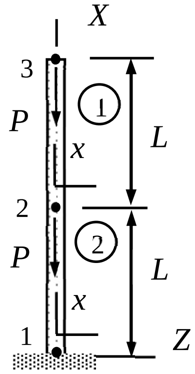
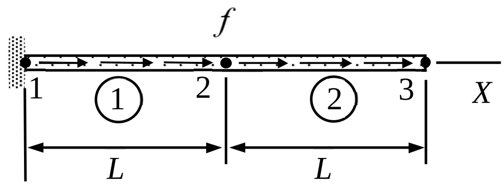
MEC-E1050 Finite Element Method in Solid, week 44/2022

1. The bar structure shown is loaded by point forces of equal magnitude P. Determine the nodal displacements u_{X2} and u_{X3}. Cross-sectional area A and Young's modulus E are constants. Use bar elements as indicated in the figure.



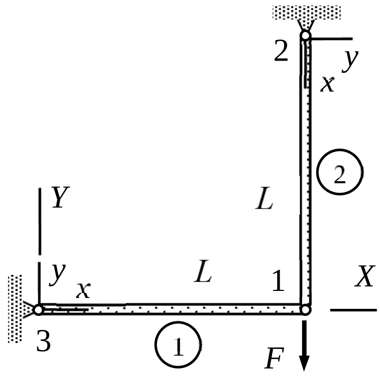
Answer $u_{X2} = -2\dfrac{PL}{EA}$, $u_{X3} = -3\dfrac{PL}{EA}$

2. Determine displacements u_{X2} and u_{X3} of nodes 2 and 3 of the bar shown. The external force per unit length is constant f and axial rigidity of the bar is EA. Use two bar elements of equal length and the bar element contribution given in the formulae collection.



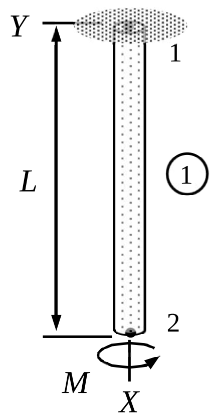
Answer $u_{X2} = \dfrac{3}{2}\dfrac{fL^2}{EA}$, $u_{X3} = 2\dfrac{fL^2}{EA}$

3. The bar structure shown is loaded by a point force at node 1. Draw the free body diagrams of the three nodes and two bars. Write down the equilibrium equations of the nodes, force-displacement relationships of the elements, and constraints on the displacements imposed by supports. Solve the nodal displacements from the equation system.



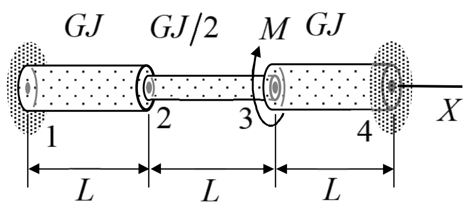
Answer $u_{X1} = 0$, $u_{Y1} = -\dfrac{FL}{EA}$

4. Consider the torsion bar of the figure loaded by torque M acting on the free end. Determine the rotation θ_{X2} at the free end if the polar moment of the cross-section J and shear modulus G are constants.



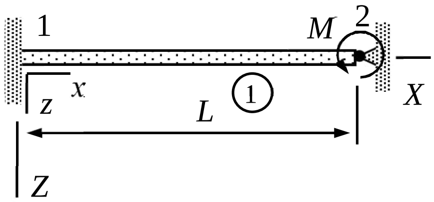
Answer $\theta_{X2} = -\dfrac{ML}{GJ}$

5. Torque M is acting in the direction of negative X-axis at node 3 of a torsion bar. Determine rotations θ_{X2} and θ_{X3} of nodes 2 and 3. Shear modulus G is constant and the polar moment of area J is piecewise constant. Use three elements of equal length.



