# Advanced Microeconomics 2

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These lecture notes are written for a research M.Sc. course in microeconomic theory covering welfare economics and competititive markets. They are meant to complement the course textbook 'Microeconomic Theory' by Mas-Colell, Whinston and Green and the material presented in the lectures. Special thanks to Mikael Mäkimattila for comments.

## Introduction

The four parts in the research M.Sc. sequence in microeconomic theory at Helsinki GSE cover: Decision Theory (Part I), Welfare Economics and Competitive Markets (Part II and this course), Game Theory (Part III) and Economics of Information (Part IV). At a very general level, the aim of Part II is to introduce formal models of economies consisting of multiple economic agents. Key concepts for analyzing economies revolve around evaluating economic outcomes (often called allocations), considering various economic institutions (in particular competitive markets) and aggregating individual behavior within the institutions. Part III extends the analysis to cover strategic aspects, Part IV concentrates on models of imperfect and incomplete information.

These lecture notes are organized as follows:

1. Economic Setup
   (a) Modeling Economies: economic agents, preferences, feasible outcomes or allocations.
   (b) Assessing Economic Outcomes: Pareto efficiency, social welfare functions, Arrow's theorem.
2. Institutions and Allocations in Discrete Economic Models
   (a) Assignment
   (b) Matching
3. Competitive Markets
   (a) Exchange Economies
   (b) Economies with Production
4. Competitive Equilibrium Analysis
   (a) Cobb-Douglas Exchange Economies
   (b) Exchange Economies under Uncertainty
   (c) Models of Trade
   (d) Assignment Markets: Housing in General Equilibrium

# 1 Economic Setup

## 1.1 Primitives for Social Choice

Microeconomic Theory I focuses on decision theory, i.e. the choice behavior of a single economic agent. In this course, we consider economic decisions and outcomes for groups of agents.

As the starting point, we take a set $\mathcal{N} = \{1, ..., n\}$ of economic agents and a set of social outcomes or alternatives $\mathcal{A}$. Each outcome contains a complete description of all relevant aspects to all economic agents.

As in Microeconomic Theory I, we assume that each $i \in N$ has a rational (i.e. complete and transitive) preference order on the set of alternatives. We denote the preference relation of agent $i$ by $\succeq_i$. We write $\succ_i$ for the strict part of $\succeq_i$ and $\sim_i$ for the indifference relation induced by $\succeq_i$.

In the first part of the course, we are mainly interested in evaluating different institutions, i.e. ways in which social outcomes are decided. In the second part of the course, we look more carefully at a particular institution, i.e. competitive markets as a means for reaching social outcomes. Microeconomic Theory III and IV adopt a different approach to decision making for groups of agents based on non-cooperative game theory. In those courses, each economic agent has to make an independent choice and the vector of choices determines the social outcome.

The starting point for our analysis is hence a *society*.

**Definition 1.1.** A *society* is a collection $(\mathcal{N}, \mathcal{A}, \{\succeq_i\}_{i=1}^n)$, where

1. $\mathcal{N}$ is a set of agents.

2. $\mathcal{A}$ is a set of social outcomes.

3. For all $i \in \mathcal{N}$, the preference relation $\succeq_i$ is a complete and transitive order on $\mathcal{A}$, i.e.

   i) for all $i$ and all $a, b \in \mathcal{A}$, either $a \succeq_i b$, or $b \succeq_i a$ or both, and

ii) for all $i$ and for all $a, b, c \in \mathcal{A}$,

$(a \succeq_i b) \wedge (b \succeq_i c) \implies a \succeq_i c$.

**Examples**

1. A society consisting of agents $\mathcal{N} = \{1, 2, 3, 4\}$ and houses $\{a, b, c, d\}$. We assume that each house is occupied by a single agent. The outcomes in this model can then be described by a bijective function $m : \mathcal{N} \to \{a, b, c, d\}$, where $m(i) \in \{a, b, c, d\}$ is the house occupied by agent $i$. We say in such cases that the function $m$ is a *matching* of the agents to houses. The set of possible social allocations is then the set $\mathcal{A}$ of all possible matchings, i.e.:

   $$\mathcal{A} = \{m : \mathcal{N} \to \{a, b, c, d\} | \, m \text{ is bijective}\}.$$

   **Exercise:** How many different matchings exist?

   Each agent $i$ has a preference order $\succeq_i$ on $\mathcal{A}$.

   We say that the model has no externalities if for all $i \in \{1, 2, 3, 4\}$ and for all $m, m' \in \mathcal{A}$, we have

   $$m(i) = m'(i) \implies m \sim_i m'.$$

   In this case, individual preferences on which house to occupy are sufficient to determine the individual preferences over outcomes.

2. A society consisting of employers $E = \{e_1, ..., e_n\}$ and workers $W = \{w_1, ...w_m\}$. An outcome is a not necessarily one-to-one function $\mu : W \to E$. The set of outcomes $\mathcal{A}$ is then the set of all functions from $W$ to $E$. All workers $i \in W$ and all employers $j \in E$ have preferences over $\mathcal{A}$.

   We say again that the model has no externalities if for all $i \in W$ and $j \in E$, and all $\mu, \mu' \in \mathcal{A}$:

   $$\mu(i) = \mu'(i) \implies \mu \sim_i \mu', \text{ and}$$

$$\mu^{-1}(j) = \mu'^{-1}(j) \implies \mu \sim_j \mu',$$

where $\mu^{-1}(j) = \{i \in W | \mu(j) = i\}$.

In words, the workers care only about the employer for whom they work and the firm only cares about the set of workers that it employs. If each firm has a single task and $n = m$, then the set of outcomes is the set of possible bijections (or matchings) from $W$ to $E$ as in the previous example. The notable difference is that now both sides of the match have preferences whereas houses in the previous example did not have preferences.

3. A society consisting of $n$ consumers $i \in \{1, ..., n\}$ and a total quantity $\bar{x}_l > 0$ of divisible good $l \in \{1, .., L\}$ to be shared between the consumers. Outcomes are vectors of non-negative consumption bundles that add up to no more than the total resources available:

$$\mathcal{A} = ((x_{11}, ..., x_{nL}), (x_{21}, ..., x_{2L}), ..., (x_{n1}, ..., x_{nL})) =: \boldsymbol{x} \in \mathbb{R}_+^{nL},$$

such that:

$$\sum_{i=1}^{n} x_{il} \leq \bar{x}_l \text{ for all } l.$$

In this case, we call the outcomes allocations. Each consumer $i$ has continuous preferences $\succeq_i$ over the consumption set $\mathbb{R}_+^L$.

We say that the model has no externalities if for all $i$ and all $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{A}$,

$$(x_{i1}, ...x_{iL}) = (x'_{i1}, ..., x'_{iL}) \implies \boldsymbol{x} \sim_i \boldsymbol{x}'.$$

4. Buyers (or consumers) $i \in \{1, .., b\} =: B$ and sellers $j \in \{1, ..., s\} =: S$ producing a homogenous discrete good. An outcome consists of a collection of non-negative integer-valued vectors $\{q(i, j)\}_{i \in B, j \in S}$ interpreted as the number of goods that consumer $i$ buys from seller $j$ and non-negative real vectors $\{p(i, j)\}_{i \in B, j \in S}$ interpreted as the payment that consumer $i$ makes to seller $j$.

Buyer $i$'s preferences are represented by the quasi-linear function

$$u_i\left(\sum_j q(i,j)\right) - \sum_j p(i,j).$$

In words, the consumer gets utility from the number of goods consumed and disutility from her payments to all sellers. Seller $j$'s preferences are represented by

$$\sum_i p(i,j) - c_j\left(\sum_i q(i,j)\right).$$

In words, the seller's preferences are determined by her profit, i.e. the sales revenue net of production costs.

## 1.2 Criteria for social choice

In a society consisting of a single decision maker, deciding how to choose is not that hard. We are given her preference order so it is quite uncontroversial to suggest that choice be consistent with preferences. With multiple members of society, individual preferences may disagree on the ranking of various alternatives. Social Choice Theory is a branch of economics that aims at arriving a rational social preference for any society. Clearly the primitives on which such social preferences may depend are the available options, i.e. the outcomes for the society and the individual preferences over those outcomes.

Following Arrow (1951), the task is to come up with a social preference function $\Phi$ that has the set of all rational preference profiles $\succeq := (\succeq_1, ..., \succeq_n)$ over a finite set of social outcomes $\mathcal{A}$ as its domain. The range of the social preference function is a subset of the set of rational preference rankings over the social alternatives. We write $\Phi(\succeq)$ for the social preference that obtains under $\Phi$ when the individual preferences are given by the profile $(\succeq_1, ..., \succeq_n)$. Often we will write the value of the social preference function as $\succeq^x$ with some superscript $x$ indicating the function operating on the profile of preferences. For example below, $\succeq^D$ indicates the dictatorial social preference on $\mathcal{A}$ induced by the individual preference profile $\succeq$.

**Examples**

1. *Dictatorial rule*

   The easiest social preference function to describe is the *dictatorial social choice function*. Pick any $i^* \in \{1, ..., n\}$ and define the dictatorial social preference function $\succeq^D$ by the following: for all $a, b \in \mathcal{A}$, and for all preference profiles $(\succeq_1, ..., \succeq_n)$,

   $$a \succ_{i^*} b \implies a \succ^D b,$$

   where we write: $a \succ^D b \iff (a \succeq^D b) \wedge \neg(b \succeq^D a)$.

   Notice that it is not necessarily the case that $\succeq^D = \succeq_{i^*}$ since the social preference is left arbitrary between for ranking of $a, b \in \mathcal{A}$ with $a \sim_{i^*} b$.

   Exercise: Show the resulting social preference is a legitimate rational order on the social outcomes for all profiles of rational individual preferences.

2. *Borda rule*

   Denote the set of outcomes by $\mathcal{A} := \{a_1, ..., a_k\}$. For each agent $i$ in the society, and for each alternative $a_j \in \mathcal{A}$, and each preference profile $\succeq$, compute

   $$r(i, j) = \#\{a_{j'} | a_{j'} \succ_i a_j\} + \frac{1}{2}\#\{a_{j'} | a_{j'} \sim_i a_j\},$$

   i.e. the number of alternatives that are better than $a_j$ in agent $i$'s ranking plus half the number of alternatives that are equally good. For each $a_j$, compute $u(a_j) = \sum_i r(i, j)$.

Consider the following binary relation $\succeq^B$ defined on $\mathcal{A}$ by: $a_j \succeq^B a_{j'} \iff u(a_j) \leq u(a_{j'})$. Notice that the binary relation depends obviously on the underlying profile of preferences.

Exercise: Show that $\succeq^B$ is a rational preference for all $\succeq$. The resulting ranking of the alternatives is called the Borda rule.

3. *Majority rule*

   One of the most popular rules for ranking alternatives is the majority rule. Continuing with the notation of the previous example for social outcomes, we let $n(a_j, a_{j'}) = \#\{i | a_j \succeq_i a_{j'}\}$, i.e. is the number of agents that consider $a_j$ at least as good as $a_j'$. Majority rule relation $\succeq^M$ is defined by the following binary relation on $\mathcal{A}$:

$$a_j \succeq^M a_{j'} \iff n(a_j, a_{j'}) \leq n(a_{j'}, a_j).$$

   Unfortunately $\succeq^M$ is not a rational ordering. To see this, consider the most famous (counter)example of social choice theory, the *Condorcet paradox*. Suppose that:

$$a_1 \succ_1 a_2 \succ_1 a_3,$$

   and

$$a_2 \succ_2 a_3 \succ_2 a_1,$$

   and

$$a_3 \succ_3 a_1 \succ_3 a_2.$$

   Then we get by pairwise comparisons of the three distinct pairs of alternatives that:

$$a_1 \succ^M a_2, \text{ and } a_2 \succ^M a_3, \text{ but } a_3 \succ^M a_1.$$

   But this contradicts transitivity of $\succeq^M$.

Here we see two possible social preference functions: the dictatorial one and the one giving rise to Borda rule. Are these functions reasonable and what criteria should one set for preference aggregation. In other words, what are desirable properties for a social preference function? We have already required that an acceptable social preference function outputs a rational preference ordering for any profile of rational preferences in the society. Let's state this as a formal assumption sometimes called the *unrestricted domain* or *universal domain* assumption.

**Assumption 1.1.** The domain of the social preference function $\Phi$ is the set of all rational preference profiles $(\succeq_1, ..., \succeq_n)$ over $\mathcal{A}$. The range of $\Phi$ is a subset of the set of rational preferences on $\mathcal{A}$.

**Definition 1.2.** A social choice function $\Phi$ satisfies *unanimity* if for any preference profile $\underline{\succeq}$ $(\succeq_1, ..., \succeq_n)$ and any pair of social outcomes $a, b \in \mathcal{A}$ such that $a \succ_i b$ for all $i \in \{1, ..., n\}$,

$$a \, \Phi(\underline{\succeq}) \, b \text{ and } \neg(b \, \Phi(\underline{\succeq}) \, a).$$

In words, unanimity just states that the if all agents in the society strictly prefer $a$ to $b$, then the social preference also strictly prefers $a$ to $b$. The requirement of unanimity for social choice functions is one of the least controversial modeling choices made in economics.

**Definition 1.3.** The social choice function $\Phi$ satisfies *independence of irrelevant alternatives* if for any two individual preference profiles $\underline{\succeq} = (\succeq_1, ..., \succeq_n)$ and $\underline{\succeq}' = (\succeq_1', ..., \succeq_n')$, and any social outcomes $a, b \in \mathcal{A}$ such that $a \succeq_i b \iff a \succeq_i' b$ for all $i \in \{1, ..., n\}$ :

$$a \, \Phi(\underline{\succeq}) \, b \iff a \, \Phi(\underline{\succeq}') \, b.$$

Notice that independence of irrelevant alternatives (IIA) is similar in spirit to weak axiom of revealed preference. Societal preferences on $\mathcal{A}$ induce preferences on all subsets of $\mathcal{A}$ and in particular on $\{a, b\}$. Societal preferences over these two alternatives "should" then depend only on how agents in the society rank $a$ versus $b$ and not on their preferences on some infeasible social outcomes $c$. This requirement is not as compelling for societal preferences as it is for individual decision theory. For example Borda rule as defined above violates IIA (can you show this?).

The society is said to have a dictator $i^* \in \mathcal{N}$ if $i^*$'s preferences determine the societal preference in the following sense.

**Definition 1.4.** A social preference function $\Phi$ is *dictatorial* if there is some $i^* \in \{1, ..., n\}$ such that for all $\succeq$, and all $a, b \in \mathcal{A}$,

$$a \succ_{i^*} b \implies (a\,\Phi(\succeq)\,b) \text{ and } \neg(b\,\Phi(\succeq)\,a).$$

Clearly, having a dictatorial rule is not a very desirable situation for the society even though it satisfies unanimity and IIA (can you show this?). With these properties, we have the ingredients for the most important result in Social Choice Theory.

## 1.3 Arrow's Theorem

**Theorem 1.1** (Arrow's Theorem)**.** Suppose that $\mathcal{A}$ has at least three elements and the social preference function $\Phi$ satisfies Assumption 1.1. Then if $\Phi$ satisfies unanimity and independence of irrelevant alternatives, it is dictatorial.

In the proof below, I denote the social preference induced by the profile $(\succeq_1, ..., \succeq_n)$ by $\succeq$ for notational convenience. It should be kept in mind that this preference depends on the underlying profile of preferences. As before, I denote strict social preference by $\succ$. The social preference induced by profile $(\succeq_1', ..., \succeq_n')$ is denoted by $\succeq'$.

*Proof.* We assume that $\Phi$ satisfies 1.1, unanimity and IIA and show that it is dictatorial. The proof is divided into four steps.

STEP 1 Fix an alternative $b$ and consider a profile $(\succeq_1, ..., \succeq_n)$ such that for all $i$, either $a \succ_i b$ or $b \succ_i a$ for all $a \neq b$. Then either $a \succ b$ or $b \succ a$ for all $a \neq b$.

**Remark.** In words, if for all $i$, alternative $b$ is either the uniquely best or uniquely worst alternative, then $b$ is either the uniquely best or uniquely worst alternative in the social ranking (even if, say halt the individuals rank $b$ at the top and half rank it at the bottom).

*Proof.* Assume to the contrary that for some $a, c \neq b$, we have $a \succeq b \succeq c$.

Consider another preference profile $(\succeq'_1, ..., \succeq'_n)$, where $\succeq_i = \succeq'_i$ if $c \succ_i a$. If $a \succeq_i c$, then modify the ranking of alternative $c$ in $\succeq_i$ to construct a new individual preference $\succeq'_i$ by requiring that $c \succ'_i a$ and $a' \succ'_i c$ for all $a' \neq a$ such that $a' \succ_i a$.

(In words, the preference of $i$ is unchanged if $c \succ_i a$, but if $a \succeq_i c$, then alternative $c$ is raised to a position immediately above $a$ (and therefore below $a'$ if $a' \succ_i a$) in the new ranking $\succeq'_i$. This change does not affect the relative ranking of $a, b$ or $b, c$ for any agent since $b$ is by assumption either at the top or at the bottom of each individual ranking.)

Let $\succeq'$ be the social preference generated by the new profile $(\succeq'_1, ..., \succeq'_n)$. By unanimity, $c \succ' a$. Since $a \succeq_i b \iff a \succeq'_i b$ for all $i$ and $a \succeq b$, IIA implies that $a \succeq' b$. Since $b \succeq_i c \iff b \succeq'_i c$ for all $i$ and $b \succeq c$, IIA implies that $b \succeq' c$. By transitivity, $a \succeq' c$ contradicting $c \succ' a$. □

STEP 2 Some individual $i^*$ is pivotal in the sense that depending on $\succeq_{i^*}$, some alternative $b$ is ranked either at the top or at the bottom of the social preference order $\succeq$ for some preference profile of other agents.

*Proof.* Suppose $b$ is ranked uniquely at the bottom for all $i$ at some fixed preference profile. Then by unanimity $b$ is ranked uniquely at the bottom for the social preference $\succeq$.

Consider alternative profiles indexed by $k$ where for agents $i \in \{1, ..., k\}$, $b$ is moved to the top of their preference (but otherwise their preferences are unchanged), and the preferences of agents $i \in \{k+1, ..., n\}$ (for $k < n$) are unchanged.

By the previous step, the social preference ranks $b$ uniquely at the top or at the bottom of all alternatives. Set $i^*$ to be the smallest $k$ such that the social preference ranks $b$ at the top. Such $i^*$ exists since unanimity implies that for $k = n$, the social preference ranks $b$ uniquely at the top. □

Denote the profile preference profile in the previous step for $k = i^* - 1$ by I and the profile for $k = i^*$ by II. the social preference then ranks $b$ uniquely at the bottom in I and uniquely at the top in II.

STEP 3 For all $a, c \neq b$, we have $a \succ c$ if $a \succ_{i^*} c$.

*Proof.* Construct profile III from II by changing outcome $a$ to the top in the ranking of $i^*$ so that $a \succ_{i^*} b \succ_{i^*} c$. Let all other agents $i \neq i^*$ have otherwise arbitrary preferences, but $b$ remains at the extreme position as in II. By IIA, $a \succ b$ at profile III since

$$a \succeq_i b \text{ at profile III} \iff a \succeq_i b \text{ at profile I} .$$

Similarly by IIA, $b \succ c$ at profile III since

$$b \succeq_i c \text{ at profile III} \iff b \succeq_i c \text{ at profile II} .$$

By transitivity, $a \succ c$. By independence of irrelevant alternatives, $a \succ c$ if $a \succ_{i^*} c$. □

STEP 4 For all $a$, we have $a \succ b$ if $a \succ_{i^*} b$ and $b \succ a$ if $b \succ_{i^*} a$.

*Proof.* Consider any profile where $a \succ_{i^*} b$. Take an arbitrary outcome $c$ and modify $i^*$'s preference (if necessary) so that $a \succ_{i^*} c \succ_{i^*} b$ and so that for the other agents, $c$ is ranked at the top. At the new profile, $c \succ b$ by unanimity.

By the previous step, we know that $a \succ_{i^*} c \implies a \succ c$. Hence by transitivity, $a \succ b$ at the new profile. Since all agents rank $a, b$ in the same way in the two profiles, we conclude by IIA that $a \succ b$ at the original profile.

The case where $b \succ_{i^*} a$ is handled similarly. □

**Remark.**

1. The proof is not terribly long, but it is not trivial either. You may want to consult Geanakoplos (2005) for different ways of proving the result.

2. Even though Arrow's Theorem has a negative message, some reasonable ways for aggregating individual preferences exist. Borda rule is often reasonable even though if fails IIA.

3. Unrestricted domain is also a strong requirement. We say that individual preferences $\succeq_i$ are *single peaked* on $\mathcal{A} \subset \mathbb{R}$ if for all $x, y, z \in \mathcal{A}$ such that $x > y > z$, either $y \succ_i x$ or $y \succ_i z$ or both. If all agents have single peaked preferences and anti-symmetric preferences, then majority rule defined in Example 3 above produces a complete and transitive social ranking. This result goes under the name of Median Voter Theorem and it is due to Black (1948).

4. If one has more information on cardinal utilities of the agent, then much more can be done. Ia a world with quasilinear preferences, the strength of individual preferences can be quantified in terms of money. If this (or other cardinal information on utilities) is available, then much more can be done.

5. A separate issue concerns the incentives that individuals have for reporting their preferences. If individual preferences are used in social decision making, then it may well be in the agents' best interest to report their preferences strategically. This issue is taken up in Microeconomic Theory III, where Gibbard-Satterthwaite Theorem plays the role of Arrow's Theorem in showing that the only non-trivial social decision processes that do not give rise to strategic manipulation are dictatorial ones (if there are three or more alternatives).

