Advanced Microeconomics 2
Helsinki GSE
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Problem Set 4, Due December 14, 2022

1. An investor has initial wealth w_0 that she must divide between two risky assets. Denote the total investment into asset i by w_i for $i \in \{1, 2\}$. There are two possible states H and L for the economy. Assume that the economy will be in state H with probability p and in L with probability $(1-p)$.The first asset pays a gross return R^H if the state of the economy is H and a gross return 0 if the state is L. I.e. in state H, an investment of w_1 in asset 1 results in a return $w_1 R^H$ and in state L, the return is 0. The second asset pays R in both states. Denote the fraction of initial wealth that is invested in asset i by α_i for $i \in \{1, 2\}$ so that $w_i = \alpha_i w_0$ and $\alpha_1 + \alpha_2 = 1$.

   (a) For an arbitrary investment portfolio $\{\alpha_1, \alpha_2\}$, determine the final wealth of the investor.

   (b) Assume the investor has a Bernoulli utility function $u(w)$ where w denotes the final wealth. Assume that $u'(w) > 0$ and $u''(w) < 0$. Write down the expected utility resulting from an arbitrary portfolio $\{\alpha_1, \alpha_2\}$ and characterize the first order conditions for the optimal portfolio. Argue also that second order conditions are also satisfied at the point that satisfies the first order conditions.

   (c) Consider now a third asset that pays 0 in H and R^L per unit invested in L. Assume further that the utility function takes the form $u(w) = \ln w$.For what values of p, R^H, R^L, R will the investor invest in assets 1 and 3 only?

2. Consider an economy with one good x and two agents $i \in \{1, 2\}$ with utility functions $u_i(x_i) = \ln(x_i)$. There are two equally probable states, $s \in \{L, H\}$. There are also two firms: firm 1 produces 1 unit in both of the states, whereas firm 2 produces 0 units in state L and 2 units in state H. Agent 1 owns initially the whole firm 1, while agent 2 owns initially firm 2. However, the agents can trade the firm shares with each other before the state is realized. The agents have no other endowment.

(a) Compute the equilibrium shareholdings and the market values of the two firms (relative to each other).

(b) Suppose there is also state 3, in which firm 1 produces 2 units and firm 2 produces 3 units. Create a call option on one of the firm shares, and use the firm shares and the call option to create an asset portfolio that returns 1 unit in any of the three states.

3. An economy consists of a continuum of mass 1 of agents and a continuum of mass 2 of houses. The agents' willingness to pay for housing quality v_i is uniformly distributed on $[1, 2]$ so that the utility for an agent of type v_i from owning a house of quality q_j is $v_i q_j$. The houses have a uniform quality distribution on $[0, 2]$.

   (a) The houses are owned by absentee landlords and their opportunity cost of renting (outside option for holding the house of quality q_j) is αq_j. Assume that the agents and the house owners have quasilinear utilities in utility from housing and money. What is the efficient allocation of houses in this market (as a function of α)?

   (b) Assume that $\alpha = \frac{1}{2}$ and solve for the competitive equilibrium price (function) $p(q)$ for the houses and describe the equilibrium allocation of houses.

4. Consider a large population of identical children. A child's preferences can be represented by a utility function $U : \mathbb{R} \longrightarrow \mathbb{R}$, which takes a number of gifts received as input. Furthermore, $U' > 0$ and $U'' < 0$.

   Christmas is coming. There are two possible states of Christmas. With a probability of $\frac{1}{2}$, Christmas is happy, in which case each child receives two Christmas gifts from Santa Claus. With the remaining probability of $\frac{1}{2}$, Christmas is miserable, in which case each child receives only one Christmas gift from Santa Claus. The state of Christmas is realized on Christmas Eve.

   Children can trade three different assets. A child who holds one *risk-free asset* on Christmas Eve is entitled to one additional Christmas gift regardless of the state of Christmas. A holder of a *procyclical asset* is entitled to 2 additional Christmas gifts if Christmas is happy, and to 0 additional gifts if Christmas is miserable. A child who holds a *countercyclical asset* on Christmas Eve is entitled to 2 additional Christmas gifts if Christmas is miserable, and to 0 additional gifts if Christmas is happy. Denote the pre-Christmas prices of these assets by

P^{RF}, P^{PC}, and P^{CC}, respectively (the prices are measured in terms of gifts). Short sales are allowed, and the assets must be in zero net supply.

Hint: Notice that in the equilibrium, there is no asset trade, and each child is indifferent between buying and selling any of the assets before Christmas.

(a) Determine the following three price ratios: P^{RF}/P^{PC}, P^{PC}/P^{CC}, and P^{RF}/P^{CC}. (The answer should be in terms of $U'(1)$ and $U'(2)$.) Which one of the assets is the most expensive one, and which asset is the cheapest one? Which asset has the highest expected rate of return?

(b) Consider an otherwise similar situation, but with a small difference regarding the miserable state of Christmas: if Christmas is miserable, half of the children receive 0 Christmas gifts from Santa Claus, whereas the remaining half of the children receive 2 gifts from Santa Claus. The children are still identical *ex ante*. Determine the price ratio P^{PC}/P^{CC} in this situation. If $U''' = 0$ (in the relevant part of the domain), is P^{PC}/P^{CC} equal, greater, or less than in Part (a)? If $U''' > 0$, is P^{PC}/P^{CC} equal, greater, or less than in Part (a)?

5. (Crossover to Macroeconomics) A single, representative agent lives forever and consumes (and holds assets) optimally given the market prices. The preferences are represented by:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where the per period utility from consumption c_t in period t is given by $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$ (CRRA).

Endowments: The agent is endowed at $t = 0$ with 1 tree. In each period, the tree yields stochastic consumption (dividend) d_t that cannot be stored. The dividend d_t evolves as follows:

- If $d_t = d_{t-1}$, then $d_{t+1} = d_t$ for all $t' > t$.
- If $d_t \neq d_{t-1}$, then $d_{t+1} = \gamma d_t$ with probability π and $d_{t+1} = d_t$ with probability $1 - \pi$, and $\gamma > 1$.

In words: d grows at rate $\gamma - 1$ until some random event occurs (with probability $1 - \pi$), at which point growth stops forever.

Markets: There are competitive markets for consumption (choose this as numeraire) and trees (price p_t). Assume that p_t is cum dividend, meaning that d_t accrues to the household who buys the tree in t and holds it into $t + 1$.

(a) State the agent's optimization problem paying particular attention to the intertemporal budget constraint:

$$p_{t+1}(k_{t+1} - k_t) + c_t = d_t k_{t+1},$$

where k_{t+1} denotes the holdings of trees from t to $t + 1$. Derive the first-order condition characterizing the optimal consumptions in consecutive periods .

(b) Define a competitive equilibrium.

(c) Characterize the stochastic process of p_t. Is p_t a Markov process? Hint: there are 2 phases: before and after dividends have stopped growing. Assume that $\frac{p_t}{d_t}$ is constant during the phase with growth.

(d) What happens to the stock market (i.e. p_t) when the economy stops growing? Does it crash (does the price decline)? Under what condition?