Answer $\theta_{X2} = -\dfrac{1}{4}\dfrac{ML}{GJ}$, $\theta_{X3} = -\dfrac{3}{4}\dfrac{ML}{GJ}$

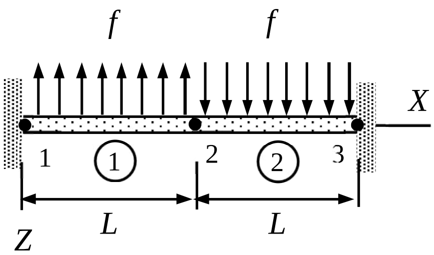
6. Determine rotation of the bending beam shown at node 2, internal forces and moments acting between the nodes and the beam element, and the constraint forces at the supports. The beam is clamped at the left end and simply supported at the right end. Young's modulus of the material E and the second moment of the cross-section $I_{yy} = I$ are constants. External distributed force $f_z = 0$.



Answer $\theta_{Y2} = \frac{1}{4}\frac{ML}{EI}$, $F^1_{Z1} = -\frac{3}{2}\frac{M}{L}$, $M^1_{Y1} = \frac{1}{2}M$, $F^1_{Z2} = \frac{3}{2}\frac{M}{L}$, $F_{Z1} = -\frac{3}{2}\frac{M}{L}$,

$M_{Y1} = \frac{1}{2}M$, $F_{Z2} = \frac{3}{2}\frac{M}{L}$.

7. External load acting on the beam shown consists of piecewise constant parts having equal magnitudes but opposite signs. Determine displacement u_{Z2} and rotation θ_{Y2} of the mid-point (point 2). Young's modulus of the material and the second moments of area are E and I, respectively. Use two beam elements of equal length.



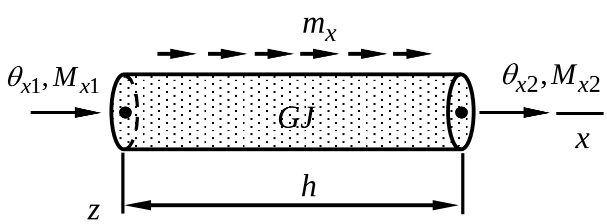
Answer $u_{Z2} = 0$, $\theta_{Y2} = -\frac{fL^3}{48EI}$

8. The boundary value problem defining the element contribution of a torsion bar consist of

$$GJ\frac{d^2\phi}{dx^2} + m_x = 0 \quad x \in]0,h[,$$

$$\phi(0) = \theta_{x1} \quad \text{and} \quad \phi(h) = \theta_{x2},$$

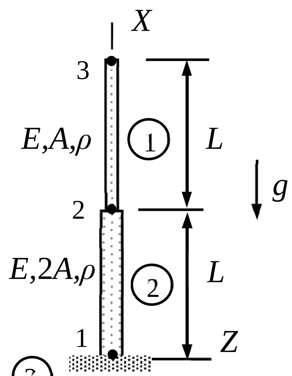
$$GJ\frac{d\phi}{dx}(0) = -M_{x1} \quad \text{and} \quad GJ\frac{d\phi}{dx}(h) = M_{x2},$$



in which the shear modulus G, cross-sectional area of the bar A, and external distributed moment per unit length m_x are constants. Derive the element contribution of a torsion bar element with the aid of the boundary value problem.

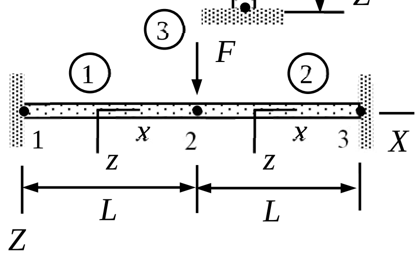
Answer $\begin{Bmatrix} M_{x1} \\ M_{x2} \end{Bmatrix} = \frac{GJ}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

9. Consider the bar structure of the figure loaded by its own weight. Determine the displacements u_{X2} and u_{X3} by using two bar elements. Acceleration by gravity g and material properties E and ρ are constants.