## 1.4 Pareto-Efficiency

**Definition 1.5** (Pareto-Efficiency)**.** Given a society with a preference profile $\succeq$ over social outcomes $\mathcal{A}$, an outcome $a$ *Pareto-dominates* $b$ if $a \succeq_i b$ for all $i \in \{1, ..., n\}$ and $a \succ_i b$ for some $i$. Outcome $a$ *strictly Pareto-dominates* $b$ if $a \succ_i b$ for all $i$. Outcome $a$ is said to be *Pareto-efficient* is there is no $b \in \mathcal{A}$ that Pareto-dominates $a$.

Pareto-domination induces an order $\succeq^P$ on $\mathcal{A}$: $a \succeq^P b$ iff $a$ Pareto-dominates $b$. It should be clear that this order is transitive (since individual preferences are transitive) but it is far from complete.

Nevertheless, we can show that the set of Pareto-efficient outcomes is non-empty whenever the set of outcomes is finite or the set is compact and all individual preferences are continuous.

**Definition 1.6** (Serial Dictatorship)**.** *Serial Dictatorship* is defined as follows: Let agent 1 choose her set of most preferred alternatives $A_1 \in \mathcal{A}$. By the results in Microeconomic Theory 1, this set is non-empty and in the case with a compact $\mathcal{A}$, it is also compact. Let agent 2 choose her set of most preferred alternatives $A_2 \subset A_1$. Continue iteratively until the last agent the process so that agent $i$ chooses her most preferred outcomes in $A_{i-1}$ for all $i$. The set $A_n$ is the outcome of the serial dictatorship.

**Proposition 1.1.** The outcome of serial dictatorship is Pareto-efficient.

*Proof.* Left as an exercise. □

In the above construction, $A_n$ clearly depends on the order in which the agents make their choices. Can you show with a simple example that there are some Pareto-efficient outcomes that are not in $A_n$ for any ordering of the agents?

Suppose now that we have a family of utility functions $u_i$ where each $u_i$ represents agent $i$'s preferences $\succeq_i$. We can then associate with each social outcome $a \in \mathcal{A}$, an $n$-dimensional real vector $u(a) = (u_1(a), ..., u_n(a))$. An outcome $a \in \mathcal{A}$ is then Pareto-efficient if and only if there is no $b \in \mathcal{A}$ such that $u_i(b) \geq u_i(a)$ for all $i$ and $u_i(b) > u_i(a)$ for some $i$. This gives a nice geometric interpretation to the set of Pareto-efficient points also often called the *Pareto-frontier*.

Consider now a strictly increasing function $W : \mathbb{R}^n \to \mathbb{R}$ and the problem:

$$\max_{a \in \mathcal{A}} W(u(a)). \tag{1}$$

**Proposition 1.2.** If $a^*$ solves Problem 1, then $a^*$ is Pareto-efficient.

*Proof.* The claim is proved by contrapositive. If $a^*$ is not Pareto efficient, then there is another $b \in \mathcal{A}$, such that $u_i(b) \geq u_i(a^*)$ for all $i$ and $u_i(b) > u_i(a^*)$ for some $i$. Since $W$ is a strictly increasing function, $W(u(b)) > W(u(a^*))$ so $a^*$ is not a solution to Problem 1. □

The converse of this Proposition is also true, but since it is not practical to work with the set of all (possibly quite complicated) strictly increasing functions $W$, it would be good if the converse (or at least something close to that) would be true for simple $W$. A linear $W$ would certainly be simple to handle. An application of the separating hyperplane theorem can be used to prove the converse for the case where the set $F = \{v \in \mathbb{R}^n | v \leq u(a) \text{ for some } a \in \mathcal{A}\}$ is convex.

**Proposition 1.3.** If $F$ is convex and $a^*$ is Pareto-efficient, then there is a $\lambda = (\lambda_1, ..., \lambda_n) \neq 0$ with $\lambda_i \geq 0$ for all $i$, such that $a^*$ solves

$$\max_{a \in \mathcal{A}} \sum_{i=1}^{n} \lambda_i u_i(a).$$

*Proof.* Let $P := \{v \in \mathbb{R}^n | v \geq u(a^*)\}$. Then $F$ and $P$ are convex sets whose intersection has an empty interior. Separating hyperplane theorem guarantees the existence of a vector $\lambda \in \mathbb{R}^n$ and a real number $\gamma$ such that $\lambda \cdot v \leq \gamma$ for all $v \in F$ and $\lambda \cdot v \geq \gamma$ for all $v \in P$. Since $u(a^*) \in F \cap P$, we conclude that $\lambda \cdot u(a^*) = \gamma \geq \lambda \cdot v$ for all $v \in F$. Furthermore, $\lambda_i \geq 0$ since $P$ includes points $u(a^*) + M\boldsymbol{e}^i$ for all positive $M$, where $\boldsymbol{e}^i$ is the $i^{th}$ unit vector. □

**Remark.**

1. F is convex if $\mathcal{A}$ is a convex set and $u_i$ is concave for all $i$.
2. If $u_i(a)$ is the Bernoulli utility function of agent $i$ for all $i$, then the set $F$ of all utility vectors for the corresponding von-Neumann - Morgenstern expected utility functions over lotteries on the outcomes is convex.

One social utility function that has attracted some attention is the Rawlsian function $w(a) := \min_{i \in \{1,\dots,n\}}\{u_i(a)\}$. The maximizers of $w(a)$ need not be Pareto-efficient, but can you find a modification for the Rawlsian function so that its maximizers are Pareto-efficient outcomes?

# 2 Institutions and Allocations in Discrete Economies

This section is a first introduction to the use of welfare economic analysis in market contexts. The first subsection gives the simplest possible example for discussing different allocation methods in a society consisting of multiple agents. It is meant to illustrate the general methodology rather than represent an important real-life market. The second subsection provides a more elaborate model that has been applied in practice to important allocation problems. In a first-year course we cannot unfortunately go very deep into the applications or extensions of the model, but I hope you get a sense of the type of research done in the relatively new paradigm of market design.

## 2.1 Assignment

We specialize the problem of choosing social outcomes to that of finding feasible housing arrangements for the agents $\mathcal{N} := \{1, ..., n\}$. The members of the society have access to a set of houses $\mathcal{H} := \{1, ..., h\}$ and the number of houses is assumed to be at least as large as the number of agents.

### 2.1.1 Allocations and efficiency

We assume that all houses are single occupancy and therefore feasible outcomes are one-to-one functions from $m : \mathcal{N} \to \mathcal{H}$. We call such functions allocations. An allocation is then identified with a vector $(m(1), ..., m(n))$ where $m(i) \in H$ denotes the house assigned to $i \in \mathcal{N}$. Hence the name assignment model.

We also assume that the housing decisions impose no externalities on occupants of other houses so that the preferences of $i \in \mathcal{N}$ are over the set of houses and hence the preferences $\succeq_i$ of $i$ over outcomes are determined by the house $m(i)$ assigned to $i$ in allocation $m$. With this in mind, we define an assignment society without externalities directly in terms of individual preferences over houses.

**Definition 2.1.** A *society without externalities* is a collection $(\mathcal{N}, \mathcal{H}, \{\succeq_i\}_{i \in \mathcal{N}})$ of agents $i \in \mathcal{N} := \{1, ..., n\}$, houses $\mathcal{H} = \{1, ..., h\}$, where the number of houses is at least as large as the number of agents, and an individual rational preference relation for each $i$ over $\mathcal{H}$. An *allocation* is a one-to-one function $m : \mathcal{N} \rightarrow \mathcal{H}$.

**Example 2.1.** Consider a society with four agents $\mathcal{N} = \{1, 2, 3, 4\}$ and five houses $\mathcal{H} = \{a, b, c, d, e\}$. The individual preferences are given in the following table where agents represent the columns and the houses are ranked in the descending order of preference within columns.

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| b | a | ⓑ | d |
| ⓒ | c | e | ⓔ |
| e | ⓓ | a | a |
| a | b | c | b |
| d | e | d | c |

Figure 1: The allocation is represented by the circled elements in the table.

**Definition 2.2.** An allocation $m(1), ..., m(N)$ is *Pareto-efficient* if there is no other allocation $m'$ such that $m(i) \succeq_i m'(i)$ for all $i$ and $m(i) \succ_i m'(i)$ for some $i$.

The allocation depicted in Figure 1 is not Pareto-efficient. House $a$ is not occupied in that allocation, but agent 2 ranks $a$ the highest. Hence $m' = (c, a, b, e)$ Pareto-dominates $m = (c, d, b, e)$. But $m'' = (c, a, b, d)$ Pareto-dominates $m'$. You should verify that $m''$ is Pareto-efficient. An allocation can be Pareto-efficient only if all agents (weakly) prefer their assigned house to all unoccupied houses. In $m''$ the only unoccupied house is $e$. Can you find another Pareto-efficient allocation where some other house is left unoccupied?

A useful observation on the set of Pareto-efficient allocations is the following: some agent is assigned her favorite house. Let $h^*(i)$ denote any house that is at the top of agent $i$'s ranking.

**Proposition 2.1.** If $m$ is a Pareto-efficient allocation, then for all $i \in \mathcal{N}$, there is a $j \in \mathcal{N}$ such that $m(j) = h^*(i)$ and for some $i^* \in \mathcal{N}$, $m(i^*) = h^*(i)$.

*Proof.* i) If $h^*(i)$ is unoccupied for some $i$ in allocation $m$, then $m$ is not Pareto-efficient.

ii) Suppose $m$ is Pareto-efficient and $m(i) \neq h^*(i)$ for all $i \in \mathcal{N}$. Consider the agents in an arbitrary order $i_1, i_2, ..., i_n$. Construct a chain $i_k \to i_{k+1}$ for all $k$ by requiring $m(i_{k+1}) = h^*(i_k)$ so that $i_{k+1}$ occupies the favorite house of $i_k$. Since the favorite house of all agents is occupied by some agent by part i), there must be a $k^*$ such that $i_{k^*+1} = i_l$ for some $l \leq k^*$. Let $m'$ be the allocation where $m'(i_k) = m(i_{k+1})$ for $l \leq k \leq k^*$, and $m'(i_k) = m(i_k)$ otherwise. Then $m'$ Pareto-dominates $m$ and the claim is proved. □

### 2.1.2 Property Rights and Market Equilibrium

For this subsection, we assume that the starting point in the society is that the houses are initially owned by the agents. The key difference to the previous discussion of Pareto-efficiency is that we now give the agents property rights to their houses. They can stay in their own house if they so decide.

An allocation in this context is a bijection from the agents to the set of houses (all houses in this section are initially occupied by some agent). We call the allocations also *matchings* We denote the *initial allocation* in the society by $\boldsymbol{e} = (e(1), ..., e(n))$, and $\mathcal{H} = \{e(i)\}_{i \in \mathcal{N}}$.

**Definition 2.3.** An *economy* is a society without externalities together with an initial allocation $\boldsymbol{e}$ denoted by $(\mathcal{N}, \mathcal{H}, \{\succeq_i\}_{i \in \mathcal{N}}, \boldsymbol{e})$ .

We are interested in allowing the agents in our economy to trade. You should notice that this is somewhat weird trading since there is no money or any other good that could be exchanged for the houses. We will take up trading with a richer set of trade-offs in sections 3 and 4 of these notes.

Nevertheless, it is instructive to see how to construct a market with prices for this very simple setup. Towards this, we assign a (positive) real number $p(h)$ to each house and interpret it as the price of the house. Agent $i$ occupies initially house $e(i)$ so we determine her budget as $p(e(i))$.

The idea is to construct a market equilibrium for the economy where all agents choose the best house that they can afford. In other words, each $i$ chooses the best house in $\{h' \in \mathcal{H}|\ p(h') \leq p(e(i))\}$.

**Definition 2.4.** A *market equilibrium* of the economy $(\mathcal{N}, \mathcal{H}, \{\succeq_i\}_{i \in \mathcal{N}}, \boldsymbol{e})$ is a house price vector $\boldsymbol{p}$ and a vector of housing demands $\boldsymbol{a} = (a(1), ..., a(n))$ with $a(i) \in \mathcal{H}$ for all $i$ such that

i) For all $i$, $a(i) \succeq_i h'$ for all $h'$ such that $p(h') \leq p(e(i))$

ii) $\boldsymbol{a}$ is an allocation (i.e. the vector of optimal demands is a matching).

Notice the structure of this definition. An equilibrium is a price and an vector of demands with the requirement that the demands are optimal within the feasible set given the prices and markets clear (in this case, this implies that the demand vectors form an allocation represented by a matching). The agents are not required to know anything about other agents' preferences or total resources in the society. It is enough that they know their own preferences and their budget set. Of course, there is no explanation of how an equilibrium might arise. Equilibrium prices depend on the individual preferences. But if individual preferences are not known to others, how can prices depend on preferences. Maybe there is a mechanism that asks individuals about their preferences? But then the issue of manipulability arises. These issues are treated (to a very limited extent) in Parts III and IV of the Microeconomic Theory sequence.

To start the analysis, we discuss how to move between two allocations.

**Definition 2.5.** From an arbitrary initial allocation, we define a *trading cycle* to be an ordered set of distinct agents $(i_1, ..., i_k)$ with the interpretation that the agents trade their houses in such a way that $i_l$ gets the house of $i_{l+1}$ for $l < k$, and $i_k$ gets the house of $i_1$.

**Definition 2.6.** A *trading partition* is a collection of $t$ trading cycles such that each agent belong to exactly one trading cycle. We say that a trading partition $\{(i_1^1, ..., i_{k_1}^1), ..., (i_1^t, ...i_{k_t}^t)\}$ transforms allocation $m$ to allocation $m'$ if for each $j \in \{1, ..., t\}$,

$$m'(i_l^j) = m(i_{l+1}^j) \text{ for all } l \in \{1, ..., k_{j-1}\} \text{ and}$$

$$m'(i_{k_j}^j) = m(i_1^j).$$

Since this is quite a complicated definition, lets see what trading partitions do in examples.

**Example 2.2.** Start with $m = (a, b, c, d, e)$. The trading partition $\{(1, 3, 4), (2, 5)\}$ transforms $m$ to $m' = (c, e, d, a, b)\}$. The trading partition $\{(1, 3, 5, 4)(2)\}$ transforms $m'$ to $m'' = (d, e, b, c, a)$.

**Example 2.3.** Consider two allocations $m = (b, c, e, d, a)$ and $m' = (a, c, d, e, b)$. Since agent 1 gets the house of agent 5, agent 5 gets the house of agent 1, agent 2 keeps her house and agents 3 and 4 swap houses to get from $m$ to $m'$, we see that $\{(1, 5), (2), (3, 4)\}$ transforms $m$ to $m'$.

The cycle decomposition theorem for permutations guarantees that for any two allocations $m$ and $m'$, there is a unique trading partition transforming $m$ to $m'$. (Sketch of a proof: Pick an arbitrary $i_1$. If $m'(i_1) = m(i_1)$, add $\{(i_1)\}$ to the trading partition $T$. If not, take $i_2$ to be defined by $m(i_2) = m'(i_1)$ and add $(i_1, i_2)$ to $T$ if $m'(i_2) = m(i_1)$. If not, define $i_3$ by $m(i_3) = m'(i_2)$ etc until agent $i_k$ such that $m'(i_k) = m(i_1)$. Since $n$ is finite, such an $i_k$ must exist (why can't we have $m'(i_k) = m(i_l)$ for some $1 < l < k$?). Then add $(i_1, ..., i_k)$ to T. Restart the process with the set $\mathcal{N} \setminus \{i_1, ..., i_k\}$ to find the next trading cycle and repeat until no agents remain.)

**Proposition 2.2.** Let $(\boldsymbol{p}, \boldsymbol{a})$ be a market equilibrium for the economy $(\mathcal{N}, \mathcal{H}, \{\succeq_i\}_{i\in\mathcal{N}}, \boldsymbol{e})$ and $T$ the trading partition transforming $\boldsymbol{e}$ to $\boldsymbol{a}$. Then the prices of all houses in any trading cycle in $T$ are equal.

*Proof.* Let $(i_1, ..., i_k)$ be a trading cycle of $T$. Then $a(i_l) = e(i_{l+1})$ for $l < k$, and $a(i_k) = e(i_1)$.This means for $l < k$ that $e(i_{l+1})$ must be in the budget set of $i_l$ or $p(e(_{l+1}) \leq p(e_l)$ and similarly $p(e_1) \leq p(e(i_k))$. But then all the prices must be equal. □

**Definition 2.7.** A trading cycle $(i_1, ..., i_k)$ is a *top trading cycle* if we set $i_{k+1} = i_1$ and we have for all $l \leq k$, $e(i_{l+1}) \succeq_{i_l} h'$ for all $h' \in \mathcal{H}$

**Proposition 2.3.** Every economy $(\mathcal{N}, \mathcal{H}, \{\succeq_i\}_{i\in\mathcal{N}}, \boldsymbol{e})$ has a top trading cycle.

*Proof.* Start with an arbitrary $i_1$. Ask $i_1$ to point at the occupants of her favorite house. If she points at herself, then $(i_1)$ is a trivial top trading cycle. Otherwise, let $i_2$ be a person that $i_1$ points at. Ask $i_2$ to point at the occupants of her favorite house. If she points at herself, there is the trivial trading cycle $(i_2)$. If she points at $i_1$, then $(i_1, i_2)$ is a top trading cycle. Otherwise, let $i_3$ be any agent that $i_2$ points. Continue inductively until some $i_k$ points at some $i_l$ with $l \leq k$. Such a $k$ must exist since $(i_1, ..., i_{n-1})$ are all distinct and $i_n$ must point at herself or some other agent. By construction, $(i_l, i_{l+1}, ..., i_k)$ is a top trading cycle. □

**Exercise:** Show that an economy can have many top trading cycles.

We are now in a position to prove the existence of a market equilibrium and also to demonstrate some of its properties.

**Theorem 2.1** (Existence of a Market Equilibrium)**.** Every economy has a market equilibrium.

*Proof.* By Proposition 2.3, $(\mathcal{N}, \mathcal{H}, \{\succeq_i\}_{i\in\mathcal{N}}, \boldsymbol{e})$ has a top trading cycle $(i_1, ..., i_k)$. Assign each of the agents in this cycle their favorite house and attach the same price $p_1 = p(e(i_l))$ for $l \leq k$. Consider a new economy consisting of $\mathcal{N}_1 := \mathcal{N} \setminus \{i_1, ..., i_k\}$, houses $H_1 := \mathcal{H} \setminus \cup_{l=1}^{k}\{e(i_l)\}$, preferences of $i \in \mathcal{N}_1$

on $H_1$ induced by the original preference, and initial allocation $(e(i))_{i \in \mathcal{N}_1}$. By Proposition 2.3, this new economy has a top trading cycle. Assign the houses to the agents according to the trading cycle and set price $p_2 < p_1$ to all houses in this second cycle. Remove the agents and the houses in the cycle to arrive at a smaller sub-economy. Continue the house assignment and price setting according to the top trading cycles recursively until no agents are left (the process ends in at most $n$ steps). This process arrives at an allocation of houses to agents and a price vector such that the assigned house is by construction at least as good as any of the houses in the agent's budget set. □

It is a good exercise to show that if the agents have strict preferences (no ties), then the equilibrium allocation is unique. Equilibrium prices are obviously not pinned down since only the ordinal prices matter. You should find an example to show that the ordinal ranking of house prices can also differ across equilibria.

The next two theorems relate equilibrium allocations to Pareto-efficient allocations when preferences are strict.

**Theorem 2.2** (First Welfare Theorem)**.** If the agents have strict preferences over houses, then every market equilibrium allocation is Pareto-efficient.

*Proof.* Let $(\boldsymbol{p}, \boldsymbol{a})$ be a market equilibrium of the economy $(\mathcal{N}, \mathcal{H}, \{\succeq_i\}_{i \in \mathcal{N}}, \boldsymbol{e})$. If $\boldsymbol{a}'$ is an allocation that Pareto-dominates $\boldsymbol{a}$, then $a'(i) \succeq_i a(i)$ for all $i$ and $a'(i) \succ_i a(i)$ for some $i$. But $a'(i) \succ_i a(i) \implies p(a'(i)) > p(a(i))$ since otherwise $a'(i)$ would be budget feasible. If $a'(i) \sim_i a(i)$, strict preferences imply that $a'(i) = a(i)$ and thus $p(a'(i)) = p(a(i))$. By summing over the agents

$$\sum_{i=1}^{n} p(a'(i)) > \sum_{i=1}^{n} p(a(i)).$$

But this is not possible if both $\boldsymbol{a}$ and $\boldsymbol{a}'$ are allocations. □

**Exercise:** Find an example of an economy and a market equilibrium that is not Pareto-efficient if preferences are not strict. Do all economies have a Pareto-efficient equilibrium if preferences are not strict?

We can also prove a converse result to the first welfare theorem showing that Pareto-efficient allocations can be supported as market equilibria.

**Theorem 2.3** (Second Welfare Theorem)**.** Suppose $\boldsymbol{a}$ is Pareto-efficient for $(\mathcal{N}, \mathcal{H}, \{\succeq_i\}_{i \in \mathcal{N}})$ and the agents have strict preferences. Then in all market equilibria $(\boldsymbol{p}, \boldsymbol{a}')$ of $(\mathcal{N}, \mathcal{H}, \{\succeq_i\}_{i \in \mathcal{N}}, \boldsymbol{a})$, we have $\boldsymbol{a} = \boldsymbol{a}'$.

