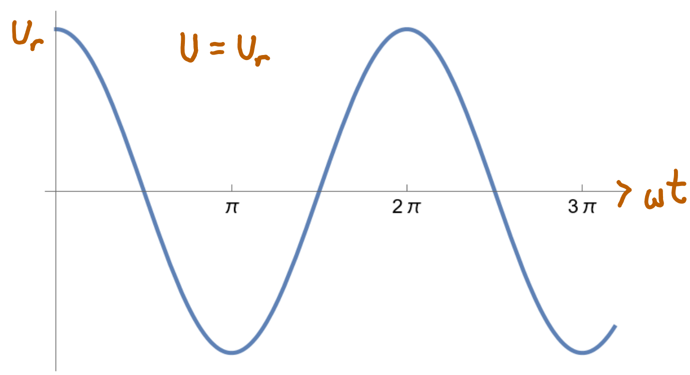
Esimerkkejä

$$U = U_r + jU_i$$

$$U(t) = Re\{(U_r + jU_i)e^{j\omega t}\}$$

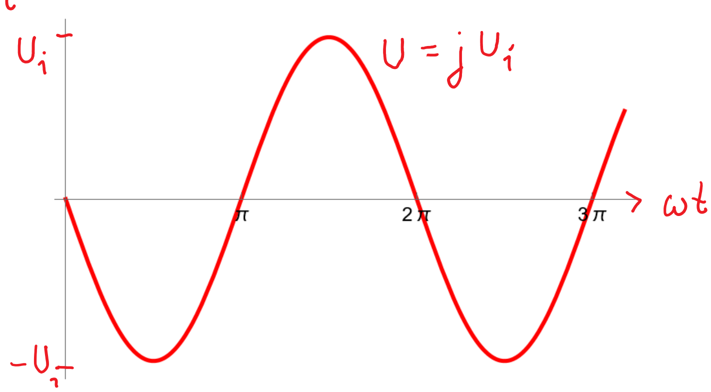
$$= Re\{(U_r + jU_i)(\cos\omega t + j\sin\omega t)\}$$

$$= \underbrace{U_r \cos\omega t} - U_i \sin\omega t$$



$$U_r = 0$$

$$\Rightarrow U(t) = -U_i \sin\omega t$$



$$U = e^{-j\pi/5} U_0$$

$$U(t) = Re\{U_0 \underbrace{e^{-j\pi/5} e^{j\omega t}}_{}\}$$

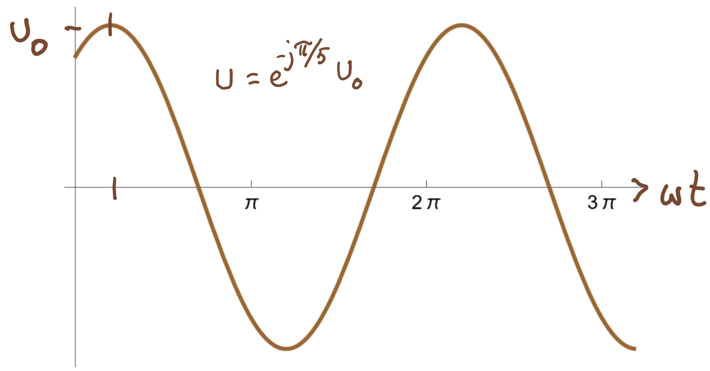
$e^{j(\omega t - \pi/5)}$

$$= U_0 \cos(\underbrace{\omega t - \pi/5})$$

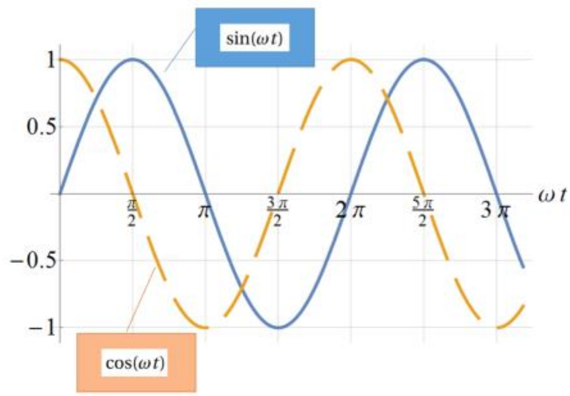
$\omega t = +\pi/5$

$$\omega t = +\pi/5$$

$$\Rightarrow \cos(\omega t - \pi/5) = 1$$



Derivointi — j:llä kertominen — edistys vai viivästys?