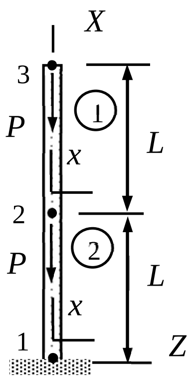


Answer $u_{X2} = -\dfrac{g\rho L^2}{E}$, $u_{X3} = -\dfrac{3}{2}\dfrac{g\rho L^2}{E}$

10. Determine displacement u_{Z2} at node 2 of the beam structure shown. Use two beam elements of equal length. Assume that rotation $\theta_{Y2} = 0$. Point force of magnitude F is acting on node 2. Young's modulus of the material E and the second moment of area I are constants.



Answer $u_{Z2} = \dfrac{1}{24}\dfrac{FL^3}{EI}$



The bar structure shown is loaded by two point forces of equal magnitude P. Determine the nodal displacements u_{X2} and u_{X3}. Cross-sectional area A and Young's modulus E are constants. Use two bar elements as indicated in the figure.

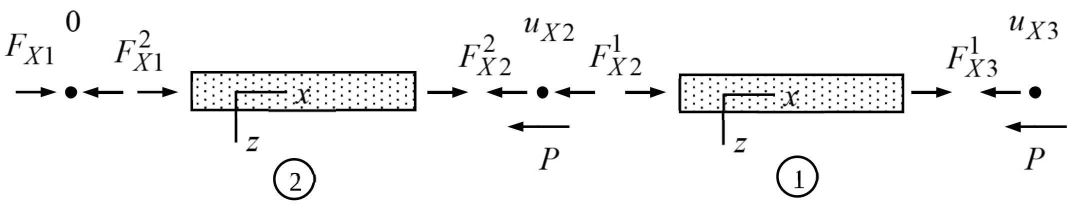
Solution

The generic force-displacement relationship of a bar element

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

depends on the cross-sectional area A, Young's modulus E, bar length h, and force per unit length of the bar f_x in the direction of the $x-$axis.

Let us start with the free body diagram of the structure consisting of two bar elements (the structure is rotated clockwise just to save space).



Element contributions (notice that $f_x = 0$ and the force components of the material and structural systems coincide here) are:

$$\text{bar 1 :}\quad \begin{Bmatrix} F_{X2}^1 \\ F_{X3}^1 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{matrix} \text{eq.1} \\ \text{eq.2} \end{matrix}$$

$$\text{bar 2 :}\quad \begin{Bmatrix} F_{X1}^2 \\ F_{X2}^2 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{matrix} \text{eq.3} \\ \text{eq.4} \end{matrix}$$

Equilibrium equations of the nodes are:

node 1: $\sum F_X = F_{X1} - F_{X1}^2 = 0$ eq.5

node 2: $\sum F_X = -F_{X2}^2 - F_{X2}^1 - P = 0$ eq.6

node 3: $\sum F_X = -F_{X3}^{1} - P = 0$ eq.7

The outcome is 7 linear equations for the 2 displacements, 4 internal forces, and 1 constraint force. As the first step toward the solution (always), the internal forces are replaced in eq.6 and eq.7 (non-constrained nodes) by their expression given by eq.1, eq.2 and eq.4, to get the equilibrium equations of the nodes in terms of displacements:

node 2: $-(\frac{EA}{L}u_{X2}) - (\frac{EA}{L}u_{X2} - \frac{EA}{L}u_{X3}) - P = 0$

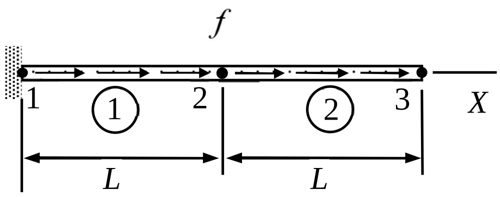
node 3: $-(-\frac{EA}{L}u_{X2} + \frac{EA}{L}u_{X3}) - P = 0$

After that, the unknown displacements follow from the system of linear equations for node 2 and 3. In matrix form

$$\begin{bmatrix} -2\frac{EA}{L} & \frac{EA}{L} \\ \frac{EA}{L} & -\frac{EA}{L} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \begin{Bmatrix} P \\ P \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} = \begin{Bmatrix} -2\frac{PL}{EA} \\ -3\frac{PL}{EA} \end{Bmatrix}.$$