*Proof.* If $\boldsymbol{a}'$ is a market equilibrium allocation, $a'(i) \succeq_i a(i)$ for all $i$ (since initial endowment is in the budget set for all $\boldsymbol{p}$). If $\boldsymbol{a}' \neq \boldsymbol{a}$, then $a'(i) \succ_i a(i)$ for some $i$ and since $\boldsymbol{a}'$ is an allocation, this contradicts the Pareto-efficiency of $\boldsymbol{a}$. □

These two welfare theorems are sometimes interpreted as showing that the market mechanism is wonderful. It is not clear to me why this would be so. The next subsection tries to make the point that many economic institutions can have welfare theorems of the above type.

In Subsection 4.4, we shall discuss how the situation changes if there is another good, money, that the agents also like. If house prices are monetary so that buying a cheaper house leaves the agent with more money, then the strength of preference becomes measurable in cardinal monetary terms. This will have implications for the efficient allocations and for the associated equilibria.

### 2.1.3 Power and the Jungle

Suppose that the agents in the society differ in terms of their power, i.e. strength, ability to influence etc. Order the agents by descending power so that $i_1$ is the most powerful agent and $i_k$ is more powerful than $i_l$ whenever $l > k$ (we assume no ties for convenience). The consequence of power for allocations is the following; a more powerful agent wins any struggle against a less powerful one and therefore a more powerful agent can forcefully take over any house assigned to a weaker agent. Let $\triangleright$ denote the complete, transitive and asymmetric binary relation on $\mathcal{N}$, where $i \triangleright j$ means that $i$ is more powerful than $j$. I will follow the colorful language of Ariel Rubinstein and Michele Piccione for the following definition.

**Definition 2.8.** A *jungle* is a society without externalities together with a power relation $\triangleright$.

If the number of houses coincides with the number of agents, an equilibrium for the jungle can be defined as follows:

**Definition 2.9.** A *jungle equilibrium* of the jungle $(\mathcal{N}, \mathcal{H}, \{\succeq_i\}_{i \in \mathcal{N}}, \triangleright)$ is an allocation $m$ such that $i \triangleright j \implies m(i) \succeq_i m(j)$.

In words, an equilibrium is an allocation where no agent wants to exert her power to claim the house of a less powerful agent. If there are more houses than agents, the same definition goes through if we add dummy agents that have the least power and that are indifferent between any houses. We are ready for the first existence and welfare theorems of this course.

**Theorem 2.4.** Every jungle has a jungle equilibrium. If the agents' preferences are strict, then the equilibrium is unique.

*Proof.* Recalling the serial dictatorship from Section 1, let $i_k$ be the $k^{th}$ most powerful agent for $k \in \{1, ..., n\}$. Denote one of the best houses in $i_1$'s ranking by $h_1$. Assign recursively a house $h_k$ that is best in $i_k$'s ranking of the houses $H_k := \mathcal{H} \setminus \{h_1, ..., h_{k-1}\}$. Since $h_l \in H_k$ for $k < l$, we conclude that $h_{i_l} \succeq_{i_l} h_{i_k}$ for all $i_l \triangleright i_k$. The uniqueness with strict preferences over houses is immediate. □

For the rest of this section, we assume that preferences over houses are strict for all agents.

**Theorem 2.5** (First Jungle Welfare Theorem)**.** With strict preferences over houses, all jungle equilibria are Pareto-efficient.

*Proof.* Let $m^*$ denote the jungle equilibrium allocation constructed by the serial dictatorship induced by $\triangleright$. Let $m$ be another allocation that Pareto dominates $m^*$. Let $i_k$ be the first agent according to $\triangleright$ such that $m(i_k) \succ_{i_k} m^*(i_k)$. Then for all $l < k$, $m(i_l) = m^*(i_l)$ since there are no ties in preferences. But then $m(i_k) \in H_k$ contradicting that $m^*(i_k)$ is the best choice for $i_k$ in $H_k$. □

**Theorem 2.6** (Second Jungle Welfare Theorem)**.** Every Pareto-efficient Allocation is a jungle equilibrium allocation for some power relation $\triangleright$.

*Proof.* If $m$ is Pareto-efficient, then by Proposition 2.1, for some $i_1 \in \mathcal{N}$, $m(i_1) = h^*(i_1)$. Give $i_1$ the highest ranking in $\triangleright$ and consider a society $S_1$, consisting of agents $\mathcal{N}_1 = \mathcal{N} \setminus \{i_1\}$ and houses $H_1 = \mathcal{H} \setminus \{h^*(i_1)\}$. Since $m$ is Pareto-efficient for the original society, $(m(i))_{i \in \mathcal{N}_1}$ is Pareto-efficient for $S_1$. Again by Proposition 2.1, there is an agent $i_2 \in \mathcal{N}_1$ such that $m(i_2)$ is a highest ranked house for $i_2$ in $H_1$. Put $i_2$ at the second highest rank of $\triangleright$. Define recursively for $1 \leq k \leq n-1$, $\mathcal{N}_{k+1} = \mathcal{N}_k \setminus \{i_k\}$, and $H_{k+1} = H_k \setminus \{h^*_k(i_k)\}$, where $h^*_k(i)$ denotes the highest ranked house for $i$ in $H_k$. Put $i_k \triangleright i_{k+1}$. By construction, $m$ is a jungle equilibrium for $(\mathcal{N}, H, \{\succeq_i\}_{i \in \mathcal{N}}, \triangleright)$. □

1. The main reason for including this subsection is to familiarize you with the fundamental concepts (Pareto-efficiency, equilibrium, etc.) in a simple context.

2. In the area of market design, assignment models, matching models of the next subsection, and the concepts arising in these (e.g. top trading cycles) play a key role. For a nice polemical article on Market Design, see Kominers (2017).

3. If individual preferences depend on the entire allocation (preference on neighbors on top of preference over own house), then equilibria may fail to exist and they are not Pareto-efficient in general. Externalities are discussed further in Advanced Microeconomics: Game Theory.

## 2.2 Matching

### 2.2.1 Setup

Two finite populations $X$ and $Y$ of equal size need to be matched in pairs. Each $x \in X$ has rational preferences $\succeq_x$ over $Y$ as match partners and similarly each $y \in Y$ has rational preferences $\succeq_y$ over $X$. Assume for simplicity that all preferences are strict. Examples are abundant: i) workers and tasks (e.g. medical students and residencies), ii) pilot and copilot, iii) marriage market. We define formally:

**Definition 2.10.** A *society* is a collection $(X, Y, \{\succeq_x\}_{x\in X}, \{\succeq_y\}_{y\in Y})$. A *matching* $\mu \in M$ for $(X, Y, \{\succeq_x\}_{x\in X}, \{\succeq_y\}_{y\in Y})$ is a bijection from $X$ to $Y$. For each $x \in X$, we call $(x, \mu(x))$ a *match*. A *matching method* is a function that assigns a matching to each preference profile of the society.

**Example 2.4.** Recall the serial dictatorship from Section 1 and fix any predetermined order on $X$ Let the members in $x$ choose their match according to this order amongst the $Y$ that were not previously chosen. With strict preferences, this produces a match so that serial dictatorship is a matching method.

Let $u(x, y)$ be the rank of $y$ in $x$'s preference order (i.e. the number of alternatives better than $y$ recalling that we assume strict preferences). Similarly let $v(y, x)$ be the rank of $x$ in $y$'s order.

**Example 2.5.** Let $g(u(x, y), v(y, x))$ be a strictly increasing function of its two arguments. Then choosing $\mu \in \arg\min_{\mu\in M} \sum_x g(u(x, \mu(x)), v(\mu(x), x))$ and selecting according to serial dictatorship among the matchings if there are multiple solutions produces a matching for all preference profiles. Hence this procedure is a matching method.

We could of course get more structure if we took cardinal representations of the preferences. For example, one could assume quasilinear preferences over match and money and maximize the surplus from the match.

The rapidly growing literature on Optimal Transport takes this route. Note that the optimization step is far from trivial here.

**Example 2.6** (The Greedy Algorithm)**.** Continuing with the previous example, at first step, choose $(x, y) \in \arg\min_{(x,y)\in X\times Y} g(u(x, y), v(y, x))$ (with multiple minimizers, choose in the order of a pre-determined order on $X$). Remove this pair from $X \times Y$ and continue recursively until all $x \in X$ are matched.

Pareto-efficiency of matchings is defined in the usual way.

**Definition 2.11.** A matching $\mu \in M$ is Pareto efficient if there is no other $\tilde{\mu} \in M$ such that $\tilde{\mu}(x) \succeq_x \mu(x), \tilde{\mu}^{-1}(y) \succeq_y \mu^{-1}(y)$ for all $x \in X, y \in Y$, and for some $x$ or some $y$, $\tilde{\mu}(x) \succ_x \mu(x)$ or $\tilde{\mu}^{-1}(y) \succeq_y \mu^{-1}(y)$.

**Exercise:** Which of the matching methods result in Pareto-efficient matchings for all strict preference profiles?

If the match partners have autonomy on agreeing to a match, it seems reasonable to think that a matching $\mu$, where $y \succ_x \mu(x)$ and $x \succ_y \mu^{-1}(y)$ would not be stable because $x$ would have an incentive to approach $y$ and suggest a pairing of $(x, y)$.

**Definition 2.12.** A matching $\mu \in M$ is *pairwise stable* if $y \succ_x \mu(x) \implies \mu^{-1}(y) \succ_y x$.

**Exercise:** Construct an example showing that serial dictatorship does not necessarily produce a pairwise stable matching.
**Exercise:** Show that every pairwise stable matching is Pareto-efficient.

### 2.2.2 The Gale-Shapley Algorithm

An extremely widely used matching method is the Gale-Shapley algorithm also known as the deferred acceptance algorithm. In this method, agents on one side of the market (without loss of generality consider $x \in X$) make offers to the other side.

In the first stage each $x$ makes an offer to the highest ranked $y$ (according to $\succeq_x$). If all $y$ receive one offer, the algorithm ends and each $x$ is matched with the $y$ that got the offer. All $y \in Y$ that receive multiple offers accepts tentatively the one they rank the highest. All other offers are rejected at the end of the first stage.

At the beginning of each stage after the first, each $y \in Y$ holds at most one offer and during the stage she may receive new ones. All $x$ that are not tentatively matched send a new offer to the highest ranked $y \in Y$ that she has not sent an offer in previous periods. At the end of the stage, each $y$ is tentatively matched to her best offer and rejects the others.

The algorithm stops after the first stage where no offers are rejected, i.e. when all $y$ have exactly one offer, and all $y \in Y$ are matched with the agents whose offer they hold.

To show that this algorithm is a matching method, we need to show that the algorithm stops after finitely many stages in a well-defined match.

More formally, the algorithm is defined as follows:

1. At the start of stage 1:
   (a) Each $x \in X$ makes an offer to her 1st choice.
   (b) Any $y \in Y$ tentatively accepts (or keeps) the best offer and rejects the others (deferred acceptance).
2. At stage $k$,
   (a) Any $x \in X$ rejected at step $k-1$ makes a new offer to its most preferred $y$ that has not rejected $x$ in any prior stage.
   (b) Every $y \in Y$ tentatively accepts her most preferred acceptable offer up to (and including) stage $k$, and rejects any others.
3. STOP: when no further proposals are made, and match each $y \in Y$ to the $x$ whose whose offer she has tentatively accepted.

**Proposition 2.4.** For any society and any profile of strict preferences, the Gale-Shapley algorithm is well-defined and results in a matching.

*Proof.* i) No $x \in X$ is ever rejected by all $y \in Y$. To see this, note that all $y$ that reject an offer are tentatively matched. All $y$ tentatively matched at some stage remain tentatively matched or matched until termination. Since the number of agents in $X$ and $Y$ is the same, all $y$ are tentatively matched only if no $x$ is rejected.

ii) The algorithm stops. At least one $x$ is rejected in each non-terminal stage and no $y$ ever gets an offer from the same $x$ more than once. Hence if the algorithm does not stop, some $x$ must be rejected by all $y \in Y$ contradicting i).

iii) The algorithm ends when nobody is rejected and hence no $x$ remains unmatched. □

Maybe the most important reason for the popularity of Gale-Shapley algorithms in practical markets is that it results in a pairwise stable matching. If a matching is not stable, the agents in the society would have incentives to search for pairwise improving opportunities to leave their current matches. It is hard to legislate against the freedom to contract in any society and therefore an unstable matching would be unlikely to remain in place.

**Proposition 2.5.** Any matching produced by the Gale-Shapley algorithm is pairwise stable.

*Proof.* Let $\mu$ be the matching and assume that $y \succ_x \mu(x)$. Then $x$ must have made an offer to $y$ in some stage prior to making an offer to $\mu(x)$. Furthermore, $y$ must have rejected $x$ and tentatively accepted some $x'$ with $x' \succ_y x$. Since $y$ rejects a tentatively accepted offer only if she gets to accept tentatively a better offer, we conclude by transitivity of $\succeq_y$ that $\mu^{-1}(y) \succ_y x$ and hence $\mu$ is pairwise stable. □

The following proposition shows that the Gale-Shapley algorithm selects the best matching for all $x \in X$ amongst the pairwise stable matchings.

**Proposition 2.6.** Let $\mu$ be the matching generated by the Gale-Shapley algorithm. Then for all $x \in X$, and all pairwise stable $\mu' \in M$, we have $\mu(x) \succ_x \mu'(x)$.

For the proof, I use the following terminology: agent $y \in Y$ is achievable for $x \in X$ if there is a stable matching $\mu'$ such that $\mu'(x) = y$.

*Proof.* Let $\mu$ be the matching produced by the Gale-Shapley algorithm and suppose no $x$ has been rejected by an achievable $y$ prior to stage $k$ of the algorithm. Assume that in stage $k$, some $y$ rejects $x$. This can happen only if $y$ tentatively accepts some $x'$. We show that $y$ is not achievable to $x$. Consider $\mu'$ with $y = \mu'(x)$ and $y' = \mu'(x')$ achievable for $x'$. Then $\mu'$ cannot be pairwise stable since by the inductive step ($y$ rejects $x$ for $x'$ in stage $k$), $x' \succ_y x$ and $y \succ'_x y'$ for all $y'$ achievable to $x'$ (by inductive step, no rejections by achievable $y$ up to stage $k$ and Gale-Shapley algorithm makes offers in descending order of preference). Hence each $x$ is matched with the highest ranked $y$ in the set of achievable $Y$. □

Unfortunately $\mu$ is similarly the worst amongst all pairwise stable matchings for the $Y$. This follows immediately from the definition of pairwise stability.

### 2.2.3 Extensions and Related Models

1. Since the manipulation of a matching mechanism is a topic for game theory, I refrain from elaborating on this issue here. Unfortunately the Gale-Shapley algorithm can be manipulated. This means that if the agents are asked to report their preferences with the understanding that the G-S algorithm is the run based on the reported

preferences, some agents may have an incentive to report a preference profile different from their true one. The G-S algorithm cannot be manipulated by $x$ (or even coalitions of agents in $X$), but unfortunately the agents in $y$ can gain from manipulation their preferences. In fact, a theorem by Al Roth proves:

**Theorem 2.7.** No pairwise stable matching mechanism exists where no agent can profit by manipulating her reported preferences.

Kominers (2017), gives references on this and a number of other related topics.

2. How essential is it that we have assumed strict preferences? Many of the results go through with weak preferences. For example, the Gale-Shapley algorithm can be run by breaking any ties in individual preferences in an arbitrary manner (e.g. assign numerical names to the agent and break ties in favor of the smaller name). The outcome of the G-S algorithm remains pairwise stable in this case as well. This amounts to adding a stage 0 to the algorithm where ties are broken. Not all results survive this, e.g. Proposition 2.6 is not true for weak preferences.

3. It is quite straightforward to allow for different numbers of agents in $X$ and $Y$ as well as allowing for the possibility that some $x$ may prefer to remain unmatched rather than be matched with some of the $y$. You will encounter such variations in the Problem Set questions.

4. An important extension of the model concerns the case where the agents on one side of the market are to be matched with groups of agents (up to a capacity constraint) on the other side. These problems are called school choice or college admission problems or many-to-one matching problems for obvious reasons. Deferred acceptance algorithms can be constructed for this case as well with straightforward modifications. Since college and school admissions are ob-

viously a very important real world problem that needs a centralized admission system, research (both theoretical and empirical) into such models is huge and still growing. A nice (and fairly recent) survey on theory developments is in Abdulkadiroğlu and Sönmez (2013). More prectical issues in school choice are covered in Cantillon (2017).

5. It is essential that there are two separate sides to be matched. The related roommate problem with a single population $X$ and where a matching is a partition of $X$ into non-overlapping pairs does not necessarily allow for any stable matchings. You may be invited to find such examples on a Problem Set.

6. The model of matching in this section is still quite special in the sense that the matching is the only endogenous variable in the model. There are no trade-offs that would allows any kind of quantification of the strength of ordinal preferences. If the model allowed for preferences over randomized allocations, the analysis would change by quite a bit. Even more dramatic would be the introduction of money in the model. Since many matching markets have monetary contracts or prices to go with the matching of the different parties, there is also a literature on matching with contracts. 'The Assignment Game I: The Core' by Shapley and Shubik (International Journal of Game Theory, 1971) (unfortunately no free copy available) started this literature and Kelso and Crawford (1982) connected matching literature with auctions. Hatfield and Milgrom (2005) gave an extra boost to this area of research. Rostek and Yoder (2020) is a recent example (with a good discussion of the area and extensive references) of theoretical work in this area. We discuss equilibria of assignment models in Subsubsection 4.4 of these notes, where some versions of assignnment models with prices are analyzed in more detail.

# 3 Competitive Markets

In this section, we consider equilibrium and efficiency in the classical setting of competitive consumer theory and producer theory.

## 3.1 Exchange economies

### 3.1.1 Allocations and Preferences: The Edgeworth Box

A society consists of $n$ consumers $i \in \{1, ..., n\}$ and a total quantity $\bar{x}_l > 0$ of divisible good $l \in \{1, .., L\}$ to be shared between the consumers. Feasible outcomes, called allocations, are collections of non-negative consumption bundles. We denote an allocation by $\boldsymbol{x}$ and the set of possible allocations by $\boldsymbol{X}$.

$$\boldsymbol{X} = \{\boldsymbol{x} \in \mathbb{R}_+^{nL}\},$$

where

$$\boldsymbol{x} := (\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n) := ((x_{11}, ..., x_{1L}), (x_{21}, ..., x_{2L}), ..., (x_{n1}, ..., x_{nL})) \in \mathbb{R}_+^{nL},$$

such that:

$$\sum_{i=1}^{n} x_{il} \leq \bar{x}_l \text{ for all } l.$$

Hence $\boldsymbol{x}_i$ is the consumption vector $(x_{i1}, ..., x_{iL})$ of consumer $i$, and $x_{il}$ is consumer $i'$s consumption of good $l$.

We assume that each consumer $i$ has a continuous rational preference relation represented by the utility function $u_i$ on the consumption set $\mathbb{R}_+^L$.

**Definition 3.1.** An allocation $\boldsymbol{x}$ is Pareto-efficient if there is no feasible allocation $\boldsymbol{x}' = (\boldsymbol{x}'_1, ..., \boldsymbol{x}'_n)$ such that for all $i \in \mathcal{N}$, $u_i(\boldsymbol{x}'_i) \geq u_i(\boldsymbol{x})$ and for some $i$, $u_i(\boldsymbol{x}'_i) > u_i(\boldsymbol{x}_i)$.

**Example 3.1.** If $n = 2$, there allocations and preferences can be displayed in an Edgeworth Box $E$ (Edgeworth Rectangle?) that consist of a rectangle with opposite corners at the origin and at $(\bar{x}_1, \bar{x}_2)$ respectively. Any point $(x_1, x_2)$ inside the rectangle allows represents a feasible allocation $((x_{11}, x_{12}), (x_{21}, x_{22}))$ by identifying $(x_{11}, x_{12}) = (x_1, x_2)$, and $(x_{21}, x_{22}) = (\bar{x}_1 - x_1, \bar{x}_2 - x_2)$.

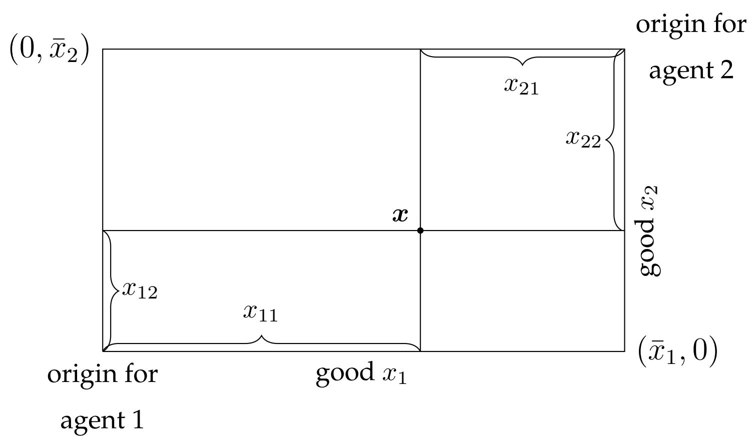


Figure 2: The Edgeworth box with consumption allocation at $\boldsymbol{x}$

We can also draw the indifference curves for the two agents positioned at $(0, 0)$ and $(\bar{x}_1, \bar{x}_2)$ respectively through allocation $\boldsymbol{x}$.