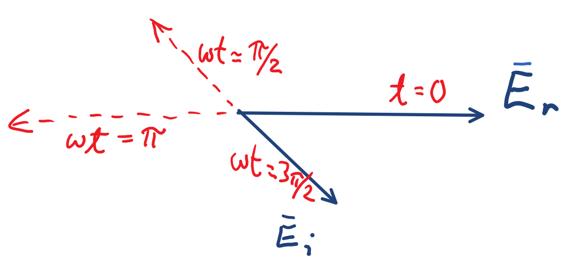
(Kosini 90 astetta siniä edellä ajassa => j:llä kertominen edistää)

Kompleksivektorit $\quad \bar{E}(\bar{r}) = \bar{E}_r(\bar{r}) + j\,\bar{E}_i(\bar{r})$

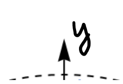
$$\bar{E}(\bar{r},t) = \mathrm{Re}\left\{\bar{E}(\bar{r})\, e^{j\omega t}\right\}$$

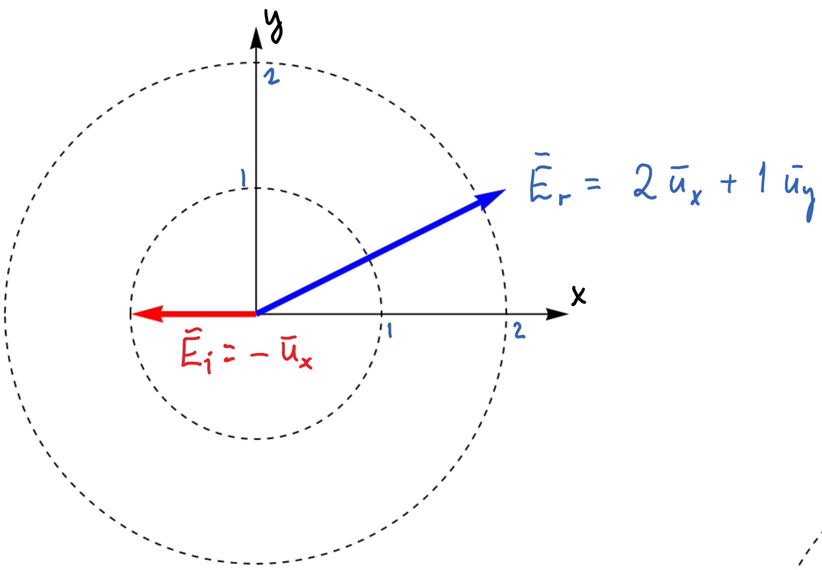
$$\bar{E} = \bar{E}_r + j\,\bar{E}_i \quad \Rightarrow \quad \bar{E}(t) = \mathrm{Re}\left\{(\bar{E}_r + j\bar{E}_i)(\cos\omega t + j\sin\omega t)\right\}$$

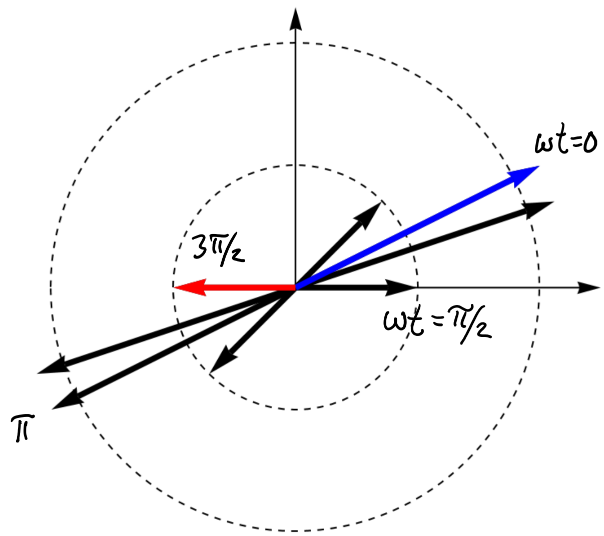
$$= \bar{E}_r \cos\omega t - \bar{E}_i \sin\omega t$$