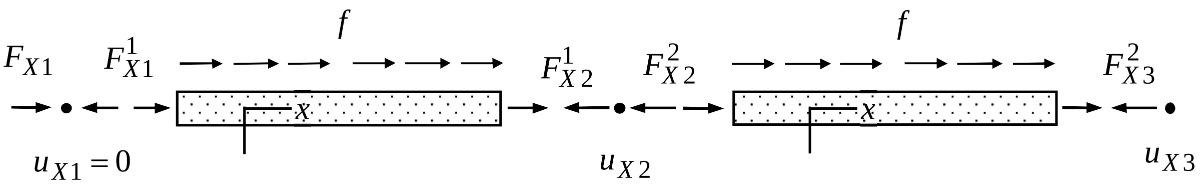
Use the code of MEC-E1050 to check the solution!

Determine displacements u_{X2} and u_{X3} of nodes 2 and 3 of the bar shown. The external force per unit length is constant f and axial rigidity of the bar is EA. Use two bar elements of equal length and the bar element contribution given in the formulae collection.



Solution

Only the displacement in the direction of the bar matters. From the figure, the non-zero displacement components are u_{X2} and u_{X3}. Free body diagram of the two bar elements and nodes 1, 2 and 3 is



Element contributions of the bar elements 1 and 2 (formulae collection) and the equilibrium equations of nodes 1, 2 and 3 are (written in terms of the force and displacement components of the structural system)

Bar 1: $\begin{Bmatrix} F_{X1}^1 \\ F_{X2}^1 \end{Bmatrix} = \dfrac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \dfrac{fL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$,

Bar 2: $\begin{Bmatrix} F_{X2}^2 \\ F_{X3}^2 \end{Bmatrix} = \dfrac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \dfrac{fL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$,

Node 1: $\sum F_X = F_{X1} - F_{x1}^1 = 0$,

Node 2: $\sum F_X = -F_{X2}^1 - F_{X2}^2 = 0$,

Node 3: $\sum F_X = -F_{X3}^2 = 0$.

Elimination of the internal forces from the two equilibrium equations of the non-constrained nodes 2 and 3 using the element contributions gives

Node 2: $-(\dfrac{EA}{L}u_{X2} - \dfrac{fL}{2}) - (\dfrac{EA}{L}u_{X2} - \dfrac{EA}{L}u_{X3} - \dfrac{fL}{2}) = 0$,

Node 3: $-(-\dfrac{EA}{L}u_{X2} + \dfrac{EA}{L}u_{X3} - \dfrac{fL}{2}) = 0$.

When the equilibrium equations are written in the "standard" matrix form

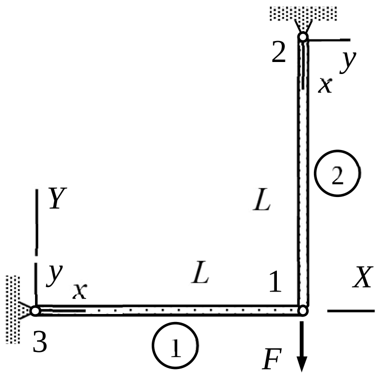
$$\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - fL\begin{Bmatrix} 1 \\ 1/2 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \left(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

$$\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} = \frac{fL^2}{EA}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{Bmatrix} 1 \\ 1/2 \end{Bmatrix} = \frac{fL^2}{EA}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ 1/2 \end{Bmatrix} = \frac{fL^2}{EA}\begin{Bmatrix} 3/2 \\ 2 \end{Bmatrix} \quad \Leftrightarrow$$

$$u_{X2} = \frac{3}{2}\frac{fL^2}{EA} \quad \text{and} \quad u_{X3} = 2\frac{fL^2}{EA}.$$

Use the code of MEC-E1050 to check the solution!

The bar structure shown is loaded by a point force at node 1. Draw the free body diagrams of the three nodes and two bars. Write down the equilibrium equations of the nodes, force-displacement relationships of the elements, and constraints on the displacements imposed by supports. Solve the nodal displacements from the equation system.