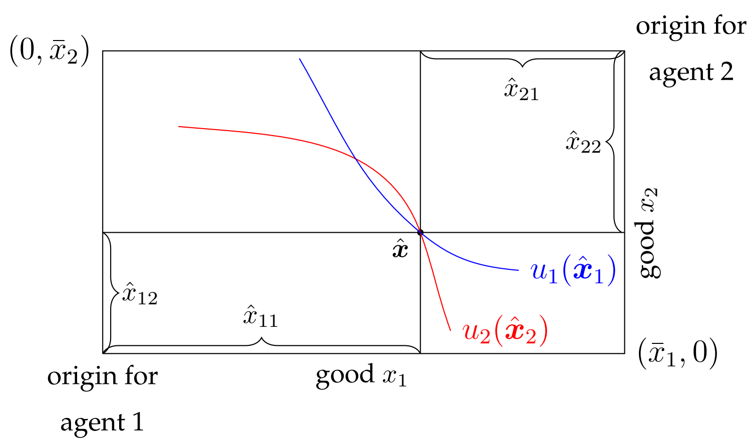


Figure 3: The Edgeworth box with consumption allocation at $\hat{\boldsymbol{x}}$

Let

$$B_1(x_{11}, x_{12}) = \{\boldsymbol{x} \in \mathbb{R}^2_+ | u_1(\boldsymbol{x}) \geq u_1(x_{11}, x_{12})\},$$

$$B_2(x_{21}, x_{22}) = \{\boldsymbol{x} \in \mathbb{R}^2_+ | u_2(\boldsymbol{x}) \geq u_2(x_{21}, x_{22})\}.$$

Then the set of Pareto-efficient allocations consists of all $((x_{11}, x_{12}), (x_{21}, x_{22}))$ such that for all

$$\boldsymbol{x}' \in E \cap B_1(x_{11}, x_{12}) \cap B_2(x_{21}, x_{22}),$$

$$u_i(\boldsymbol{x}'_i) = u_i(\boldsymbol{x}_i) \text{ for } i \in \{1, 2\}.$$

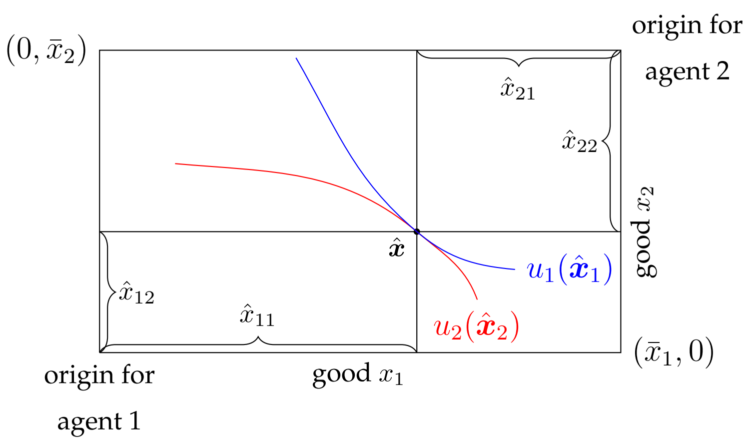


Figure 4: Pareto-efficient allocation at $\hat{\boldsymbol{x}}$

Observe that if the indifference curves intersect at an interior allocation $\hat{\boldsymbol{x}} \in \mathbb{R}^4_{++}$ in the Edgeworth Box, then $\hat{\boldsymbol{x}}$ is not Pareto-efficient.

### 3.1.2 Initial endowments and competitive budget sets

An *exchange economy* is a society $(\mathcal{N}, \{\succeq_i\}_{i \in \mathcal{N}})$ together with an *initial endowment* vector $\boldsymbol{\omega} \in \mathbb{R}^{nL}_+$, where $\boldsymbol{\omega} := (\boldsymbol{\omega}_1, ..., \boldsymbol{\omega}_n)$, and $\boldsymbol{\omega}_i := (\omega_{i1}, ..., \omega_{iL})$.

In an exchange economy, the agents have property rights over their initial endowment and they are competitive price takers. In other words, they may exchange goods in the economy at market prices $\boldsymbol{p} \in \mathbb{R}^L_+$ as in the classical consumer theory part of Microeconomic Theory I.

The budget set $B(\boldsymbol{p}, \boldsymbol{\omega}_i)$ of agent $i$ is given by

$$B(\boldsymbol{p}, \boldsymbol{\omega}_i) = \{\boldsymbol{x}_i \geq \boldsymbol{0} | \, \boldsymbol{p} \cdot \boldsymbol{x}_i \leq \boldsymbol{p} \cdot \boldsymbol{\omega}_i\}.$$

Agent $i$ chooses $\boldsymbol{x}_i \in B(\boldsymbol{p}, \boldsymbol{\omega}_i)$ to maximize a utility function $u_i$ representing her preferences. The resulting *demand correspondence* is denoted by $\boldsymbol{x}_i(\boldsymbol{p}, \boldsymbol{\omega})$.

**Definition 3.2.** The *excess demand* correspondence $\boldsymbol{z}_i(\boldsymbol{p}) \in \mathbb{R}^L$ of agent $i$ is given by:

$$\boldsymbol{z}_i(\boldsymbol{p}) = \boldsymbol{x}_i(\boldsymbol{p}, \boldsymbol{\omega}_i) - \boldsymbol{\omega}_i.$$

Negative excess demands are called *excess supplies* and since demands are positive, excess supplies are bounded from above and excess demands are bounded from below.

I have suppressed the dependence on the initial endowment for notational convenience since $\boldsymbol{p}$ is an endogenous variable, but $\boldsymbol{\omega}$ is kept fixed in the analysis.

We can illustrate the budget line at slope $\frac{-p_1}{p_2}$ through $\boldsymbol{\omega}$ and optimal demands in the Edgeworth Box.

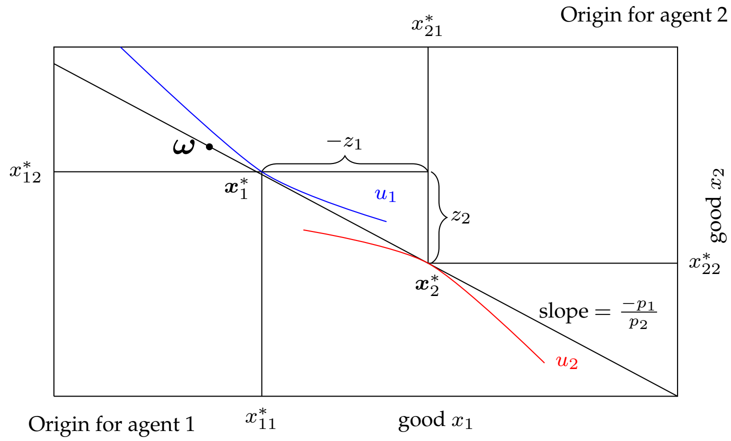


Figure 5: Consumer choice and aggregate excess demands for the two goods in the Edgeworth Box

**Exercises:**

1. A society consists of two agents and two goods. Suppose the agents have linear preferences, i.e. their indifference curves have a constant slope everywhere in their consumption set. Illustrate in an Edgeworth Box the Pareto efficient allocations for the society with fixed resources $(\bar{x}_1, \bar{x}_2)$.

2. Suppose an agent with initial endowment $(\omega_1, \omega_2)$ can trade the two goods at prices $(p_1, p_2) \in \mathbb{R}_+^2$. Find the optimal demand correspondence for the agent.

We recall some properties of optimal demands from Advanced Microeconomics 1.

**Theorem 3.1.** Consider an agent with strictly convex and strictly increasing preferences with a strictly positive intial endownment $\boldsymbol{\omega} \in \mathbb{R}_{++}^L$. Then her optimal demand correspondence is a function that is:

i) Continuous in $\boldsymbol{p}$,

ii) Homogenous of degree 0 in $\boldsymbol{p}$,

iii) Satisfies Walras' law: $\boldsymbol{p} \cdot \boldsymbol{x}(\boldsymbol{p}, \boldsymbol{\omega}) = \boldsymbol{p} \cdot \boldsymbol{\omega}$,
iv) If $\boldsymbol{p}^n \to \boldsymbol{p}$ and $\boldsymbol{p} \neq 0$ and $p_l = 0$ for some $l$, then

$$\max_l \{x_1(\boldsymbol{p}, \boldsymbol{\omega}), ..., x_L(\boldsymbol{p}, \boldsymbol{\omega})\} \to \infty.$$

**Remark.** These are familiar results, but the last property deserves a comment. If the last property is violated, then the sequence of optimal demands is bounded and therefore must have a convergent subsequence. Since the preferences are continuous, this would imply the optimality of bounded demand at zero prices for some goods. This contradicts the assumption of strictly increasing preferences.

These same properties hold obviously for individual excess demands as well. Our next task is to aggregate the individual excess demands across all agents.

**Definition 3.3.** The *aggregate excess demand* $\boldsymbol{z}(\boldsymbol{p})$ of an economy is given by

$$\boldsymbol{z}(\boldsymbol{p}) = \sum_{i=1}^{n} \boldsymbol{z}_i(\boldsymbol{p}).$$

I will comment on the positive and normative meaning of aggregate demand at the end of this section. For now, it is enough to observe that the above properties on individual demands carry over to aggregate excess demands.

**Theorem 3.2.** Suppose all agents in the economy have strictly convex and strictly increasing preferences and that the aggregate intial endownment $\sum_{i=1}^{n} \boldsymbol{\omega}_i \in \mathbb{R}^L_{++}$. Then the aggregate excess demand function $\boldsymbol{z}(\boldsymbol{p})$ satisfies:
i) $\boldsymbol{z}(\boldsymbol{p})$ is continuous in $\boldsymbol{p}$,
ii) $\boldsymbol{z}(\boldsymbol{p})$ is homogenous of degree 0 in $\boldsymbol{p}$,
iii) Walras' law: $\boldsymbol{p} \cdot \boldsymbol{z} = 0$,
iv) There is an $s < \infty$ such that $z_l(\boldsymbol{p}) \geq -s$ for all $l$ and all $\boldsymbol{p}$,
v) If $\boldsymbol{p}^n \to \boldsymbol{p}$ and $\boldsymbol{p} \neq 0$ and $p_l = 0$ for some $l$, then

$$\max_l \{z_1(\boldsymbol{p}, \boldsymbol{\omega}), ..., z_L(\boldsymbol{p}, \boldsymbol{\omega})\} \to \infty.$$

Property iv) follows from the fact that the individual excess supplies are bounded above by the initial endowment and by summing over agents.

### 3.1.3 Competitive equilibrium: existence and properties

We start with the definition of a competitive equilibrium for an exchange economy.

**Definition 3.4.** A *competitive equilibrium* of the exchange economy $(\mathcal{N}, \{\succeq_i\}_{i\in\mathcal{N}}, \{\boldsymbol{\omega}_i\}_{i\in\mathcal{N}})$ is a price vector $\boldsymbol{p} \in \mathbb{R}_+^L$ and an allocation $\boldsymbol{x} \in \mathbb{R}_+^{nL}$ such that:

i) For all $i \in \mathcal{N}$, $\boldsymbol{x}_i$ solves:

$$\max_{\boldsymbol{y}\in\mathbb{R}^L} u_i(\boldsymbol{y}) \text{ subject to } \boldsymbol{p}\cdot\boldsymbol{y} \leq \boldsymbol{p}\cdot\boldsymbol{\omega}_i.$$

ii) Markets clear, i.e. for all $l$,

$$\sum_{i=1}^{n} x_{il} \leq \sum_{i=1}^{n} \omega_{il}.$$

An equilibrium consists then of a price and an allocation such that all agents maximize their preferences (represented by $u_i$) in the budget set determined by the price vector $\boldsymbol{p}$ and their initial endowment. Individual decisions must be compatible in the sense that demand not exceed supply. This is the requirement of market clearing: prices are such that the aggregate demand at those prices must stay below the aggregate supply.

Notice that the market clearing condition allows for possibility of not using up the entire aggregate endowment. This is known as the assumption of free disposal. For the cases that are covered in these lectures, we could also insist on equality in ii) above.

We can summarize the conditions for a competitive equilibrium as finding a price vector $\boldsymbol{p}$ such that $\boldsymbol{z}(\boldsymbol{p}) \leq 0$. Walras' law implies immediately that $p_l = 0$ for all $l$ such that $z_l(\boldsymbol{p}) < 0$. Our search for a competitive equilibrium is then equivalent to the search of a price vector $\boldsymbol{p}$ such that

$\boldsymbol{z}(\boldsymbol{p}) \leq 0$. This is a non-trivial problem. In this subsection, we will show the existence of such a vector when the conditions for Theorem 3.2 hold.

To show this, we notice first that by the homogeneity of degree zero of the aggregate excess demand, we can normalize the prices to be in the simplex $\Delta := \{\boldsymbol{p} \in \mathbb{R}^L | \sum_{l=1}^{L} p_l = 1\}$. We construct a correspondence $f : \Delta \rightrightarrows \Delta$ that satisfies the conditions for Kakutani's fixed point theorem and where

$$\boldsymbol{p} \in f(\boldsymbol{p}) \implies \boldsymbol{z}(\boldsymbol{p}) = 0.$$

Denote the interior of the price simplex, i.e. $\boldsymbol{p} \in \Delta$ such that $\boldsymbol{p} >> 0$ by $\Delta^o$. Let $\partial\Delta := \Delta \setminus \Delta^o$ be the boundary of the simplex. For $\boldsymbol{p} \in \Delta^o$, we let:

$$f(\boldsymbol{p}) = \arg\max_{\boldsymbol{q} \in \Delta} z(\boldsymbol{p}) \cdot \boldsymbol{q}.$$

For $\boldsymbol{p} \in \partial\Delta$, we let:

$$f(\boldsymbol{p}) = \arg\max_{q \in \Delta} -\boldsymbol{p} \cdot \boldsymbol{q}.$$

**Exercise:** Show that if $\boldsymbol{p} \in \partial\Delta$, then $\boldsymbol{p} \notin \boldsymbol{f}(\boldsymbol{p})$. Show also that if $z_l(\boldsymbol{p}) > z_{l'}(\boldsymbol{p})$ for some $l, l'$, then $f(\boldsymbol{p}) \subset \partial\Delta$. Use Walras' law to argue that

$$\boldsymbol{p} \in f(\boldsymbol{p}) \implies \boldsymbol{z}(\boldsymbol{p}) = 0.$$

Recall that Kakutani's fixed point theorem (Theorem M.I.2 in MWG) guarantees the existence of a fixed point for a correspondence $f$, i.e. a point such that $\boldsymbol{p} \in f(\boldsymbol{p})$ if the domain of $f$ is compact and convex and if $f$ is nonempty-valued, upper hemi-continuous and convex-valued.

**Theorem 3.3** (Existence of Competitive Equilibrium)**.** Suppose that the aggregate demand $\boldsymbol{z}$ of the exchange economy $(\mathcal{N}, \{\succeq_i\}_{i\in\mathcal{N}}, \{\boldsymbol{\omega}_i\}_{i\in\mathcal{N}})$ satisfies the conditions i)-v) of Theorem 3.2. Then a competitive equilibrium $(\boldsymbol{p}, \boldsymbol{x})$ exists.

*Proof.* In light of the Exercise above, we only need to show that the correspondence $f$ has a fixed point. We have two steps:

i) The domain of $f$ is compact and convex (as shown in Advanced Microeconomic 1).

ii) Since $\Delta$ is compact and the functions to be maximized in the definition of $f$ are linear, $f(\boldsymbol{p})$ is nonempty-valued and convex-valued for all $\boldsymbol{p}$. (In fact $f(\boldsymbol{p})$ is a vertex, edge or a face of $\Delta$).) Hence the only property that needs to be verified is that $f$ is u.h.c. Consider $\boldsymbol{p}^n \to \boldsymbol{p}$. If $\boldsymbol{p} \in \Delta^o$, then the claim follows from the Berge's theorem of the maximum (M.K.6 in MWG), i.e. if the objective function is continuous in $\boldsymbol{x}$, the set of maximizers is an u.h.c. correspondence in $\boldsymbol{x}$.

If $\boldsymbol{p} \in \partial\Delta$, then property v) of Theorem 3.2 shows the existence of an unbounded excess demand $z_{l'}$ for some good $l'$ with $p_{l'} \to 0$ whenever $\boldsymbol{p}^n \to \boldsymbol{p}$. But this shows that $\lim f(\boldsymbol{p}^n) \subset f(\boldsymbol{p})$ and as a result, $f$ is u.h.c. $\square$

Notice that the hypothesis is Theorem 3.2 gives a nice set of sufficient conditions for $\boldsymbol{z}(\boldsymbol{p})$ to satisfy i)-v) of that theorem. We will use the same existence theorem again when discussing production economies, but in that case, we will have different primitive assumptions on the economy (playing the role of the hypothesis in Theorem 3.2).

The constructed correspondence $f$ in not particularly intuitive, but it makes the proof simple. A further advantage of this proof is that it makes the inclusion of firms in the economy quite straightforward. As long as the aggregate excess demand satisfies the conditions of Theorem 3.2, we have the existence of a competitive equilibrium.

An alternative and somewhat more natural approach to proving the existence is given in Proposition 17.C.2 of MWG, but unfortunately this approach does not handle the case where some $p_l \to 0$ very well.

The existence proof can be extended to the case where the individual demands (and hence excess demands and aggregate excess demands) are convex valued correspondences. This covers the case of linear utility functions below, but also other types of linear models. See 'Theory of Value' by Debreu (1960) for details.

**Example 3.2.** We can demonstrate the competitive equilibria in the Edgeworth-Box for an economy consisting of two agents.

For the Edgeworth-Box, the existence proof is quite a bit simpler. Starting with a Pareto-inefficient allocation such as $\hat{\boldsymbol{x}}$ in the next figure, we can see that $\hat{\boldsymbol{x}}_1$ is optimal for agent $i$ if $\frac{p_1}{p_2} = MRS_i(\hat{\boldsymbol{x}}_i)$, i.e. the slope of agent $i$'s indifference curve through $\hat{\boldsymbol{x}}$.

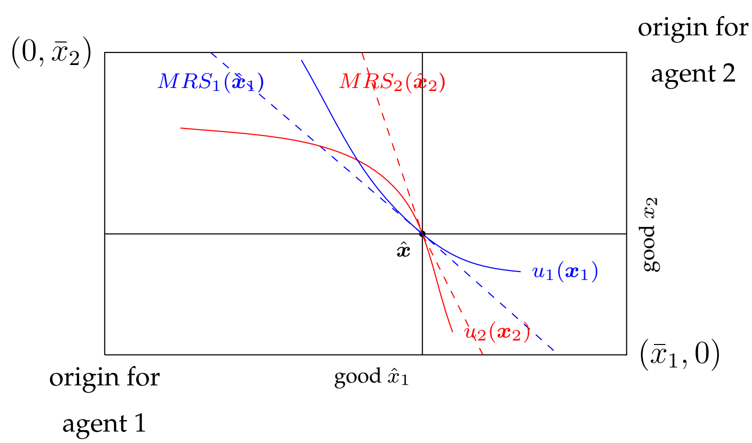


Figure 6: The Edgeworth box with initial endowment at $\hat{\boldsymbol{x}}$

Since $MRS_2(\hat{\boldsymbol{x}}_2) > MRS_1(\hat{\boldsymbol{x}}_1)$, we see (for example by the law of compensated demand) that the aggregate excess demand for good 1 is positive for budget lines through $\hat{\boldsymbol{x}}$ at slope $\frac{p_1}{p_2} = MRS_1(\hat{\boldsymbol{x}}_1)$ and negative for price ratio $\frac{p_1}{p_2} = MRS_2(\hat{\boldsymbol{x}}_2)$. Since the excess demands are continuous, there must be a price vector $(\hat{p}_1, \hat{p}_2)$ such that:

$$MRS_1(\hat{\boldsymbol{x}}_1) < \frac{\hat{p}_1}{\hat{p}_2} < MRS_2(\hat{\boldsymbol{x}}_2),$$

and

$$z_{11}(\hat{p}_1, \hat{p}_2) + z_{21}(\hat{p}_1, \hat{p}_2) = 0.$$

Walras' law implies that market clears for good 2 as well at those prices.

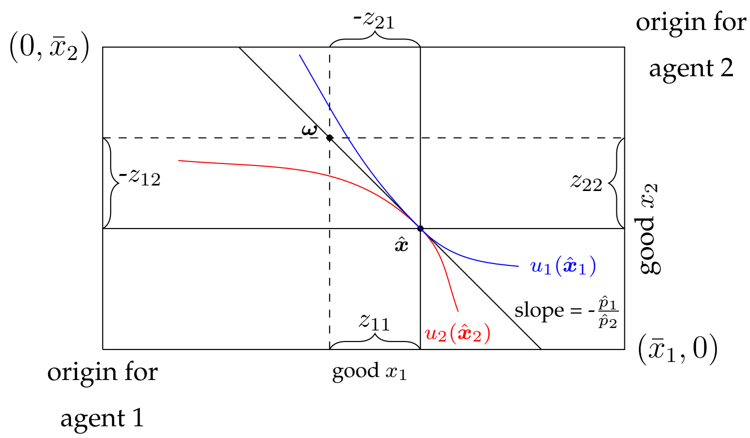


Figure 7: Competitive equilibrium: $((\hat{p}_1, \hat{p}_2), (\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2))$

We show next that all competitive equilibria are Pareto-efficient.

**Theorem 3.4** (First Welfare Theorem for Exchange Economies)**.** Suppose all agents in an exchange economy $(\mathcal{N}, \{\succeq_i\}_{i \in \mathcal{N}}, \{\boldsymbol{\omega}_i\}_{i \in \mathcal{N}})$ have locally non-satiated preferences. Then all competitive equilibrium allocations of the economy are Pareto-efficient.