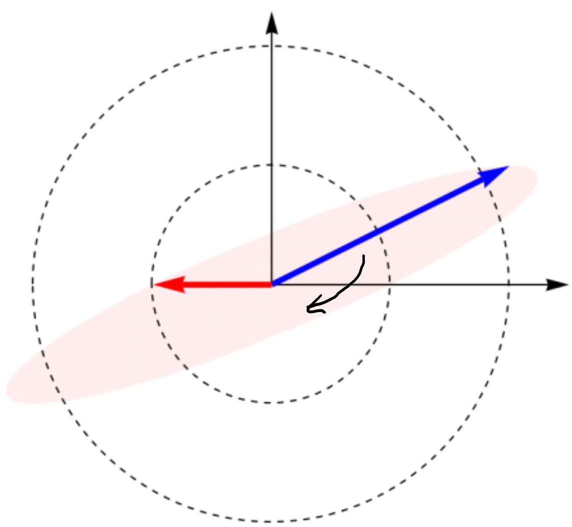


ELLIPSI !









Maxwellin yhtälöt (kompleksivektoreilla)

$$\nabla \times \bar{E} = -j\omega \underbrace{\bar{B}}_{\mu_0 \bar{H}} \qquad \nabla \times \bar{H} = \underbrace{\bar{J}}_{0} + j\omega \underbrace{\bar{D}}_{\varepsilon_0 \bar{E}}$$

$$\nabla \cdot \underbrace{\bar{D}}_{\varepsilon_0 \bar{E}} = \underbrace{\rho}_{0} \qquad \nabla \cdot \underbrace{\bar{B}}_{\mu_0 \bar{H}} = 0$$

Lähteetön alue

$\bar{J} = 0,\ \rho = 0$

$\varepsilon_0 \quad \mu_0$

Lähteetön alue

$\bar{J} = 0, \rho = 0$

$\bar{E} = ?$ $\quad \bar{H} = ?$ $\quad \varepsilon_0 \; \mu_0$

$\bar{E} = \bar{E}(\bar{r})$

$$\nabla \times (\nabla \times \bar{E}) = -j\omega\mu_0 \nabla \times \bar{H} = -j\omega\mu_0 (j\omega\varepsilon_0 \bar{E}) = +\omega^2\mu_0\varepsilon_0 \bar{E}$$

$$= \nabla(\underbrace{\nabla \cdot \bar{E}}_{=0}) - \nabla^2 \bar{E}$$

$$\nabla \cdot \bar{D} = \rho = 0$$

$$\varepsilon_0 \underbrace{\nabla \cdot \bar{E}}_{=0} = 0$$

Helmholtzin yhtälö

$$\nabla^2 \bar{E}(\bar{r}) + \underbrace{\omega^2\mu_0\varepsilon_0}_{k^2} \bar{E}(\bar{r}) = 0$$

$$\bar{E}(\bar{r}) = \bar{u}\, E(z)$$

$$\nabla^2 = \frac{d^2}{dz^2}$$

$$\Rightarrow \quad \bar{u}\underbrace{\left(\frac{d^2}{dz^2} E(z) + k^2 E(z)\right)}_{=0} = 0$$

$$E''(z) + k^2 E(z) = 0$$

$$E(z) = e^{jkz}$$

$$E'(z) = jk\, e^{jkz}$$

$$E''(z) = (jk)^2 e^{jkz} = -k^2 e^{jkz}$$

$$\bar{E}(z) = \bar{u}\left(E_+ e^{-jkz} + E_- e^{+jkz}\right)$$

$$E(z) = e^{-jkz}$$

$$E''(z) = (-jk)^2 e^{-jkz} = -k^2 e^{-jkz}$$

$$Re\left\{e^{-jkz} \cdot e^{j\omega t}\right\} = Re\left\{e^{j(\omega t - kz)}\right\}$$

$$= \cos(\omega t - kz)$$