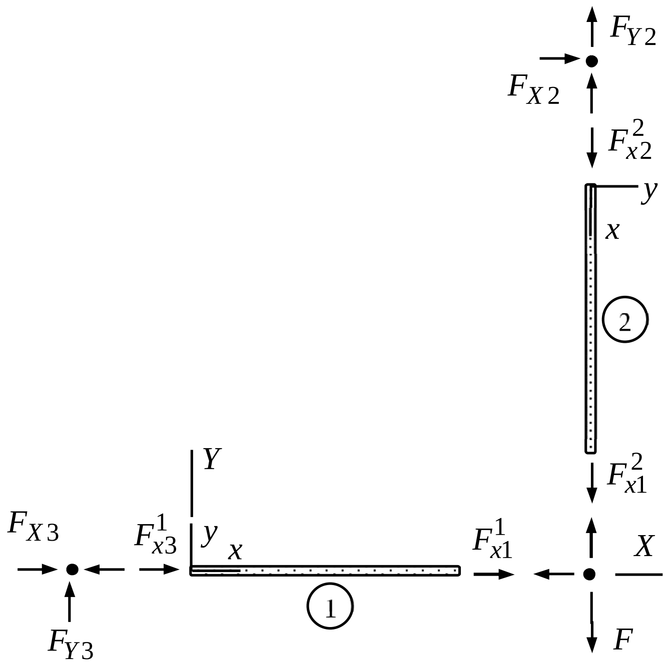
Solution

The generic force-displacement relationship of a bar element

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

depends on the cross-sectional area A, Young's modulus E, bar length h, and force per unit length of the bar f_x in the direction of the $x-$axis. In the present case, the distributed force $f_x = 0$.

Let us start with the free body diagram of the structure consisting of two bar elements. A bar takes only forces acting in its direction. The external point force acts on node 1. Supports are replaced by reaction forces which they impose on the structure.



Element contributions are written in terms of the force and displacement components in the structural system. All the components of the elementwise material coordinate systems need to be expressed in terms of those of the structural system before writing the equilibrium equations. In this case, the relationships between the material and structural system can easily be seen from the free body diagram ($F_{x1}^1 = F_{X1}^1,\ \ F_{x1}^2 = -F_{Y1}^2,$ etc.)

bar 1 : $$\begin{Bmatrix} F_{X3}^1 \\ F_{X1}^1 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X1} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$ eq.1 eq.2

$$\text{bar 2}: \begin{Bmatrix} -F_{Y2}^{2} \\ -F_{Y1}^{2} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ -u_{Y1} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{matrix} \text{eq.3} \\ \text{eq.4} \end{matrix}$$

Force equilibrium equations of the nodes in the *X*- and *Y*- directions are:

$$\text{node 1:} \quad \sum \begin{Bmatrix} F_X \\ F_Y \end{Bmatrix} = \begin{Bmatrix} -F_{X1}^{1} \\ -F_{Y1}^{2} - F \end{Bmatrix} = 0\,, \qquad \begin{matrix} \text{eq.5} \\ \text{eq.6} \end{matrix}$$

$$\text{node 2:} \quad \sum \begin{Bmatrix} F_X \\ F_Y \end{Bmatrix} = \begin{Bmatrix} F_{X2} \\ F_{Y2}^{2} + F_{Y2} \end{Bmatrix} = 0\,, \qquad \begin{matrix} \text{eq.7} \\ \text{eq.8} \end{matrix}$$

$$\text{node 3:} \quad \sum \begin{Bmatrix} F_X \\ F_Y \end{Bmatrix} = \begin{Bmatrix} F_{X3} - F_{X3}^{1} \\ F_{Y3} \end{Bmatrix} = 0\,. \qquad \begin{matrix} \text{eq.9} \\ \text{eq.10} \end{matrix}$$