*Proof.* Let $(\boldsymbol{p}, \boldsymbol{x})$ be a competitive equilibrium and $\boldsymbol{y}$ an allocation that Pareto-dominates $\boldsymbol{x}$. By local non-satiation, $\boldsymbol{p} \cdot \boldsymbol{y}_i \geq \boldsymbol{p} \cdot \boldsymbol{x}_i$ for all $i$ and $\boldsymbol{p} \cdot \boldsymbol{y}_i > \boldsymbol{p} \cdot \boldsymbol{x}_i$ for some $i$. Summing over all $i$, we get $\sum_{i=1}^n \boldsymbol{p} \cdot \boldsymbol{y}_i > \sum_{i=1}^n \boldsymbol{p} \cdot \boldsymbol{x}_i = \sum_{i=1}^n \boldsymbol{p} \cdot \boldsymbol{\omega}_i$, where the last equality follows from Walras' law. Hence, there must be an $l$ such that $\sum_{i=1}^n y_{il} > \sum_{i=1}^n \omega_{il}$ showing that $\boldsymbol{y}$ is not feasible. □

Notice that this theorem is little more than just the observation that at optimal consumptions, budget constraint binds for locally non-satiated preferences.

There is also a sense in which almost all Pareto-efficient allocations can be made competitive equilibrium allocations if preferences are convex.

**Theorem 3.5** (Second Welfare Theorem for Exchange Economies)**.** Suppose all agents in a society $(\mathcal{N}, \{\succeq_i\}_{i \in \mathcal{N}}, (\bar{x}_1, ..., \bar{x}_L))$ have continuous, strictly monotone and convex preferences on $\mathbb{R}^L_+$ and $\boldsymbol{x}^*$ is a Pareto-efficient allocation. Then the there is a price vector $\boldsymbol{p}^* > 0$ such that $(\boldsymbol{p}^*, \boldsymbol{x}^*)$ is a competitive equilibrium of the exchange economy $(\mathcal{N}, \{\succeq_i\}_{i\in\mathcal{N}}, \{\boldsymbol{x}_i^*\}_{i\in\mathcal{N}})$

The proof is essentially an application of the supporting hyperplane theorem.

*Proof.* STEP 1 Let $S_i(\boldsymbol{x}_i^*) := \{\boldsymbol{x}_i \in \mathbb{R}^L_+ | \ \boldsymbol{x}_i \succeq_i \boldsymbol{x}_i^*\}$. Set $S(\boldsymbol{x}^*) := S_1(\boldsymbol{x}_1^*) + S_2(\boldsymbol{x}_2^*) + ... + S_n(\boldsymbol{x}_n^*)$, where $S$ is the direct sum of the sets $S_i$, i.e.

$$\boldsymbol{x}_{\sum} \in S(\boldsymbol{x}^*) \subset \mathbb{R}^L_+ \text{ if for all } i,\ \exists \boldsymbol{x}_i' \in S_i(\boldsymbol{x}_i^*) \text{ such that } \boldsymbol{x}_{\sum} = \sum_{i=1}^n \boldsymbol{x}_i'.$$

Each of the $S_i$ is closed and convex and since the direct sum of convex sets is also convex, we conclude that $S(\boldsymbol{x})$ is a non-empty closed and convex subset of $\mathbb{R}^L$.

STEP 2 Let $\boldsymbol{x}_{\sum}^* = \sum_{i=1}^n \boldsymbol{x}_i^*$ Since $\boldsymbol{x}^*$ is Pareto-efficient and since the preferences are strictly monotone, we conclude that $\boldsymbol{x}_{\sum}^* \notin S^o(\boldsymbol{x}^*)$, where $S^o(\boldsymbol{x}^*)$ denotes the interior of the set $S(\boldsymbol{x}^*)$. Hence by the Supporting hyperplane theorem (MWG M.G.3), there exist $\boldsymbol{p}^* \neq 0$ and $w^*$ such that $\boldsymbol{p}^* \cdot \boldsymbol{x}_{\sum}^* = w^*$ and $\boldsymbol{p}^* \cdot \boldsymbol{x}_{\sum} \geq w^*$ for all $\boldsymbol{x}_{\sum} \in S(\boldsymbol{x}^*)$.

STEP 3 To show that $\boldsymbol{p}^* > 0$, let $\boldsymbol{e}_k$ denote the kth unit vector and $\mathbf{1}$ denote the vector of 1's. Then by strict monotonicity, there is an $\epsilon > 0$ such that $\boldsymbol{x}_{\sum}^* + \boldsymbol{e}_k - \epsilon\mathbf{1} \in S(\boldsymbol{x}^*)$. By Step 2, $\boldsymbol{p}^* \cdot (\boldsymbol{x}_{\sum}^* + \boldsymbol{e}_k - \epsilon\mathbf{1}) \geq w^*$. Since $\boldsymbol{p}^* \cdot \boldsymbol{x}_{\sum}^* = w^*$ by step 1, we have

$$p_k^* \geq \epsilon \sum_{l=1}^L p_l^* > 0,$$

where the last step follows again from Step 2 showing that $\boldsymbol{p}^* \neq 0$. Since $k$ is arbitrary, we have $\boldsymbol{p}^* > 0$.

STEP 4 The remaining task is to show that $\boldsymbol{x}_i \succ_i \boldsymbol{x}_i^* \implies \boldsymbol{p}^* \cdot \boldsymbol{x}_i > \boldsymbol{p}^* \cdot \boldsymbol{x}_i^*$ showing the optimality of $\boldsymbol{x}_i^*$ in the budget set of $i$ with initial endowment $\boldsymbol{x}_i^*$.

If $\boldsymbol{x}_i \succ_i \boldsymbol{x}_i^*$, then by continuity of preferences, $\alpha\boldsymbol{x}_i \succ_i \boldsymbol{x}_i^*$ for some $0 < \alpha < 1$. Then $\alpha\boldsymbol{x}_i + \sum_{j\neq i}\boldsymbol{x}_j^* \in S(\boldsymbol{x}^*)$ so that:

$$\boldsymbol{p}^* \cdot (\alpha\boldsymbol{x}_i + \sum_{j\neq i}\boldsymbol{x}_j^* \in S(\boldsymbol{x}^*)) \geq w^* = \boldsymbol{p}^* \cdot (\boldsymbol{x}_i^* + \sum_{j\neq i}\boldsymbol{x}_j^*).$$

But then we have $\alpha\boldsymbol{p}^*\boldsymbol{x}_i \geq \boldsymbol{p}^*\boldsymbol{x}_i^*$ and since $\boldsymbol{p}^* > 0$ and $\boldsymbol{x}_i \in \mathbb{R}_+^L \setminus \{0\}$, we have $\boldsymbol{p}^*\boldsymbol{x}_i > \alpha\boldsymbol{p}^*\boldsymbol{x}_i \geq \boldsymbol{p}^*\boldsymbol{x}_i^*$ proving the claim. □

For historical reasons, these two theorems are viewed as fundamental to understanding the welfare properties of competitive economies. Hence they are sometimes called 'Fundamental Theorems of Welfare Economics'. In my opinion, they are interesting, but have little to do with how we typically understand welfare in a society.

**Example 3.3.** In order to have concrete examples of how to find competitive equilibria, it is extremely useful to consider agents with linear preferences, i.e. $u_i(x_{i1}, x_{i2}) = \alpha_{i1}x_{i1} + \alpha_{i2}x_{i2}$. Consider the case with two agents and assume that $\frac{\alpha_{11}}{\alpha_{12}} \neq \frac{\alpha 21}{\alpha_{22}}$ and draw the Edgeworth Box.

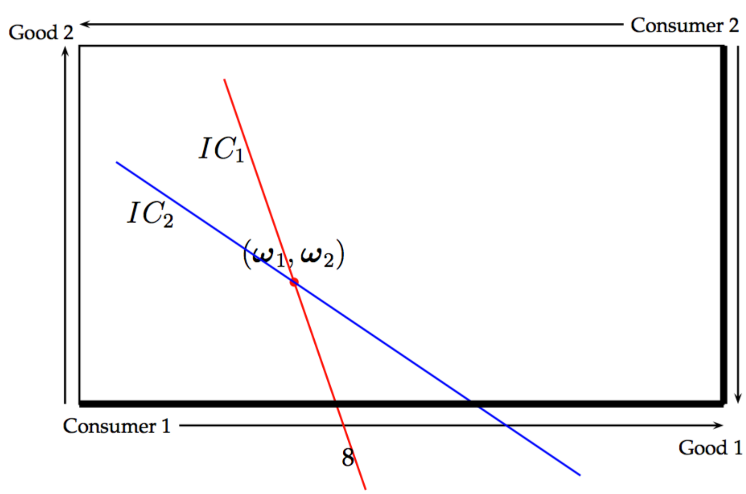


Figure 8: Linear indifference curves and Pareto-efficient points

Since $MRS_1 \neq MRS_2$ at all points, there cannot be interior Pareto-efficient allocations. Hence all Pareto-efficient allocations are located on a side or at a corner of the box. Make sure that you understand the economics behind this (think about profitable trades when both agents own both of the goods and the $MRS$ are different).

By first welfare theorem, we know that the competitive equilibrium allocation is Pareto efficient. On the side of the box, one of the agents has an interior consumption vector and hence optimality requires that the $MRS$ of this agent be equal to the price ratio.

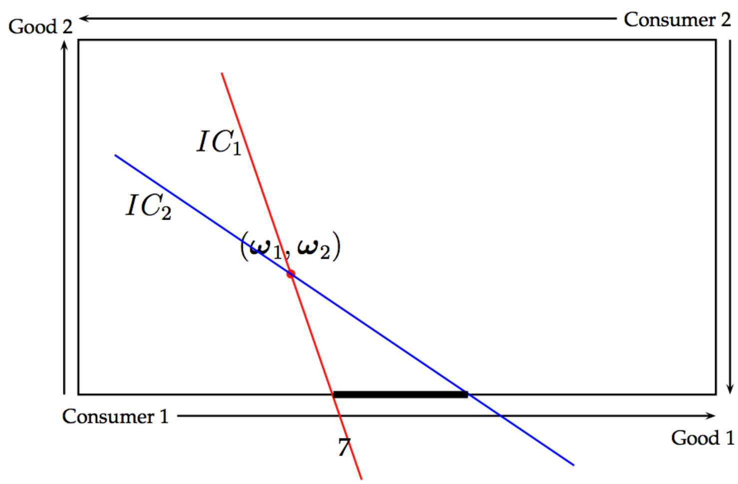


Figure 9: Linear indifference curves and some Pareto-efficient points

At a corner of the Edgeworth Box, both agents are at the boundary of their consumption set and hence an equilibrium price can be different from both $MRS$'s.

**Exercise (with the linear utility specification of this Example):**

1. Show that if $\boldsymbol{x}^* := (\omega_{11} + \omega_{21}, 0), (0, \omega_{12} + \omega_{22})$ Pareto-dominates $\boldsymbol{\omega}$, then $((\frac{\omega 12}{\omega 21}, 1), \boldsymbol{x}^*)$ is a competitive equilibrium for this economy.

2. Show that if for all Pareto efficient $\boldsymbol{x}^*$ that Pareto-dominate $\boldsymbol{\omega}$, we have $x^*_{12} = 0$, then $(\frac{\alpha_{21}}{\alpha_{22}}, 1)$ is the price vector in the competitive equilibrium of the economy.

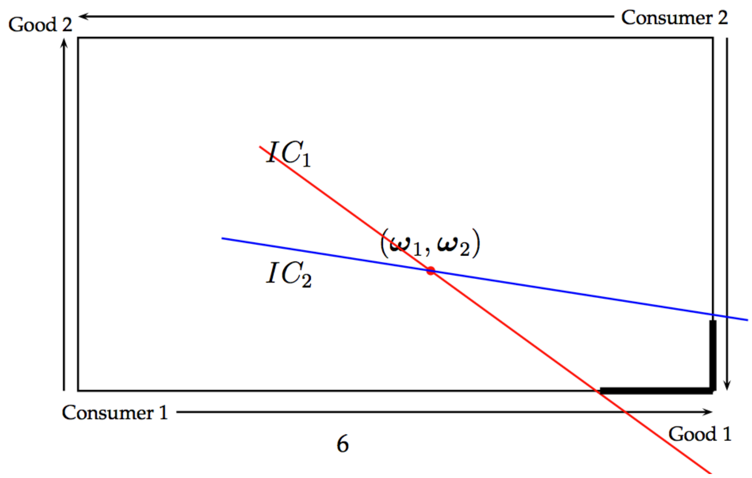


Figure 10: Linear indifference curves and some Pareto-efficient points

#### 3.1.4 The Core of and Exchange Economy

We discussed pairwise stability in the context of the matching model. If two agents can improve their situation by rematching with each other, we do not consider the original matching to be stable. Similarly we could ask about the stability of an allocation in an exchange economy against deviations by groups of agents. The idea is simply that a group can destabilize or *block* an allocation if it can find a better feasible allocation. An allocation is feasible for a group of agents if it distributes no more than the aggregate endowment of the group amongst its members. An allocation is stable if no group can find another feasible allocation that Pareto dominates the original allocation for the group members.

Pareto-efficient allocations are those allocations that are stable against deviations by the group consisting of all agents. We say that an allocation is in the *core* of the exchange economy if it is stable against all group deviations. In an Edgeworth box, the set of Pareto-efficient allocations is sometimes called the contract curve. The core is the part of the contract curve that is individually rational, i.e. Pareto-efficient allocations where each agent is at least as well off as at the initial endowment (since the only deviating groups in an economy consisting of two agents are individual deviations and deviations by the entire society).

**Proposition 3.1.** Suppose that $(\boldsymbol{p}, \boldsymbol{x})$ is a competitive equilibrium of an economy $(\mathcal{N}, \{\succeq_i\}_{i\in\mathcal{N}}, \{\boldsymbol{\omega}_i\}_{i\in\mathcal{N}})$ with locally non-satiated preferences. Then $\boldsymbol{x}$ is in the core of the exchange economy.

*Proof.* The proof is almost identical to the proof of the first welfare theorem. Let $I$ be a group of deviating agents. If $\boldsymbol{x}_i' \succeq_i \boldsymbol{x}_i$, then by local non-satiation and optimality of the equilibrium choice $\boldsymbol{x}_i$, $\boldsymbol{p} \cdot \boldsymbol{x}_i' \geq \boldsymbol{p} \cdot \boldsymbol{x}_i$ and $\boldsymbol{x}_i' \succ_i \boldsymbol{x}_i \implies \boldsymbol{p} \cdot \boldsymbol{x}_i' > \boldsymbol{p} \cdot \boldsymbol{x}_i$. Hence if $\boldsymbol{x}'$ Pareto-dominates $\boldsymbol{x}$ for $I$, then

$$\sum_{i\in I} \boldsymbol{p} \cdot \boldsymbol{x}_i' > \sum_{i\in I} \boldsymbol{p} \cdot \boldsymbol{x}_i = \sum_{i\in I} \boldsymbol{p} \cdot \boldsymbol{\omega}_i,$$

and this contradicts feasibility for group $I$. □

The more surprising result is that the core of an exchange economy shrinks to the competitive equilibrium allocation as the economy grows large.

I will not give a full proof of this result in the notes since it is well covered in MWG. I will just outline the steps in the proof and discuss the interpretation of the result. In the remainder of this section, we will maintain the following assumptions on individual preferences.

**Assumption 3.1.** Each agent $i$ has preferences $\succeq_i$ that satisfy:

1. $\succeq_i$ is continuously differentiable,
2. $\succeq_i$ is strictly increasing,
3. $\succeq_i$ is strictly convex,
4. for all $i$, $x_{il} = 0$ for some $l \in \{1, ..., L\}$ implies that $\boldsymbol{\omega}_i \succ_i \boldsymbol{x}_i$

The society consists of $H$ types of agents $i \in \{1, ..., H\}$. Each type is characterized by her utility function $u_h$ and her initial endowment $\boldsymbol{\omega}_h$. The N-replica economy is one where there are $N$ agents of each type. (How should we deal with the case where there are different numbers of different types of agents?)

**Lemma 1.** If $\boldsymbol{x}$ is in the core of the N-replica of the economy, then all agents of the same type have the same consumption vector.

*Proof.* Sketch: i) If agents of the same type have different consumption allocations at a core allocation, then they are not indifferent between the allocations (use Pareto-efficiency and strict convexity).

ii) Form a deviating group by joining together one of the worst treated agents of each type. One of them must be strictly worse off by part i) if the agents of the same type do not all have the same consmuption vector. By the strict convexity of preferences, the average consumption vector of each type is at least as good as the original consumption vector for all types in the deviating group and strictly better than the original for the type with unequal consumption vectors.

iii) The average consumption vector of the N-replica economy is feasible for this deviating group (an exercise in accounting) showing that $\boldsymbol{x}$ cannot be in the core of the N-replica economy. □

You should treat the proof sketch below as an exercise, or alternatively you may want to check the proof in MWG Proposition 18.B.3.

**Theorem 3.6.** If $\boldsymbol{x}$ is in the core for all N-replica of the economy, then $\boldsymbol{x}$ is a competitive equilibrium allocation.

*Proof.* Since only Pareto-efficient allocations can be in the core, we assume that $\boldsymbol{x}$ is Pareto-efficient. Bullet 4 of Assumption 3.1 implies that $\boldsymbol{x}_{h,n} > 0$ for all $h, n$ since core allocations must be at least as good as the initial endowment for all agents.

By Lemma 1, all agents of the same type have the same consumption vectors at any core allocation and we let $\boldsymbol{x}_h$ denote the consumption vector of all agents of type $h$.

By second welfare theorem and strict convexity of preferences, there exists a $\boldsymbol{p} \geq 0$ such that $\boldsymbol{x}$ is the competitive equilibrium allocation with initial endowments given by $\boldsymbol{x}$.

We show the claim by contrapositive, i.e. if $\boldsymbol{x}$ is not a competitive equilibrium allocation, then it is not in the core for all N-replica of the economy. First we note that if $\boldsymbol{x}$ is not a competitive equilibrium allocation for the N-replica economy with initial endowment $\boldsymbol{\omega}$, then for some type $h'$, we have

$$\boldsymbol{p} \cdot (\boldsymbol{x}_{h',N} - \boldsymbol{\omega}_{h',N}) > 0.$$

Consider a deviating group formed of all agents of types other than $h'$ and all but one agent of type $h'$. To maintain feasibility, distribute the excess trade of the excluded agent evenly to all agents in the deviating group. Use next the fact that at an interior Pareto-efficient allocation, $\boldsymbol{p}$ is proportional to $\nabla u_h(\boldsymbol{x}_h)$ for all $h$.

Finally, compute the effect of the distributed excess demand on the utility functions of the other agents by Taylor's approximation to $u_h$ around $\boldsymbol{x}_h$ when N is large (and therefore the per agent change in consumption is small). □

**Remark.** 1. The core formalizes the stability of an allocation against group deviations. From this perspective, the above theorem tells us that in large economies, the only stable allocations are competitive equilibrium allocations.

2. The core is not entirely unproblematic. Why do the agents not in the deviating group not react to the deviation? If there are many blocking allocations for a deviating group and the preferences amongst the deviators differ on which is the best deviation, how should this be resolved?

3. On a more positive note, here is another interpretation for competitive prices. Start with the quasilinear setting, where

$$u_h(\boldsymbol{x}_h) = \psi_h(x_1, ..., x_{L-1}) + x_L.$$

Let $\boldsymbol{\mu} = (\mu_1, ..., \mu_H)$ be the vector representing the fractions of agents of type $h$ in the economy.

With quasi-linear utilities, Pareto-efficient points can now be found by solving:

$$\max_{\boldsymbol{x}} \sum_{h=1}^{H} \mu_h u_h(\boldsymbol{x}_h)$$

subject to the feasibility constraint

$$\boldsymbol{\mu} \cdot \boldsymbol{x} \leq \boldsymbol{\mu} \cdot \sum_h \boldsymbol{\omega}_h.$$

Let $v(\boldsymbol{\mu})$ denote the value function of this problem. An allocation $\boldsymbol{x}$ is a competitive allocation of the economy if and only if

$$\frac{\partial v(\boldsymbol{\mu})}{\mu_h} = u_h(\boldsymbol{x}_h).$$

To see how to derive this, you should just apply the envelope theorem to the above value function and maximization problem and note that the competitive equilibrium price vector arises as the Lagrange multiplier in the first-order condition of the constrained optimization problem.

In words, an allocation is a competitive allocation if and only if the agent's utility is equal to her marginal contribution to the social value function. You may want to recall this when considering VCG-mechanisms in Part IV of he course.

Subsection 18.E of MWG contains more details on this.

### 3.1.5 Appendix: Aggregate Demand, Excess Demand, and the Set of Equilibria

Let $\boldsymbol{x}_i(\boldsymbol{p}, w_i)$ denote the Walrasian demand of consumer $i$. In Chapter 4 of MWG, you saw that it is not easy to find conditions that give a nice interpretation to the aggregate demand

$$\boldsymbol{x}(\boldsymbol{p}, w) = \sum_{i \in \mathcal{N}} \boldsymbol{x}_i(\boldsymbol{p}, w_i).$$

If you want $\boldsymbol{x}$ to depend on prices and aggregate wealth ($w = \sum_i w_i$) only, then you are restricted to utility functions giving rise to the Gorman form indirect utility:

$$v_i(\boldsymbol{p}, w_i) = a_i(\boldsymbol{p}) + b(\boldsymbol{p}) w_i,$$

i.e. wealth affects the indirect utility of all agents in the same linear fashion.

