Microeconomic Theory II
Helsinki GSE
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Problem Set 1, Due November 15, 2022

1. 3 students must split a pizza that was cut to 24 equally large slices at the restaurant. Their preferences are strongly monotonic and there are no externalities.
   (a) Formulate the preference relation induced by the majority rule on the feasible sharings of the pizza. Show that this relation is complete.
   (b) Show that for every allocation of pizza, there exists another one that is strictly preferred by a majority of agents.
   (c) What does this imply about the transitivity of the induced preference relation?

2. Consider a society with three agents (1, 2, 3) and three alternative social outcomes x, y, z. The following table shows three preference profiles (A), (B), (C) for the agents (the columns represent the agents and the outcomes are ranked in the descending order of preference within the columns):

(A)			(B)			(C)		
1	2	3	1	2	3	1	2	3
x	x	x	x	y	z	x	x	z
y	y	y	y	z	x	y	y	x
z	z	z	z	x	y	z	z	y

   Recall that by Arrow's theorem we know that the only rule that maps any rational preference profile of the agents over x, y, z into a social preference relation satisfying (i) completeness and transitivity, (ii) independence of irrelevant alternatives, and (iii) unanimity is a dictatorial one.

   (a) Write the societal preference for each of these preference profiles induced by the majority rule and by the Borda rule.
   (b) Demonstrate the failure of (i), (ii) or (iii) for the preference relation derived from Borda rule for some preference profile.

(c) Demonastrate the failure of (i), (ii) or (iii) for the preference relation derived from the majority rule for some preference profile.

3. This problem considers housing allocation in a society without externalities (as in Definition 2.1 of the Lecture Notes) where the agents have strict preferences over houses.

   (a) The society has a nepotistic leader: an agent who wants to occupy a house currently occupied by another agent can do so if the nepotistic leader likes her more than the current occupant. The leader's liking ranking over the agents is complete and transitive, with no ties. An equilibrium of the society under the nepotistic regime is then an allocation such that for no agents i, j is it the case that the leader likes i more than j but i would prefer the house of j to the house allocated to herself. Show that the equilibrium is Pareto efficient.

   (b) The nepotistic leader is replaced by a leader that praises free markets. The new leader gives the agents property rights to the houses they occupied in the equilibrium under the nepotistic leader, and trade is now allowed. Show that no trade occurs in the market equilibrium.

4. Consider an economy with five agents $\mathcal{N} = \{1, 2, 3, 4, 5\}$ and five houses $\mathcal{H} = \{a, b, c, d, e\}$. The individual preferences are given in the following table where columns represent the agents and the houses are ranked in the descending order of preference within the columns. The initial allocation is represented by the boxed elements in the table.

1	2	3	4	5
$\boxed{\text{a}}$	a	d	a	a
b	c	a	b	$\boxed{\text{e}}$
c	d	$\boxed{\text{c}}$	$\boxed{\text{d}}$	c
d	$\boxed{\text{b}}$	e	e	d
e	e	b	c	b

   (a) Find a market equilibrium of this economy.

   (b) Which conditions do the house prices have to satisfy in *any* equilibrium of this economy?

5. Consider an economy with n agents. Let X be the set of alternatives available in this economy. For each pair $(x, y) \in X \times X$, define the variable d_i for each $i \in \{1, ..., n\}$ as follows:

$$d_i = \begin{cases} 1 \text{ if } x \succ y, \\ 0 \text{ if } x \sim y, \\ -1 \text{ if } y \succ x. \end{cases}$$

A social choice function is a function $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ (with the same interpretation as above). Let $d = (d_1, ..., d_n)$. The majority decision rule is defined as follows:

$$f(d_1, ..., d_n) = \begin{cases} 1 \text{ if } \Sigma_{i=1}^n d_i > 0, \\ 0 \text{ if } \Sigma_{i=1}^n d_i = 0, \\ -1 \text{ if } \Sigma_{i=1}^n d_i < 0. \end{cases}$$

Let $n^+(d) = \#\{i \text{ such that } d_i = 1\}$ and $n_-(d) = \#\{i \text{ such that } d_i = -1\}$. A social choice function is said to be *anonymous* if $f(d) = f(d')$ whenever $n^+(d) = n^+(d')$ and $n_-(d) = n_-(d')$. In other words, the rule treats all individuals in the same manner. A social choice function is *neutral* if $f(-d) = -f(d)$. A social choice function is *responsive* if $f(d) \geq 0$ and $d' > d$ imply that $f(d') = 1$.

(a) Show that the majority rule is anonymous, neutral and responsive.

(b) Show that whenever f is anonymous and neutral, $n^+(d) = n_-(d)$ implies that $f(d) = 0$.

(c) Prove that whenever f is anonymous, neutral and responsive, it is given by the majority rule.

6. Consider the single-dimensional spatial model where the set of alternatives is given by the interval $X = [0, 1]$ and there are an odd number of voters $i \in \{1, ..., n\}$. Each voter has rational preferences over X. Assume further that for each i, there is an ideal alternative $x_i^* \in [0, 1]$ and that the preferences are single-peaked, i.e.

$$x < x' < x_i^* \implies x_i^* \succ x' \succ x \text{ and } x > x' > x_i^* \implies x_i^* \succ x' \succ x.$$

(a) Show that the societal preference induced by majority voting between pairs of alternatives is complete and transitive.

(b) Show that the ideal point of the median voter (i.e. the median of the set $\{x_1^*, ..., x_n^*\}$) is strictly preferred to any other alternative in the social preference induced by majority voting.

(c) Is the median voter a dictator in the sense of Arrow's theorem?

7. (Optional) Consider an economy with a countable infinity of generations. Each generation can obtain a utility $x_i \in \{0, 1\}$. An outcome in this economy is then a sequence $x = (x_1, x_2, ...)$ with $x_i \in \{0, 1\}$ for each i. Thus $X = \{0, 1\}^{\mathbb{N}}$ Suppose that a social planner for the economy has preferences that satisfy two properties:

   i) Pareto-principle: For all $x, y \in X$, $x \geq y \Rightarrow x \succ y$.

   ii) Intergenerational Equity: For all $x, y \in X$, if $\exists i, j$ such that $x_i = y_j$ and $y_i = x_j$ and $x_k = y_k$ for $k \notin \{i, j\}$, then $x \sim y$.

   It can be shown that such rational preferences do exist. Show that the planner's preferences cannot have a utility representation. Hint: can you relate this to lexicographic individual preferences?