The outcome is a set of 10 linear equations for 2 displacement components, 4 internal force components, and 4 constraint force components. As the first step toward the solution (always), the internal forces are replaced in eq.5 and eq.6 (non-constrained directions) by their expression given in eq.2 and eq.4, to get the equilibrium equations of the nodes in terms of displacements:

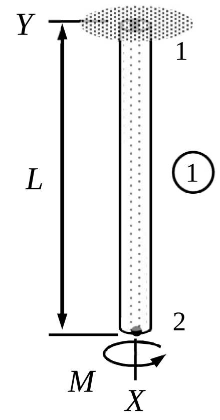
$$\text{node 1:} \quad \sum \begin{Bmatrix} F_X \\ F_Y \end{Bmatrix} = \begin{Bmatrix} -\dfrac{EA}{L} u_{X1} \\ -\dfrac{EA}{L} u_{Y1} - F \end{Bmatrix} = \begin{bmatrix} -\dfrac{EA}{L} & 0 \\ 0 & -\dfrac{EA}{L} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0\,.$$

After that, the unknown displacements are solved from the system of linear equations

$$\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\dfrac{FL}{EA} \end{Bmatrix}.$$

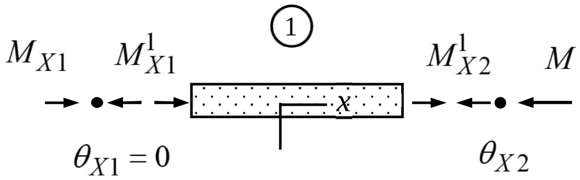
Use the code of MEC-E1050 to check the solution!

Consider the torsion bar of the figure loaded by torque M acting on the free end. Determine the rotation θ_{X2} at the free end if the polar moment of the cross-section J and shear modulus G are constants.



Solution

Only the rotation in the direction of the bar matters. From the figure, only the rotation component θ_{X2} may not be zero. Free body diagrams of the torsion bar and nodes 1 and 2 are (the structure is rotated just to save space)



Element contribution of the torsion bar and the equilibrium equations of nodes 1 and 2 are (the distributed moment vanishes here) written in terms of the rotation and moment components of the structural system:

Bar 1: $$\begin{Bmatrix} M^1_{X1} \\ M^1_{X2} \end{Bmatrix} = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \theta_{X2} \end{Bmatrix}, \qquad \begin{matrix}\text{eq.1}\\ \text{eq.2}\end{matrix}$$

Node 1: $$\sum M_X = M_{X1} - M^1_{X1} = 0, \qquad \text{eq.3}$$

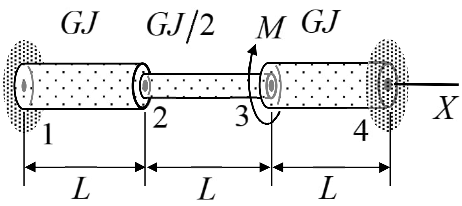
Node 2: $$\sum M_X = -M^1_{X2} - M = 0. \qquad \text{eq.4}$$

Elimination of the internal forces from the equilibrium eq.4 for node 2 using the element contribution eq.2 gives

$$-\frac{GJ}{L}\theta_{X2} - M = 0 \quad \Leftrightarrow \quad \theta_{X2} = -\frac{ML}{GJ}.$$

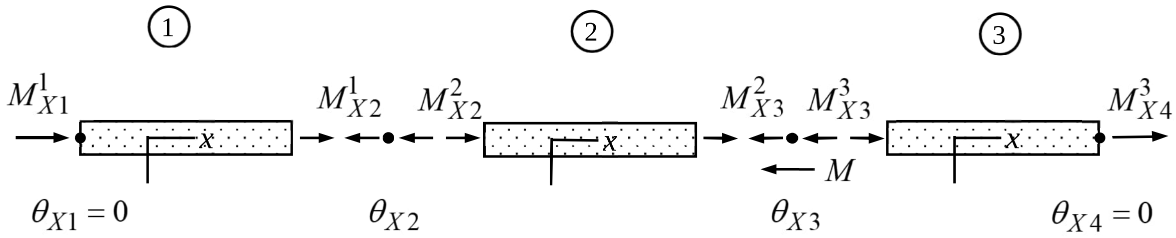
Solution to the unknown rotation was obtained from the equilibrium equation of a non-constrained node 2. The equilibrium equation of the constrained node 1 contains the constraint moment and is useful if that is needed too.