It is not enough that the individual demand functions satisfy the compensated law of demand (or WARP) for the aggregate demand to satisfy WARP. If individual demands satisfy uncompensated law of demand, then the aggregate demand also satisfies uncompensated law of demand and hence also WARP.

On a more positive note, aggregate demand has some properties that individual demands do not have. In particular aggregating over individual consumers reduces non-convexities in demand. This is very important for the analysis of general equilibrium models with e.g. discrete goods (such as the houses in Section 2 of these notes). In order to show the full power of aggregation in this respect would necessitate talking about economies with a continuum of non-atomic agents. See Kreps 13.3 for some details on this (if interested).

With regard to aggregate excess demand functions, classical consumer theory does not put many restrictions.

**Theorem 3.7.** Let $\zeta : \Delta^L \to \mathbb{R}^L$ be a continuous and homogenous of degree 0 function satisfying Walras' law. The for all $\epsilon > 0$, there are $L$ agents with continuous, strictly convex and monotone preferences $\succeq_i$ and initial endowments $\boldsymbol{\omega}_i$ such that $\zeta$ is the aggregate excess demand of the economy $(\{1, ..., L\}, \{\succeq_i\}_{i\in\{1,...,L\}}, \{\boldsymbol{\omega}_i\}_{i\in\{1,...,L\}})$ on the set $\{\boldsymbol{p} \in \Delta |\, p_i > \epsilon \text{ for all } i\}$.

Finally, here is a very disappointing result concerning the predictive power of general equilibrium theory.

**Theorem 3.8.** Let $Q$ be a non-empty closed subset in the interior of $\Delta$. Then there is an exchange economy $(\mathcal{N}, \{\succeq_i\}_{i\in\mathcal{N}}, \{\boldsymbol{\omega}_i\}_{i\in\mathcal{N}})$ with continuous, strictly convex and monoton preferences such that $Q$ is the set of competitive equilibrium prices of that economy.

MWG contains a lot more information on the mathematical properties of competitive equilibria. For example, in the typical case the competitive equilibria are locally isolated and therefore in principle, comparative statics analysis is possible. Typically also the set of competitive equilibria is finite. In my view, these results are of limited interest since there are no constructive ways for computing the equilibria.

The message of these last two results is that if you want to get real results, you must be willing to make some assumption. This is what we will do after dealing with the productions side of the economy in the next subsection.

## 3.2 Economies with Production

### 3.2.1 Theory of the Competitive Firm

I should start by noting that we are not really talking about firms as organizational entities consisting of many individuals with possibly differing views on the best way to operate. We take the black-box approach where a productive unit is characterized by its ability to transform some goods (inputs and intermediate products) into final goods. We call these units competitive firms because we emphasize profit maximization as the behavioral objective for the units.

There are $K$ firms and they operate in an economy with $L$ goods. The production possibilities of firm $j$ are summarized by a *production set* $Y_j \subset \mathbb{R}^L$. We assume that $Y_j$ is a closed and convex set for each $j$, and $\{0\} \in Y_j$ for all $j$. This last assumption is interpreted as giving each firm the option of not operating.

To interpret the meaning of the $Y_j$, negative components in $\boldsymbol{y}_j \in Y_j$ are understood to be inputs and positive components are the outputs of the technology. Convexity rules out increasing returns to scale. You should check MWG chapter 5 for additional details.

It is also assumed that $Y_j \cap \mathbb{R}^L_+ = \{0\}$ so that it is not possible to positive amounts of any good without using some inputs. Another typical assumption is that the sets $Y_j$ are downward comprehensive, i.e.

$$\boldsymbol{y}_j \in Y_j \text{ and } \boldsymbol{y}'_j \leq \boldsymbol{y}_j \implies \boldsymbol{y}'_j \in Y_j.$$

This is essentially an assumption guaranteeing free disposal.

The boundary points $\{\partial Y_j\}$ of $Y_j$ form the *efficient boundary* of the production set. This means that:

$$\boldsymbol{y}_j \in \partial Y_j \iff \{\{\boldsymbol{y}_j\} + \mathbb{R}^L_+\} \cap Y_j = \{\boldsymbol{y}_j\}.$$

The objective of firm $j$ is to find a profit maximizing vector $\boldsymbol{y}^*_j \in Y_j$. The profit depends on the price vector $\boldsymbol{p} \geq 0$ prevailing in the economy. A competitive firm acts as a price taker. In other words,

$$\boldsymbol{y}^*_j \in \arg\max_{\boldsymbol{y}_j \in Y_j} \boldsymbol{p} \cdot \boldsymbol{y}_j.$$

Notice that the firm's problem has no (price dependent) budget set and as a result, the comparative statics of the firm's optimal decisions are very easy in comparison to consumer choice. If the production sets are strictly convex, then the profit maximizing production vector is single-valued and continuous in prices. Notice also that the optimal production vector is homogenous of degree zero in prices.

It is a useful exercise to recall the production Edgeworth-Box where two firms produce a single output from two inputs or factors (say labor and capital). Place the input allocations to the firms on the two axes and let the total factor endowments determine the size of the box. Put the firms at the corners of the box and draw the isoquants inside the box. An input allocation is *production-efficient* if there is no other feasible factor allocation that results in at least as high production of both final goods and strictly more production for at least one final good.

In order to understand general equilibrium with production, it is a good idea to recall from Intermediate Microeconomics that for competitive firms, marginal revenue product of each factor is equalized to the factor price. All firms (using positive amounts of both factors) have the same marginal rate of technical substitution (i.e. their isoquants are tangent to each other in the production Edgeworth Box).

A final consequence of competitive profit maximization is that the marginal rate of transformation (MRT) between the final goods is equal to their price ratio (for strictly positive levels of production). Since optimality on the consumer side for strictly positive consumption levels is achieved where marginal rate of substitution (MRS) equals the price ratio, we get the familiar characterization of competitive equilibrium as the requirement that MRS = MRT.

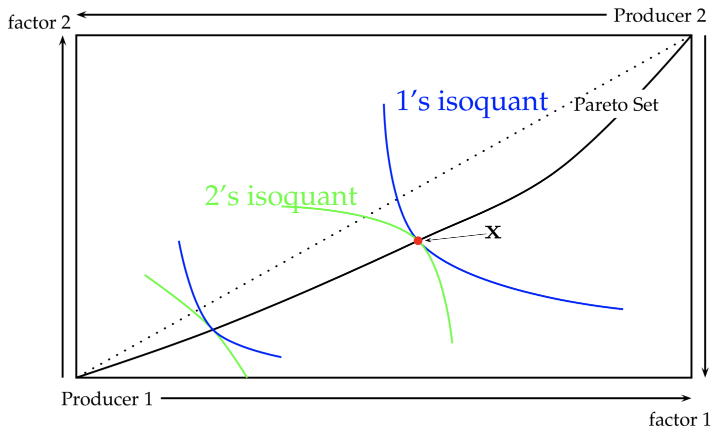


Figure 11: Production Edgeworth Box

### 3.2.2 Competitive Equilibrium in Production Economies

In order to complete the description of an economy with production, we need to take a stance on what happens to the profits that the firms generate. We denote the set of firms in the economy by $\mathcal{K} := \{1, ..., k\}$ and we assume that the agents (i.e. the consumers) own the firms. Let $\theta_{ij} \geq 0$ be agent $i$'s ownership share in firm $j$ (so that $\sum_{i \in \mathcal{N}} \theta_{ij} = 1$ for all $j$). The idea is that the budget set of each agent is determined by her income from her endowment (including labor endowment) and her capital income (dividends) from the ownership of the firms.

With these primitives, we can define formally a production economy.

**Definition 3.5.** A *production economy* is a collection of agents, firms, preferences, initial endowments, ownership shares and production technologies

$$(\mathcal{N}, \mathcal{K}, (\succeq_i)_{i \in \mathcal{N}}, (\boldsymbol{\omega}_i)_{i \in \mathcal{N}}, (\theta_{ij})_{i \in \mathcal{N}, j \in \mathcal{K}}, (Y_j)_{j \in \mathcal{K}}).$$

A production vector for the entire economy is $\boldsymbol{y} \in \mathbb{R}^{kL}$ so that $\boldsymbol{y} = (\boldsymbol{y}_1, ..., \boldsymbol{y}_k)$ and $\boldsymbol{y}_j = (y_{j1}, ..., y_{jL})$. The definition of a competitive equilibrium for a production economy extends the definition for exchange economies as follows.

**Definition 3.6.** A *competitive equilibrium* of the production economy

$$(\mathcal{N}, \mathcal{K}, (\succeq_i)_{i\in\mathcal{N}}, (\boldsymbol{\omega}_i)_{i\in\mathcal{N}}, (\theta_{ij})_{i\in\mathcal{N}, j\in\mathcal{K}}, (Y_j)_{j\in\mathcal{K}})$$

is a triple $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{y})$, where $\boldsymbol{p} \in \mathbb{R}^L_+$ is a price vector, $\boldsymbol{x} \in \mathbb{R}^{nL}_+$ is a consumption allocation and $\boldsymbol{y} \in \mathbb{R}^{kL}_+$ is a production vector such that:

i) For all $i \in \mathcal{N}$, $\boldsymbol{x}_i$ solves:

$$\max_{\boldsymbol{x}'_i \in \mathbb{R}^L_+} u_i(\boldsymbol{x}'_i) \text{ subject to } \boldsymbol{p} \cdot \boldsymbol{x}'_i \leq \boldsymbol{p} \cdot \boldsymbol{\omega}_i + \sum_{j\in\mathcal{K}} \theta_{ij} \boldsymbol{p} \cdot \boldsymbol{y}_j.$$

ii) For all $j \in \mathcal{K}$, $\boldsymbol{y}_j$ solves:

$$\max_{\boldsymbol{y}'_j \in Y_j} \boldsymbol{p} \cdot \boldsymbol{y}'_j.$$

iii) Markets clear, i.e. for all $l$,

$$\sum_{i=1}^{n} x_{il} \leq \sum_{i\in\mathcal{N}} \omega_{il} + \sum_{j\in\mathcal{K}} y_{jl}.$$

The first question to settle is whether economies with production have competitive equilibria. Luckily enough, existence follows from our previous existence theorem for exchange economies if the aggregate excess demand function of the production economy satisfies the properties in Theorem 3.2.

**Definition 3.7.** The *aggregate excess demand with production* $\boldsymbol{z}^P(\boldsymbol{p})$ of the production economy $(\mathcal{N}, \mathcal{K}, (\succeq_i)_{i\in\mathcal{N}}, (\boldsymbol{\omega}_i)_{i\in\mathcal{N}}, (\theta_{ij})_{i\in\mathcal{N}, j\in\mathcal{K}}, (Y_j)_{j\in\mathcal{K}})$ is defined as:

$$\boldsymbol{z}^P(\boldsymbol{p}) = \sum_{i\in\mathcal{N}} \boldsymbol{x}_i(\boldsymbol{p}, \boldsymbol{p} \cdot \boldsymbol{\omega}_i + \sum_{j\in\mathcal{K}} \theta_{ij} \boldsymbol{p} \cdot \boldsymbol{y}_j) - \sum_{i\in\mathcal{N}} \boldsymbol{\omega}_i - \sum_{j\in\mathcal{K}} \boldsymbol{y}_j(\boldsymbol{p}).$$

**Proposition 3.2.** The aggregate excess demand with production $z^P(\boldsymbol{p})$ satisfies properties i)-v) for of Theorem 3.2 for $\boldsymbol{p} > 0$ if the preferences are strictly convex and locally non-satiated, the production sets are strictly convex and bounded from above.

*Proof.* The proof is similar to that of Theorem 3.2 and left as an exercise. □

With this result, we get the main existence theorem for production economies by showing that there are prices $\boldsymbol{p}$ where the aggregate excess demand vanishes.

**Theorem 3.9** (Existence of Competitive Equilibrium)**.** Suppose that the aggregate demand $\boldsymbol{z}^P$ of the production economy

$$(\mathcal{N}, \mathcal{K}, (\succeq_i)_{i\in\mathcal{N}}, (\boldsymbol{\omega}_i)_{i\in\mathcal{N}}, (\theta_{ij})_{i\in\mathcal{N}, j\in\mathcal{K}}, (Y_j)_{j\in\mathcal{K}})$$

satisfies the conditions i)-v) of Theorem 3.2. Then a competitive equilibrium $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{y})$ exists.

The proof is the same as for exchange economies and therefore not repeated.

I will not go over the entire formalism for the two welfare theorems since the essential idea shown for exchange economies carries over to production economies as well. I will provide short sketches of the proofs for the theorems. Let $Y := Y_1 + Y_2 + ... + Y_k \subset \mathbb{R}^L$ be the aggregate production set of the economy (where the summation is the direct sum as in the definition of the aggregate consumption set before).

**Theorem 3.10** (First Welfare Theorem for Production Economies)**.** Suppose all agents in an a production economy have locally non-satiated preferences. Then all competitive equilibrium consumption allocations of the economy are Pareto-efficient.

*Proof.* (Sketch:) If $(\boldsymbol{p}^*, \boldsymbol{x}^*, \boldsymbol{y}^*)$ is a competitive equilibrium and consumption allocation and $\boldsymbol{x}$ Pareto-dominates $\boldsymbol{x}^*$, then $\boldsymbol{p}^* \cdot \sum_i \boldsymbol{x}_i^* < \boldsymbol{p}^* \cdot \sum_i \boldsymbol{x}_i$.

If $\boldsymbol{x}$ is feasible, then $\sum_i \boldsymbol{x}_i = \sum_i \boldsymbol{\omega}_i + \sum_j \boldsymbol{y}_j$ for some $\boldsymbol{y} \in Y$.

Taking together these two facts gives:

$$\boldsymbol{p}^* \cdot \sum_j \boldsymbol{y}_j > \boldsymbol{p}^* \cdot \sum_j \boldsymbol{y}_j^*$$

contradicting profit maximization by all firms at $\boldsymbol{y}^*$. □

The second welfare theorem generalizes in the same way. To make the statement of the theorem simpler, I include the initial endowment point in the aggregate production setso that $Y^{\boldsymbol{\omega}_\Sigma} = \{\sum_i \boldsymbol{\omega}_i\} + Y$. As before, let $\boldsymbol{x}_\sum = \sum_i \boldsymbol{x}_i$ and $\boldsymbol{y}_\sum = \sum_j \boldsymbol{y}_j$.

**Theorem 3.11** (Second Welfare Theorem for Production Economies)**.** Suppose that $\boldsymbol{x}^* \in \mathbb{R}^L_{++}$ is a Pareto-efficient consumption allocation in a production society with aggregate production set $Y^{\boldsymbol{\omega}_\Sigma}$ and strictly convex and strictly monotone preferences. Then there is a non-zero price vector $\boldsymbol{p} \geq 0$ and a $\boldsymbol{y}^* \in Y^{\boldsymbol{\omega}_\Sigma}$ such that $(\boldsymbol{p}^*, \boldsymbol{x}^*, \boldsymbol{y}^*)$ is a competitive equilibrium for some initial endowments and ownership shares.

*Proof.* (Idea:) Both $S(\boldsymbol{x}^*)$ (as defined in the proof for exchange economies) and $Y^{\boldsymbol{\omega}_\Sigma}$ are closed convex sets. Since $\boldsymbol{x}^*$ is Pareto-efficient, their intersection has no interior points. Hence the separating hyperplane theorem guarantees the existence of non-zero $\boldsymbol{p}^*$ such that for all $\boldsymbol{x}_\sum \in S(\boldsymbol{x}^*)$ and all $\boldsymbol{y}_\sum \in Y^{\boldsymbol{\omega}_\Sigma}$,

$$\boldsymbol{p}^* \cdot \boldsymbol{x}_\sum \geq \boldsymbol{p}^* \cdot \boldsymbol{x}_\sum^* \geq \boldsymbol{p}^* \cdot \boldsymbol{y}_\sum.$$

Strict monotonicity of preferences (and the fact that $\boldsymbol{x}^* \in \mathbb{R}^L_{++}$) can be used again to show that $\boldsymbol{p}^* > 0$. The rest is an exercise in accounting to distribute the ownership shares and endowments in a way that balances the individual budgets for all agents. □

**Remark.** 1. The existence proof can be extended to cover unbounded $Y$ as in the case with constant returns to scale. In that case, optimal profit must be zero and this helps in the analysis. The general existence result in Debreu (1960) covers this case as well.

2. The same disappointing conclusions from exchange economies regarding the predictive power of the production model remain valid here.

3. Firms do not make decisions, their owners do. In the next section, we see how to incorporate decisions under uncertainty into the competitive model. If the owners have different views regarding the underlying probabilities, the objective of profit maximization is not well defined (whose probability assessments should be reflected)?

4. The modern theory of the firm as an organization where workers may have differing objectives and different information is very different from the model of price taking profit maximization analyzed here. The tools of information economics as developed in Microeconomic Theory IV are more relevant for the modern view of the firm.

# 4 Competitive Equilibrium Analysis

## 4.1 Cobb-Douglas Exchange Economies

An exchange economy with Cobb-Douglas preferences consists of $n$ agents, each with an initial endowment vector $\boldsymbol{\omega}_i \in \mathbb{R}^L_+$. The preferences of agent $i$ have the utility represenation:

$$u_i(\boldsymbol{x}_i) = \sum_{j=1}^{L} \alpha_{ij} \ln(x_{ij}),$$

where for all $i$ and $j$, $\alpha_{ij} \geq 0$ and $\sum_j \alpha_{ij} = 1$.

A competitive equilibrium in a pair $(\boldsymbol{p}^*, \boldsymbol{x}^*)$ such that the agents maximize their utility in $B(\boldsymbol{p}^*, \boldsymbol{\omega}_i)$ and markets clear:

$$\sum_i \boldsymbol{x}_i^* \leq \sum_i \boldsymbol{\omega}_i.$$

Since competitive equilibria are Pareto-efficient, the inequality must be binding at competitive allocations $\boldsymbol{x}^*$.

From Advanced Microeconomics 1, you will recall that the optimal demands for good $j$, $x_{ij}(\boldsymbol{p}, \boldsymbol{\omega}_i)$, for strictly positive $\boldsymbol{p}$ are given by:

$$x_{ij}\left(\boldsymbol{p}, \omega_i\right) = \frac{\alpha_{ij}\left(\boldsymbol{p} \cdot \boldsymbol{\omega}_i\right)}{p_j}.$$

Market clearing for $x_j$ is written as:

$$\sum_{i=1}^{n} \frac{\alpha_{ij}\left(\boldsymbol{p} \cdot \boldsymbol{\omega}_i\right)}{p_j} = \sum_{i=1}^{n} \omega_{ij}.$$

Multiplying both sides by $p_j$ transforms this into a linear equation in $\boldsymbol{p}$:

$$\sum_{i=1}^{n} \alpha_{ij} \boldsymbol{p} \cdot \boldsymbol{\omega}_i = \sum_{i=1}^{n} \omega_{ij} p_j. \tag{2}$$

Hence we have a market clearing price at a non-zero solution $\boldsymbol{p}^*$ to the system of equations:

$$\mathcal{A}\boldsymbol{p}^* = 0,$$

where $\mathcal{A}$ is given by:

$$\mathcal{A} = \begin{pmatrix} \sum_{i=1}^n \omega_{i1} - \sum_{i=1}^n \alpha_{i1}\omega_{i1} & \cdots & -\sum_{i=1}^n \alpha_{i1}\omega_{iL} \\ \vdots & \ddots & \vdots \\ -\sum_{i=1}^n \alpha_{iL}\omega_{i1} & \cdots & \sum_{i=1}^n \omega_{iL} - \sum_{i=1}^n \alpha_{iL}\omega_{iL} \end{pmatrix}.$$

**Exercise:** How do you see that the system has a non-zero solution? What is the economic reason for having multiple equilibrium prices? Can you show that the system has rank $L-1$? Can you see this as a problem of finding an eigenvector?

For the case with only two goods, you can compute from Equation (2) the condition for market clearing for $x_1$ :

$$\sum_{\iota=1}^{n} \frac{\alpha_{i1}\left(p_1\omega_{i1}+p_2\omega_{i2}\right)}{p_1}=\sum_{i=1}^{n}\omega_{i1}.$$

Rearranging this yields:

$$\sum_{i=1}^{n}\left(1-\alpha_{i1}\right)p_1\omega_{i1}=\sum_{i=1}^{n}\alpha_{i1}p_2\omega_{i2},$$

so that:

$$\frac{p_1}{p_2}=\frac{\sum_{i=1}^{n}\alpha_{i1}\omega_{i2}}{\sum_{i=1}^{n}\alpha_{i2}\omega_{i1}}.$$

Walras' Law implies that the market for $x_2$ clears as well. Finally, you can find the equilibrium allocation by substituting the equilibrium prices $\boldsymbol{p}^*$ into the agents' optimal demand functions to get:

With two goods, it is easy to calculate the effects of e.g. increasing the aggregate endowment of one good on the prices and equilibrium consumption and to do other comparative statics exercise based on this simple model. The model with more goods forms the basis of many computational equilibrium models used in policy analysis.

## 4.2 Exchange Economies under Uncertainty

### 4.2.1 Contingent Good markets and Equilibrium Risk Sharing

One of the most prominent applications of the competitive equilibrium model is into markets under uncertainty and financial markets. In this subsection, we cover the first steps into modeling markets under uncertainty often called markets for *contingent commodities*.

The economy consists of $n$ agents facing uncertainty regarding the future state of the economy. As discussed in Advanced Microeconomics 1, these situations can be modeled with a state space $\mathcal{S}$ consisting of $S$ states: $s \in \{1, ..., S\}$ . At the outset, the agents are uncertain about which state $s$ is going to be realized.

There is a single physical good, $x$, and the agents have strictly increasing preferences in $x$. Consumption of $x$ in state $s$ by agent $i$ is denoted by $x_{is}$. We interpret $x_{is}$ as the consumption in t he contingency that state $s$ is realized and hence the name contingent goods and contingent markets.

The initial endowment of agent $i$ in state $s$ is denoted by $\omega_{is}$ and the agents are assumed to have well-defined von Neumann-Morgenstern utility functions. Let $u_i$ denote the (same) utility function for consumption in each state, and by assuming separability across states. Let $\pi = (\pi_1, ..., \pi_S)$ denote the objective probability distribution on the set of states, we write:

$$U_i\left(x_{i1}, ..., x_{iS}\right) = U_i\left(\boldsymbol{x}_i\right) = \sum_{s=1}^{S} \pi_s u_i\left(x_{is}\right),$$

where $u_i : \mathbb{R}_+ \to \mathbb{R}$ is a strictly increasing, strictly concave and continuously differentiable function.

For the first analysis of the model, assume that there is no aggregate risk, i.e.

$$\sum_{i=1}^{n} \omega_{is} = \overline{\omega} \text{ for all } s.$$

The relevant consumption set for the agents in this example is $\mathbb{R}_+^S$.

Financial markets are represented by trades across the states. The agents decide how to trade away from their initial endowment by choosing a vector of consumptions $\boldsymbol{x}_i$. This assumes the existence of a perfect financial market that allows for trades that shift wealth across the different states. With this assumption choosing $x_{is}$ does not differ conceptually from the choice of $x_{il}$ in the previous subsection. In the next subsection, we discuss how asset markets can be modeled to facilitate these trades across states.

A competitive equilibrium for this economy is $(\boldsymbol{p}, \boldsymbol{x})$ , where $\boldsymbol{p} \in \mathbb{R}_+^S$ and $\boldsymbol{x} = (\boldsymbol{x}_1, ..., x_n) \in \mathbb{R}^{nS}$ such that

$i)$ For all $i,\ \boldsymbol{x}_i$ maximizes $U\left(\boldsymbol{x}_i\right)$ s.t. $p \cdot \boldsymbol{x}_i \leq p \cdot \boldsymbol{\omega}_i$.

$$ii) \text{ For all } s, \ \sum_{i=1}^{n} x_{is} = \sum_{1=1}^{n} \omega_{is} = \overline{\omega}.$$

To characterize the competitive equilibria of the model, use again the welfare theorems. By the first welfare theorem, competitive allocations are Pareto-efficient. The first characterization of Pareto-efficient points follows almost immediately from the assumed concavity of the utility functions.

Consider an arbitrary feasible allocation $\boldsymbol{x}$. Suppose that $x_{is} \neq x_{is'}$ for some $i$ and for some $s, s'$. Compare this allocation to $\boldsymbol{x}_i^*$, where $(x_{is})^* = \sum_s \pi_s x_{is}$.

In this allocation, each consumer consumes her average allocation in all states. Then

$$U_i(\boldsymbol{x}_i^*) = \sum_s \pi_s u_i \left( \sum_s \pi_s x_{is} \right) = u_i \left( \sum_s \pi_s x_{is} \right)$$
$$\geq \sum_s \pi_s u_i (x_{is}) = U_i (\boldsymbol{x}_i) .$$

The inequality follows from the strict concavity of $u^i$ and it is strict if $x_{is} \neq x'_{is}$ for some $s, s'$. Thus $\boldsymbol{x}^*$ Pareto dominates $\boldsymbol{x}$ if we can show that $\boldsymbol{x}^*$ is also feasible.

To see this, note that

$$\sum_{i=1}^{N} x_{is}^* = \sum_i \sum_s \pi_s x_{is} = \sum_s \pi_s \sum_i x_{is} = \sum_s \pi_s \overline{\omega}_s = \overline{\omega}.$$

Hence the only Pareto optimal allocations have all individuals perfectly insured. By first welfare theorem, we know that all competitive allocations must then have $x_{is} = x_{is'}$ for all $i, s, s'$.

First-order conditions for optimal consumer demand imply:

$$\frac{\pi_s u'(x_{is})}{\pi_{s'} u'(x_{is'})} = \frac{p_s}{p_{s'}}.$$

But then, we must have:

$$\frac{p_s}{p_{s'}} = \frac{\pi_s}{\pi_{s'}}.$$

Obviously, full insurance for all agents is not possible if $\overline{\omega}_s \neq \overline{\omega}_{s'}$ for some $s, s'$. Let's see, how efficient risk sharing works with two agents and two states. For interior choices $x_{is} > 0$ for $i \in \{1, 2\}, s \in \{1, 2\}$, we have:

$$\frac{u_1'(x_{11})}{u_1'(x_{12})} = \frac{u_2'(\overline{\omega}_1 - x_{11})}{u_2'(\overline{\omega}_2 - x_{12})}.$$

**Exercise:**

1. What happens if agent 2 is risk-neutral? Draw the Edgeworth box for the case with $u_1(x) = \ln(x), u_2(y) = y$, and initial endowments $\boldsymbol{\omega}_1 = (1, 1),\ \boldsymbol{\omega}_2 = (2, 1)$, and $\boldsymbol{\pi} = (\frac{2}{5}, \frac{3}{5})$ and solve for the competitive equilibrium prices and allocation.

2. Do the same exercise except that $u_2(y) = \ln(y)$.

### 4.2.2 Financial Markets

In order to explain in the simplest possible terms how financial markets can be introduced to the general equilibrium framework, we start with a model with two dates $t = 0, 1$. Date $t = 0$ is just a planning stage where a decision regarding consumption in $t = 1$ has to be taken. Just like in the previous subsection, there is a single consumption good.

The state of the economy (in particular, the initial endowments) in $t = 1$ is uncertain at $t = 0$ and there are $S$ possible states. The objective probability distribution over the states is given by a non-negative vector $\boldsymbol{\pi} = (\pi_1, ..., \pi_S)$ whose components sum to 1.

An *asset* is a legal title to receive an amount $r_s$ of the good in state $s$. Let $\boldsymbol{r} := (r_1, ..., r_S)$ denote the return vector characterizing the asset. A financial market is a market where a collection of assets is traded.

**Example 4.1.** 1. An asset is called the safe asset if $\boldsymbol{r} = (1, 1, ..., 1)$, i.e. its return does not depend on state. This is essentially a commodity future and it represents a riskless asset in a world where the utility function is the same for all states and where we have a single consumption good.

2. An asset $\boldsymbol{r} = (0, ..., 0, 1, 0, ..., 0)$, where $r_s = 1$ for one state and zero for all others. The contingent commodity market in the previous subsection essentially involved trades in the $S$ different assets of this type. They are called *Arrow Securities*.

3. A *Call Option* on a *primary asset* with return vector $\boldsymbol{r}$ at the strike price $c \in \mathbb{R}$ is a *derivative asset* has a return vector:

$$\boldsymbol{r}(c) = (\max\{0, r_1 - c\}, ..., \max\{0, r_S - c\}).$$

   The interpretation is that the option gives the right to buy at the strike price the returns of the underlying primary asset after the state has been realized. Hence the option is exercised only if $r_i > c$. For example for the primary asset $(3, 5, 9)$, the call option at different strike prices has the return vectors:

$$\boldsymbol{r}(4) = (0, 1, 5),\ \boldsymbol{r}(6) = (0, 0, 3).$$

Assume next that there are $K$ different assets available. For each asset $k \in \{1, ..., K\}$, the return vector is $\boldsymbol{r}_k \in \mathbb{R}^S$. When moving towards an asset market, we assume that the assets are in zero net supply (i.e. there are no initial endowments of assets) and we allow all trades in the assets (in particular, short sales are allowed). Assets $k$ is traded in $t = 0$ at price $q_k$ so that the vector of *asset prices* for the economy is $\boldsymbol{q} := (q_1, ..., q_K)$.

Each agent can trade freely in these assets and the vector of asset demands $\boldsymbol{z} \in \mathbb{R}^K$ is called the *portfolio* of the agent. Let $\mathbf{R}$ denote the $S \times K$ matrix whose column $k$ is the return vector of asset $k$. By holding a portfolio $\boldsymbol{z} = (z_1, ..., z_k)$, an agent is entitled to consumption $\sum_k r_{ks} z_k$ in state $s$, i.e. a vector $\mathbf{R}\boldsymbol{z}$ of consumption goods across the states.

With strictly increasing utilities in the consumption good, the portfolio $\boldsymbol{z}_i$ fixes the period $t = 1$ consumptions as:

$$x_{is} = \omega_{is} + \sum_{k=1}^{K} r_{ks} z_{ik} \text{ for all } s,$$

or in vector form:

$$\boldsymbol{x}_i = \boldsymbol{\omega}_i + \mathbf{R}\boldsymbol{z}_i.$$

Hence the agent $i$'s problem is to:

$$\max_{\boldsymbol{z}_i} \sum \pi_s u_i(x_{is})$$

subject to:

$$\boldsymbol{q} \cdot \boldsymbol{z}_i \leq 0,$$

$$\boldsymbol{x}_i = \boldsymbol{\omega}_i + \mathbf{R}\boldsymbol{z}_i.$$

A portfolio $\boldsymbol{z}$ is called an *arbitrage portfolio* if $\boldsymbol{q} \cdot \boldsymbol{z} \leq 0$ and $\mathbf{R}\boldsymbol{z} \geq 0$ with $\mathbf{R}\boldsymbol{z} \neq 0$. The agent's problem does not have a solution if $u_i$ is strictly increasing and if an arbitrage portfolio exists. Therefore we assume that no arbitrage portfolios exist. We call asset prices $\boldsymbol{q}$ *arbitrage free* if no arbitrage portfolio exist for $\boldsymbol{q}, \mathbf{R}$.

**Proposition 4.1.** Assume that $r_{ks} \geq 0$ for all $k, s$, $\boldsymbol{r}_k \neq 0$ for all $k$ and that for all $s$, there is a $k$ such that $r_{ks} > 0$. Then if $\boldsymbol{q}$ is arbitrage free, there is a vector $\boldsymbol{\mu} \geq 0$ such that

$$\boldsymbol{q}^\top = \boldsymbol{\mu} \cdot \mathbf{R}.$$

*Proof.* Let $V = \{v|\ v = \mathbf{R}\boldsymbol{z} \text{ for some } \boldsymbol{z} \text{ such that } \boldsymbol{q} \cdot \boldsymbol{z} \leq 0\}$. Since $\boldsymbol{q}$ is arbitrage free, $V \cap \mathbb{R}^S_+ \setminus \{0\} = \emptyset$.

Since the origin is both in $V$ and in the closure of $\mathbb{R}^S_+ \setminus \{0\}$, the separating hyperplane theorem implies the existence of $\boldsymbol{\mu}' \neq 0$ such that $\boldsymbol{\mu}' \cdot v \leq 0$ for all $v \in V$ and $\boldsymbol{\mu}' \cdot v \geq 0$ for all $v \in \mathbb{R}^S_+$. Since $v \in V \implies -v \in V$, we have $\boldsymbol{\mu}' \cdot v = 0$ for all $v \in V$. Hence $V$ is a linear subspace of dimension at most $K - 1$.

If $\boldsymbol{q}^\top$ is not proportional to $\boldsymbol{\mu}' \cdot \mathbf{R}$, then there is a $\boldsymbol{z}'$ with $\boldsymbol{q} \cdot \boldsymbol{z}' = 0$, but $\boldsymbol{\mu}' \cdot \mathbf{R}\boldsymbol{z}' > 0$. Setting $v = \mathbf{R}\boldsymbol{z}'$, we have a contradiction. □

This result is one of the most fundamental building blocks for any theory of financial markets. The interpretation for $\mu_s$ is as the value of consumption in state $s$ and this allows one to calculate the value of an asset $k$ by simply computing $\sum_s r_{ks}\mu_s$. No arbitrage requires that this value equal the price of the asset.

Note that $\boldsymbol{q}$ need not be uniquely determined unless $\mathbf{R}$ has full rank. If $\mathbf{R}$ has full rank, then $V = \mathbb{R}^S$ and therefore all distributions of consumption across states are possible and the consumption choices and therefore also the competitive equilibria of the model coincide with those obtained in the model with free trade of contingent commodities.

The lesson from this section is that for a financial market, the important concept is $V$, i.e. the set of consumption vectors in the span of the matrix $\mathbf{R}$.

**Remark.**

1. If $V$ does not have full rank, we say that the market is *incomplete*. In this case, competitive equilibrium consumption allocations will not coincide with the equilibria of the full contingent commodity model in general. The asset structure just does not allow all feasible reallocations of consumptions. As a result, one cannot expect full Pareto-effciciency of the allocations in incomplete market models.

2. The results go through in models with multiple goods for consumption in $t = 1$. One difference that does arise is that assets with returns in fiat money are in general different from assets with real returns (Lucas trees). Many of the problems discovered in the analysis of incomplete models with real returns vanish with financial asset markets. These issues are quite subtle, but they are suitable material for the course essay.

3. The basic structure of the model is also unchanged if we have many periods of consumption. The notion of equilibrium has to be extended to include all dates and all states and rational expectations about future prices. In other words, the agents optimize given the correctly anticipated current and future prices and markets clear at these prices.

#### 4.2.3 Price Reactions to Information with Heterogenous Priors

In this last subsection on markets under uncertainty we touch upon the possibility that agents might have different subjective beliefs on the likelihood of the different states. We consider a betting market where agents can take bets on a binary state of the world. The presentation follows very closely Ottaviani and Sörensen (2015).

**Events.** Agents take positions on whether or not a binary event, $A$, is realized (e.g., Olli Rehn is the next President). There are two Arrow securities corresponding to the two possible realizations: one asset pays out 1 unit of cash if event $A$ is realized and 0 otherwise, while the other asset pays out 1 cash unit if the complementary event $A^C$ is realized and 0 otherwise.

**Wealth.** There is a unit mass of competitive, risk-neutral agents. Wealth in this market is bounded, as each agent $i$ initially holds a fixed endowment $w_{i0}$ of each asset. Agents trade with other traders in a competitive market for the Arrow securities. Short sales of the assets are banned, so there is an endogenous upper bound on the number of asset units that each individual can can purchase and eventually hold.

**Priors.** Initially, agent $i$ has subjective prior belief $q_i$ on event $A$. The initial distribution of assets over individuals is described by the cumulative distribution function $G$. Thus $G(q) \in [0, 1]$ denotes the share of all assets initially held by individuals with subjective prior belief less than or equal to $q$. We assume that $G$ is continuous, and strictly increasing (no gaps, no atoms).

**Information.** Before trading, all agents observe the realization of a public signal $s$ with likelihood ratio $L(s) := \frac{f(s|A)}{f(s|A^C)} \in (0, 1)$ for event $A$, where $f(s|A)$ is the density of signal $s$ conditional on state $A$. By Bayes' rule, the subjective posterior belief $\pi_i$ of $i$ for state $A$ satisfies:

$$\frac{\pi_i(s)}{1 - \pi_i(s)} = \frac{q_i}{1 - q_i} L(s). \tag{3}$$

**Equilibrium.** Competitive agents take asset prices as given. We normalize the sum of the two asset prices to one, and focus on the price $p$ of the asset paying in event $A$. Agent $i$ chooses a feasible asset position $(w_i(A), w_i(A^c))$ to maximize subjective expected value

$$\pi_i w_i(A) + (1 - \pi_i) w_i(A^c).$$

With Arrow securities, $w_i(A), w_i(A^c)$ also denotes the event-dependent cash payout. Markets clear when the aggregate demand for each asset equals the aggregate endowment.

**Competitive Equilibrium.** Solving the competitive demand problem of the risk-neutral agents is straightforward. Let the public information be realized with likelihood ratio $L$, and consider agent $i$ with posterior belief $\pi_i$ the posterior as computed above for $L(s) = L$.

Given market price $p$, the subjective expected return on the asset that pays out in event $A$ is $\pi_i - p$ while the other asset's expected return is $(1 - \pi_i) - (1 - p) = p - \pi_i$. With the given bound on trades, risk-neutral demand thus satisfies the following:

i) If $\pi_i > p$, agent $i$ exchanges the entire endowment of the $A^C$ asset into $\frac{(1-p)w_{i0}}{p}$ units of the $A$ asset. The final portfolio is then $(w_i(A), w_i(A^C)) = (\frac{w_{i0}}{p}, 0)$.

ii) Conversely, when $\pi_i < p$, the agent's final portfolio is $(w_i(A), w_i(A^C)) = (0, \frac{w_{i0}}{(1-p)})$.

iii) Finally, when $\pi_i = p$, the agent is indifferent over all feasible trades.

Aggregate demand for the $A$ asset is then given by $\frac{1}{p}$ times the cumulative wealth of agents with posterior belief above $p$. Markets clear when this equals the aggregate endowment 1.

**Proposition 4.2.** The competitive equilibrium price, $p$, is the unique solution to the equation

$$p = 1 - G\left(\frac{p}{(1-p)L+p}\right), \tag{4}$$

and is a strictly increasing function of the information realization $L$.

**Underreaction to Information.** Inverting Bayes' rule in equation 3 after public information realization $L$, we can always interpret the price $p$ as the posterior belief of a hypothetical individual with initial belief $\frac{p}{[(1-p)L+p]}$. According to equation (4), this hypothetical individual is the marginal agent, and this initial belief might be interpreted as an aggregate of the heterogeneous subjective prior beliefs of the individual agents. However, this way of aggregating subjective priors cannot be separated from the realization of information.

The main result states that this initial belief of the marginal agent moves systematically against the public information available to agents. This systematic change in the market prior against the information implies that the market price underreacts to information.

Consider the inference of any outside observer with a fixed prior belief $q$. The observer's posterior probability, $\pi(L)$, for the event $A$ satisfies Equation (3), or

$$\log\frac{\pi(L)}{1-\pi(L)} = \log\frac{q}{1-q} + \log L.$$

The expression on the left-hand side is the posterior log-likelihood ratio for event $A$, which clearly moves one-to-one with changes in $\log L$. Part (ii) of the following Proposition notes that the corresponding expression for the market price, $\log\frac{p(L)}{1-p(L)}$ does not possess this property, but rather moves less than one-for-one with the publicly observable $\log L$.

**Proposition 4.3.** Suppose that beliefs are truly heterogeneous, i.e., the distribution $G$ is non- degenerate.

(i) The marginal trader moves opposite to the information, i.e., the implied ex ante market belief

$$\frac{p}{[(1-p)L+p]}$$

is strictly decreasing in L.

(ii) The market price underreacts to initial information: for any pair $L' > L$ we have

$$\log L' - \log L > \log \frac{p(L')}{1-p(L')} - \log \frac{p(L)}{1-p(L)} > 0.$$

To understand the intuition for part (i), consider what happens when public information is more favorable to event $A$ (corresponding, say, to Olli Rehn winning). Naturally, by (4) the price $p$ for asset $A$ is higher when $L$ is higher. The trading bound forces optimists (with high prior $q_i$) to purchase fewer units of asset $A$: the amount of $A$ assets which can be obtained through selling all the $A^C$ endowment is $\frac{(1-p)w_{i0}}{p}$, decreasing in $p$. If the marginal trader were unchanged at the higher price that results with higher $L$, there would be insufficient demand for the $A$ assets sold out by pessimists.

To balance the market it is necessary that some traders who were betting against Olli Rehn before now change sides and put their money on Rehn. In the new equilibrium, the price must thus move traders from the pessimistic to the optimistic side. Although $p$ rises with $L$, it rises more slowly than the posterior belief, because of this negative effect on the prior belief of the marginal trader.

The underreaction result hinges on the fact that the endogenous upper bound (equal to $\frac{w_{i0}}{p}$) on the individual position in asset $A$ is inversely related to its price. Later sections in the paper show that similar effects are at play when the trades are not bounded and have DARA preferences.

**Application to Prediction Markets.** Prediction markets are trading mechanisms that target unique events, such as the outcome of a presidential election or the identity of the winner in a sport contest. Because the realized outcomes are observed, these simple markets are useful laboratories for testing asset pricing theories.

Prediction markets typically set a maximum trade as the model above. According to the following corollary of the Proposition above, underreaction implies that $\pi(L) > p(L)$ when $p(L)$ is high (so that event A is a favorite) and $\pi(L) < p(L)$ when $p(L)$ is low (longshot).

**Corollary 4.1.** The market price exhibits a favorite-longshot bias, as there exists a price $p^* \in [0, 1]$ such that $p(L) > p^*$ implies $\pi(L) > p(L)$, and $p(L) < p^*$ implies $\pi(L) < p(L)$.

Thus, the favorite-longshot bias results, with longshot outcomes occurring less often than indicated by the price, while the opposite is true for favorites. The favorite-longshot bias is widely documented in the empirical literature on betting and prediction markets when comparing winning frequencies with market prices.

**Remark.** Additional topics in this area include Fully Revealing Rational Expectations Equilibrium, where privately informed agents learn the state of the world from price observations. In a sense, this literature tries to get at the Hayekian view that that prices are sufficient to aggregate all relevant information needed in the decision making of competitive agents. This approach runs to multiple difficulties:

1. Problems with existence of equililibrium: If trades reveal information, equilibrium trades may be such that prices do not reveal information, if prices do not reveal information, optimal trades may result require prices that reveal the state.

2. Costly information acquisition: Why pay if prices reveal information, but if no information is acquired, there is nothing for the prices to convey.