Torque M is acting in the direction of negative X-axis at node 3 of a torsion bar. Determine rotations θ_{X2} and θ_{X3} of nodes 2 and 3. Shear modulus G is constant and the polar moment of area J is piecewise constant. Use three elements of equal length.



Solution

Only the rotation in the direction of the bar matters. From the figure, the non-zero rotation components are θ_{X2} and θ_{X3}. Free body diagrams of the three torsion bar elements and nodes 2 and 3 are (nodes 1 and 4 are constrained and do not contribute to the system equations)



Element contributions of the torsion bar elements 1, 2 and 3 (formulae collection) and the equilibrium equations of nodes 2 and 3 are (notice that the distributed moment vanishes here)

Bar 1: $$\begin{Bmatrix} M_{X1}^1 \\ M_{X2}^1 \end{Bmatrix} = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \theta_{X2} \end{Bmatrix},$$

Bar 2: $$\begin{Bmatrix} M_{X2}^2 \\ M_{X3}^2 \end{Bmatrix} = \frac{GJ}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{X2} \\ \theta_{X3} \end{Bmatrix},$$

Bar 3: $$\begin{Bmatrix} M_{X3}^3 \\ M_{X4}^3 \end{Bmatrix} = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{X3} \\ 0 \end{Bmatrix},$$

Node 2: $\sum M_X = -M_{X2}^1 - M_{X2}^2 = 0,$

Node 3: $\sum M_X = -M_{X3}^2 - M_{X3}^3 - M = 0.$

Elimination of the internal forces from the two equilibrium equations of the nodes using the element contributions gives the forms

Node 2: $-(\frac{GJ}{L}\theta_{X2}) - (\frac{GJ}{2L}\theta_{X2} - \frac{GJ}{2L}\theta_{X3}) = 0,$

Node 3: $-(-\frac{GJ}{2L}\theta_{X2} + \frac{GJ}{2L}\theta_{X3}) - (\frac{GJ}{L}\theta_{X3}) - M = 0.$

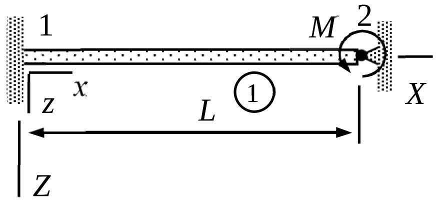
Matrix representation of the two equilibrium equations, containing the rotations of nodes 2 and 3 as unknowns, is

$$\frac{GJ}{2L}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{Bmatrix} \theta_{X2} \\ \theta_{X3} \end{Bmatrix} - M\begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \left(\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}^{-1} = \frac{1}{8}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\right)$$

$$\begin{Bmatrix} \theta_{X2} \\ \theta_{X3} \end{Bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}^{-1} 2\frac{ML}{GJ}\begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = \frac{1}{4}\frac{ML}{GJ}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = -\frac{1}{4}\frac{ML}{GJ}\begin{Bmatrix} 1 \\ 3 \end{Bmatrix} \quad \Leftrightarrow$$

$$\theta_{X2} = -\frac{1}{4}\frac{ML}{GJ} \quad \text{and} \quad \theta_{X3} = -\frac{3}{4}\frac{ML}{GJ}.$$

Determine rotation of the bending beam shown at node 2, internal forces and moments acting between the nodes and the beam element, and the constraint forces at the supports. The beam is clamped at the left end and simply supported at the right end. Young's modulus of the material E and the second moment of the cross-section $I_{yy} = I$ are constants. External distributed force $f_z = 0$.