3. No-trade theorems starting with Milgrom and Stokey (1982) show common knowledge of equilibrium trades makes it impossible to have trades based on differential information (if investors have common priors on the underlying uncertainty). The literature on market microstructure has developed game theoretic models where trading is possible.

4. All of these topics are suitable for the course essay. A proper understanding of the issues alluded to needs the tools of information economics and game theory developed in Advanced Microeconomics 3 and 4 so it may make sense to wait until these courses.

## 4.3 Models of Trade

$2 \times 2$ **Production Economy**

We start with the model behind the production Edgeworth Box. Two output products $y_1$ and $y_2$ are produced using two factors: $z_1, z_2$. The production technologies are summarized by the production functions:

$$y_i = f^i (z_{i1}, z_{i2}) \text{ for } i = 1, 2,$$

where $z_{ij}$ denotes the amount of factor $j$ used in the production of output $i$. We will assume throughout that the production technologies have constant returns to scale, i.e. the production is homogenous of degree 1 in the inputs.

Total endowment of the factors to be allocated into the two production processes are given by:: $\overline{z}_j$ for $j = 1, 2$. Recalling the definition of productive efficiency from the previous section, we have:

**Definition 4.1.** A vector $y = (y_1, y_2)$ is *output efficient* if there is no other vector $y'$ such that $y' > y$ and such that

$$y'_i = f^i (z'_{i1}, z'_{i2}) \text{ for } i = 1, 2 \text{ and } z_{1j} + z_{2j} \leq \overline{z}_j \text{ for } j = 1, 2.$$

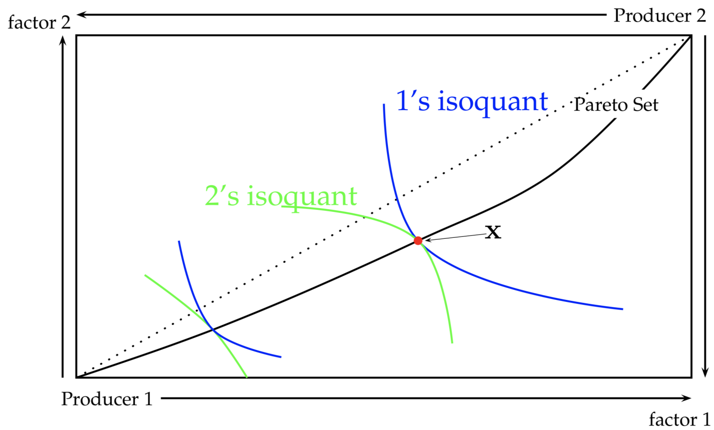


Figure 12: Production Edgeworth Box

**Equilibrium with production**

Assume that the economy in question is a small open economy. This means that output prices $p_1, p_2$ are fixed in the world market and they are exogenous to the production decisions in the small economy.

An assumption often made in old-style international trade models is that factors are not mobile across borders. This means that factor prices, $w_1$and $w_2$ are determined endogenously in the small economy.

Denote the optimal factor demands from optimal production taking both factor prices and output prices as given by:

$$z_{ij}\left(p_1, p_{2,} w_1, w_2\right) \text{ for } i = 1, 2 \text{ and } j = 1, 2.$$

Since the first welfare theorem implies efficient production and since the production functions are strictly increasing, the equilibrium factor prices $w_1, w_2$ are found where factor demands equal the factor resource:

$$\sum_i z_{ij}\left(p_1, p_{2,} w_1, w_2\right) = \overline{z}_j \text{ for } j = 1, 2.$$

Let's compute the equilibrium for the case of CES-production functions

$$y_i = z_{i1}^{\alpha^i} z_{i2}^{1-\alpha^i} \text{ for } i = 1, 2.$$

With constant returns to scale, only the ratio of factor demands for each output is determined for each firm. We also know that the optimal profit must equal zero if the optimal production quantity is strictly positive.

From the first-order conditions in the firms' problem, we get:

$$r_i = \frac{z_{i1}}{z_{i2}} = \frac{\alpha^i w_2}{(1-\alpha^i)\, w_1}.$$

The zero profit condition is:

$$p_i y_i - w_1 z_{i1} - w_2 z_{i2} = 0 \text{ for } i = 1, 2.$$

Dividing both sides by $z_{i2}$, we get:

$$p_i r_i^{\alpha^i} - w_1 r_i - w_2 = 0 \text{ for } i = 1, 2.$$

Substituting from the first-order conditions, we get:

$$p_i \left( \frac{\alpha^i w_2}{(1-\alpha^i)\, w_1} \right)^{\alpha^i} = \left( \frac{1}{1-\alpha^i} \right) w_2 \text{ for } i = 1, 2.$$

Solving from this we get the equilibrium ratio of factor prices as:

$$\frac{w_2}{w_1} = \left[ \left( \frac{1-\alpha^1}{1-\alpha^2} \right) \left( \frac{p_1}{p_2} \right) \left( \frac{\alpha^1}{1-\alpha^1} \right)^{\alpha^1} \left( \frac{\alpha^2}{1-\alpha^2} \right)^{-\alpha^2} \right]^{\frac{1}{\alpha^2-\alpha^1}}.$$

From this expression, we see that the factor endowments have no effect on the relative factor prices as long as both products are produced in strictly positive quantities so that the first-order conditions above are valid.

The comparative statics of the model with respect to i) output price increases and ii) factor endowments can be analyzed graphically in the Production Edgeworth Box. The first comparative statics result goes under the name of Samuelson-Stolper Theorem. The second is called Rybcszynski Theorem.

To see how to derive the Samuelson-Stolper Theorem, recall that for any CRS technology, the cost functions $\tilde{c}_i(w_1, w_2; q)$ for $i \in \{1, 2\}$ take the form:

$$\tilde{c}_i(w_1, w_2; q) = c_i(w_1, w_2)q,$$

and therefore the zero profit condition for strictly positive output quantities is simply:

$$c_i(w_1, w_2) = p_i \text{ for } i \in \{1, 2\}.$$

Recall from basic producer theory that the conditional factor demand for $z_{ji}(w_1, w_2; q = 1)$ at unit production of good $y_i$ is given by:

$$z_{ji}(w_1, w_2; q_i = 1) := z_{ji}(w) = \frac{\partial c_i(w_1, w_2)}{\partial w_j}.$$

We say that production of output $y_1$ is *relatively more intensive in factor 1* if for all $w$, we have:

$$\frac{z_{11}(w)}{z_{21}(w)} > \frac{z_{12}(w)}{z_{22}(w)}.$$

If we assume that output $y_1$ is relatively more intensive in factor 1, then we will have a single vector $(w_1^*, w_2^*)$, where the first-order conditions and the zero-profit conditions hold simultaneously for the two outputs. As long as both products are produced in strictly positive quantities, the factor prices are determined by the production functions and the output prices and hence the endowments do not play a role. Notice that this implies that as long as the countries do not specialize, factor prices across different countries are equalized (even though factors are not traded). This is known as the *factor price equalization theorem*.

Applying the implicit function theorem to the system:

$$c_1(w_1, w_2) = p_1,$$

$$c_2(w_1, w_2) = p_2,$$

at $(w_1^*, w_2^*)$ gives the Samuelson-Stolper Theorem:

**Theorem 4.1.** If output $y_1$ is relatively more intensive in factor 1, then an increase in $p_1$ increases $w_1$ and decreases $w_2$ assuming that the production levels for both goods are strictly positive before and after the change.

*Proof.* Differentiate the system at $(w_1^*, w_2^*)$ and use the relative intensity condition to sign the Jacobian determinant of the endogenous variables. Use Cramer's rule to get the signs of $dw_1, dw_2$. □

Rybcszynski's Theorem is even more immediate from the factor equalization theorem.

**Theorem 4.2.** If output $y_1$ is relatively more intensive in factor 1, then an increase in $\bar{z}_1$ increases the production of $y_1$ and decreases the production of $y_2$ assuming that productions before and after the change in the factor endowment are strictly positive for both goods..

*Proof.* Just draw the Production Edgeworth Boxes with the optimal rays of factor demands for the two different total endowments of factor 1. □

## 4.4 Assignment Markets: Housing in General Equilibrium

We will start with some simple examples where we add a second good, money, to the economies that we discussed in Section 2.1. The difference to that section is that now we assume that we have cardinal information on the values that agents assign to various houses. For concreteness, we assume that the agent $i$ has a willingness to pay (in monetary units) of $v_{i,h'} > 0$ for living in house $h' \in \{1, ..., H\}$. The agents have an endowment of money and the set of available houses is exogenously determined.

### 4.4.1 Identical Houses: Recalling Intermediate Microeconomics

Perhaps the simplest model for housing is one where all house $h'$ has the same value for occupant $i$, and the houses are put on the rental market by absentee landlords, with a cost $c_{h'}$ of renting (maintenance cost etc.). The tenants differ in terms of their willingness to pay for housing (maybe because of having different incomes) so that $v_{i,h'} = v_i$ for all $h'$ and for all $i$. We assume quasi-linear preferences so that negative money holdings are allowed.

In a competitive market, the houses have a price and $\boldsymbol{p} = (p_1, ..., p_h)$ is the price vector for the market. Both the tenants and the landlords are price takers. Tenant $i$'s payoff from renting house $h'$ if her wealth is $w_i$ is then $w_i + v_i - p_{h'}$ and if she does not rent a house, it is $w_i$. The landlord owning house $h'$ gets a payoff $p_{h'} - c_{h'}$ from giving it to rent and 0 otherwise. Since the houses are identical, only houses with the lowest price have positive demand and hence there is little loss of generality in concentrating on equilibria with a uniform price for all houses.

Markets clear at price $p^*$ when the demand equals supply at that price. Demand is given by the number of tenants with $v_i \geq p^*$ and the supply by the number of landlords with $c_{h'} \leq p^*$. Note that the equilibrium price is not uniquely pinned down in a discrete economy. You may want to show that an equilibrium exists even though the formal existence theorem does not apply. You should also think about the uniqueness of equilibrium prices and quantities in this model. Finally, since the payoffs are strictly increasing in money, all equilibria are Pareto-efficient (the absentee landlords' are also included in the Pareto calculation).

### 4.4.2 Houses with different qualities

Assume next that the houses differ in their quality and all tenants agree on the quality ranking. They differ in their willingness to pay for quality, i.e. they have different (but constant) marginal utility of money. To capture this situation, assume that the houses $h'$ are ranked in descending order of quality $q_{h'}$ so that

$$q_1 > q_2 > ... > q_h,$$

and the willingnesses to pay are similarly ranked:

$$v_1 > v_2 > ... > v_n.$$

We maintain for a moment the assumption of quasi-linear preferences so that the payoff for agent $i$ from renting an apartment of quality $q_{h'}$ at rental price $p_{h'}$ is:

$$v_i q_{h'} - p_{h'}.$$

For simplicity, let's assume that $c_{h'} = 0$ for all $h'$. This implies that the houses with the smallest index $h'$ generate the largest surplus. We assume also that the number of houses is at least as large as the number of tenants.

How can we find an equilibrium for such a market? We can use the first welfare theorem to find the equilibrium allocation. Since competitive equilibrium allocations are Pareto-efficient and since the models has quasi-linear utilities, we should look for a one-to-one function $r : \mathcal{N} \to \mathcal{H}$ solving:

$$\max_{(r(i))_{i=1}^{n}} \sum_{i=1}^{n} v_i q_{r(i)}.$$

It is a very nice exercise if you have not done this before to show that the solution $r^*$ is the identity function $r^*(i) = i$. (To show this assume not, then there exists a pair $i < i'$ with $r^*(i) > r^*(i')$. Evaluate the gain in the objective function from swapping the houses.)

Hence we know what the equilibrium allocation must be. It remains to figure out equilibrium prices supporting this allocation as a competitive equilibrium allocation. Since the house with $n + 1$ highest quality is not rented in the hypothetical equilibrium allocation, we must have $p_{n+1} = 0$ to have market clearing. Since agent $n$ is better off renting the house of quality $q_n$ at price $p_n$ rather than renting $q_{n+1}$ at zero price, we have:

$$v_n q_n - v_n q_{n+1} \geq p_n - p_{n+1} = p_n.$$

Hence the highest price $p_n$ compatible with the equilibrium allocation is:

$$p_n = v_n(q_n - q_{n+1}).$$

The lowest price compatible with equilibrium is $p_n = 0$. This same argument shows that for all $n' \in \{1, ..., n\}$,

$$v_n' q_n' - v_n' q_{n'+1} \geq p_n' - p_{n'+1}.$$

In the problem set, you are asked to show some basic facts about the possible equilibrium price vectors.

It is actually easier to see how the model works if we allow for a unit mass of agents with different willingness to pay and a continuum of different qualities for the houses. Denote the set of agents by their willingness to pay $v \in [\underline{v}.\overline{v}]$ and the set of house qualities $q \in [\underline{q}, \overline{q}]$.

In light of the Pareto-efficiency of the equilibria, we consider only the highest quality houses. The price $p(\underline{\mathrm{q}})$ is pinned down to zero if we assume (as before) that lower quality houses would also be available but not rented in equilibrium.

Let the c.d.f. of the willingness to pay for the agents be $F(v)$ and the c.d.f. of the houses on $[\underline{q}, \overline{q}]$ be $G(q)$. Assume that $F, G$ are strictly increasing and continuous so that they have well defined inverses.

Positive assortative matching (PAM) means that we assign the agent with the w.t.p at percentile $z$ in $F$ the the house quality at percentile $z$ in $G$. You may want to prove that social surplus is maximized by PAM. (This is actually a consequence of another inequality by Chebyshev and the result holds whenever the surplus from matching $v$ with $q$ is supermodular in $(v, q)$).

With PAM, we match the agent with w.t.p. $v$ with the house of quality $q(v)$ with:

$$q(v) = G^{-1}(F(v)). \tag{5}$$

Suppose $p(q)$ is the pricing function for the houses of different qualities. Since agents maximize, the agent with w.t.p. $v$ chooses to:

$$\max_q vq - p(q).$$

Assuming differentiability of $p(q)$, the first order condition for maximum is that $p'(q) = v$. PAM then implies that:

$$v = F^{-1}(G(q)).$$

Together with $p(\underline{q}) = 0$, we have:

$$p(q) = \int_{\underline{q}}^{q} F^{-1}(G(z))dz. \tag{6}$$

Hence we have solved the equilibrium for the market to be allocation by PAM and the housing prices are given by equation 6.

### 4.4.3 Housing Market with Income Effects

We have said before that one way to explain differences in the willingness to pay is by taking them to be a reduced form way to express differences from different wealth levels. Housing markets are actually a good example of markets where income effects are likely to play a role. I now sketch a model where the differences in willingness to pay arise from income differences.

Let $u(q,w)$ be the utility function of a representative consumer, where $q$ is the quality of the house occupied and $w$ is the monetary amount used on a composite good taking into account all other consumption.

Let the agents have an exogenous outside income $y$ and denote the c.d.f. of the income distribution by $H(y)$. The agent with income $y$ now solves:

$$\max_q u(q, y - p(q)).$$

The first order condition for this is that:

$$u_q(q, y - p(q)) - p'(q)u_w(q, y - p(q)) = 0. \tag{7}$$

In words, the MRS of the consumer at optimal $q(y)$ must equal the derivative of the price function. Draw the picture to see how this corresponds to the usual consumer choice problem.

An increase in income shifts the budget line of the consume vertically upwards. PAM between income and housing quality implies that $q(y) < q(y')$ whenever $y < y'$. I leave it as an exercise for you to show that if the MRS (i.e. $\frac{u_q(q,y-p(q))}{u_w(q,y-p(q))}$) is increasing in $y$, then the optimal choice $q(y)$ is increasing in $y$. (To this effect, differentiate the first-order condition, and use the second order condition. Or alternatively. Monotone comparative statics results by Milgrom and Shannon (1994) give a more modern and comprehensive approach to problems of this type).

If PAM holds, then $y$ and $q$ are related as follows::

$$q = G^{-1}(H(y)), \quad y = H^{-1}(G(q)).$$

Equilibrium prices are given by the differential equation 7 with $p(\underline{q}) = 0$:

$$p(q) = \int_{\underline{q}}^{q} \frac{u_q(z, G^{-1}(H(z)) - p(z))}{u_w(z, G^{-1}(H(z)) - p(z))} dz.$$

This is a nonlinear first-order differential equation, but with the initial condition, it has a unique solution. You may want to see how this simplifies e.g. in the case of a Cobb-Douglas utility function.

Terviö and Määttänen (2014) extend this model to cover the interesting case where all houses are initially occupied. This means that the budget set of the agents $i$ depends on their outside income as well as the endogenous equilibrium house price $p(q^e(i))$, where $q^e(i)$ is the quality of the house that agent $i$ owns. Hence the wealth of $i$ is $w(i) = y(i) + p(q^e(i))$.

In this model, PAM holds in $w(i), q(i)$ under the assumption that MRS increases in $y$, but not necessarily in the exogenous income $y(i)$ and $q(i)$. Even the existence of equilibrium in this model is not trivial. See the cited paper for details on this. Unfortunately, it is not easy to characterize the trades in equilibrium, but the paper conducts comparative statics exercises on the impact of changes in $H$ on housing prices and the full wealth distribution.

## 4.5 Houses with Idiosyncratic Qualities

Let $p_h$ denote the price of house $h \in H$. Each agent $i$ has a willingness to pay $v_{i,h'}$ for house $h'$ and in contrast to the previous subsections, we allow now each agent to have arbitrary preferences over the houses, i.e. the agents can have different opinions on the qualities of the houses. We assume again quasi-linear preferences and hence we also allow for negative monetary holdings. Then agent $i$ optimizes her housing choice given price vector $\boldsymbol{p} := (p_1, ..., p_h)$ by choosing:

$$\max_{h' \in \mathcal{H}} v_{i,h'} - p'_h.$$

Markets clear if the vector of optimal demands $\boldsymbol{a} = (a_1, ..., a_n)$ is an allocation (i.e. no house is demanded by more than a single agent).

**Example 4.2.** Consider a society with four agents $\mathcal{N} = \{1, 2, 3, 4\}$ and five houses, where for the sake of clarity, we give letter names to the houses: $\mathcal{H} = \{a, b, c, d, e\}$. The individual willingness of the agents to pay for the different houses is given in the table below:.

| $v_{i,h}$ | a | b | c | d | e |
|---|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 4 | 6 |
| 2 | 6 | 2 | 5 | 3 | 4 |
| 3 | 7 | 4 | 3 | 8 | 5 |
| 4 | 2 | 7 | 4 | 6 | 3 |

Figure 13: Agents' willingness to pay.

We can give a formal definition of a competitive equilibrium for this model along the lines of Section 2.1:

**Definition 4.2.** A *competitive equilibrium* of the economy $(\mathcal{N}, \mathcal{H}, (v_{i,h})_{i\in\mathcal{N},h\in\mathcal{H}})$ is a house price vector $\boldsymbol{p}$ and a vector of housing demands $\boldsymbol{a} = (a(1), ..., a(n))$ with $a(i) \in \mathcal{H}$ for all $i$ such that

i) For all $i$, $v_{i,a(i)} - p_{a(i)} \geq v_{i,h'} - p_{h'}$ for all $h'$.

ii) $\boldsymbol{a}$ is an allocation (i.e. the vector of optimal demands is a matching).

How could we find a competitive equilibrium for our housing economy? Since demands in the model are integer valued, the existence proofs from the previous section cannot be directly used here. An alternative method for showing the existence is as follows. Since the agents have locally non-satiated preferences, the first welfare theorem shows that if an equilibrium exists, it is Pareto-efficient. With quasi-linear utilities, this involves simply finding

$$\max_{\boldsymbol{a}} \sum_i v_{i,a(i)},$$

subject to the constraint that $\boldsymbol{a}$ be an allocation. Again because of the integer constraints, this is not an easy problem. It can be however relaxed by allowing for the possibility of random allocations

$$\boldsymbol{x} := ((x_{1,1}, ..., x_{1,h}), ..., (x_{n,1}, ..., x_{n,h})),$$

where $x_{i,h'}$ is the probability that $i$ is allocated house $h'$. The resulting optimization problem:

$$\max_{\boldsymbol{x}} \sum_h \sum_i v_{i,h} x_{i,h}$$

subject to

$$\sum_i x_{i,h} \leq 1, \text{ for all } h',$$

$$\sum_{h'} x_{i,h'} \leq 1, \text{ for all } i.$$

This is a linear programming problem and a solution exists by standard arguments. The optimal solution also includes an extreme point of the feasible set (by linearity) and as a result, the optimal solution contains an integer valued solution $\boldsymbol{a}$ that solves the original problem.

The next question is if $\boldsymbol{a}$ is a competitive equilibrium allocation. For this, we would need to find a price vector $\boldsymbol{p}$ such that the optimal aggregate demand given $\boldsymbol{p}$ coincides with $\boldsymbol{a}$.

The dual of the above linear program is to minimize the non-negative utilities $u_i$ of the agents and the house owners $w_{h'}$ subject to the constraint that for all $i, h'$:

$$u_i + w_{h'} \geq v_{i,h'}.$$

The solution to the dual can then be used to deduce the prices for the houses: the price is simply the optimal $w_{h'}$ in the dual problem.

Another approach to solving the problem is given by a modification of the Gale-Shapley Algorithm presented in Crawford and Knoer (1981).

The idea of the algorithm is that the landlords offer houses to potential buyers and buyers hold offers tentatively as in Gale-Shapley. Offers now increase a price and after an offer is rejected, a new offer at a lower price can be made to the same potential tenant. The process ends at landlord optimal equilibrium prices for the houses. See the cited paper for details. Hatfield and Milgrom (2005) extends the model further.