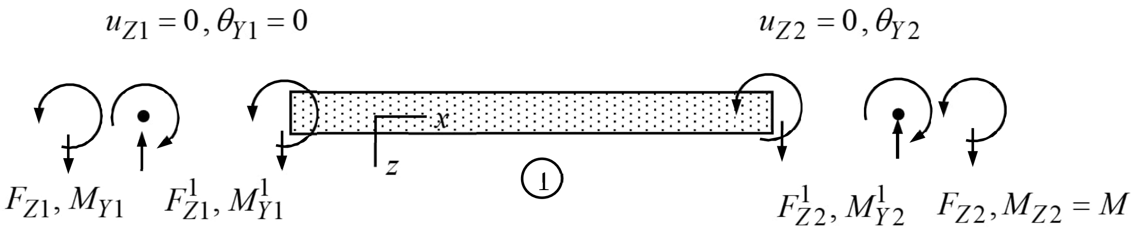


Solution

The generic force-displacement relationship of a bending beam element

$$\begin{Bmatrix} F_{z1} \\ M_{y1} \\ F_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$$

depends on the second moment of area I_{yy}, Young's modulus E, beam length h, and force per unit length f_z in the direction of the $z-$axis. Let us start with the free body diagram of the beam and the two nodes. As the axis of the material and structural system coincide, the displacement, rotation, force, and moments components of the two systems are the same



When written in terms of displacement, rotation, force, and moment components in the structural system, the beam element contribution becomes (as the orientation of the material and structural coordinate system is the same, the components are the same)

Beam: $$\begin{Bmatrix} F_{Z1}^1 \\ M_{Y1}^1 \\ F_{Z2}^1 \\ M_{Y2}^1 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6h & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

eq. 1
eq. 2
eq. 3
eq. 4

Equilibrium equations of the nodes are

Node 1 : $\sum F_Z = F_{Z1} - F_{Z1}^1 = 0$ eq. 5

$\sum M_Y = M_{Y1} - M_{Y1}^1 = 0$ eq. 6

Node 2 : $\sum F_Z = F_{Z2} - F_{Z2}^1 = 0$ eq. 7

$\sum M_Y = M - M_{Y2}^1 = 0$ eq. 8

The outcome is a set of 8 linear equations for 1 rotation, 4 internal forces, and 3 constraint forces/moments. As the first step toward the solution (always), the internal forces in the node equilibrium equations are replaced by their expressions given by eq.1, eq.2 , eq.3 and eq.4. After that, the unknown displacements and rotations follow from the corresponding equilibrium equations. Eq.4 and eq.8 imply first

$$\sum M_Y = M - \frac{EI}{L^3} 4L^2 \theta_{Y2} = 0 \quad \Leftrightarrow \quad \theta_{Y2} = \frac{1}{4}\frac{ML}{EI}. \quad \leftarrow$$

Use the code of MEC-E1050 to check the solution! Knowing the rotation angle, the remaining eq.1, eq.2 , and eq.3 of the beam element contribution give the internal forces

$$F_{Z1}^1 = -6L\frac{EI}{L^3}\frac{1}{4}\frac{ML}{EI} = -\frac{3}{2}\frac{M}{L}, \quad \leftarrow$$

$$M_{Y1}^1 = \frac{EI}{L^3} 2L^2 \frac{1}{4}\frac{ML}{EI} = \frac{1}{2}M, \quad \leftarrow$$

$$F_{Z2}^1 = \frac{EI}{L^3} 6L \frac{1}{4}\frac{ML}{EI} = \frac{3}{2}\frac{M}{L}. \quad \leftarrow$$

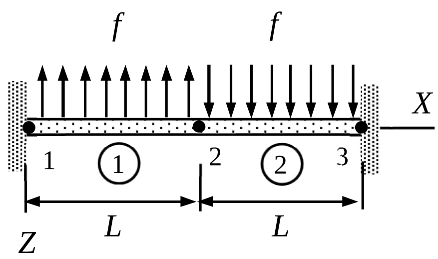
Constraint forces, due to the clamping at node 1 and simple support at node 2, follow from the remaining equilibrium eq.5, eq.6, and eq.7 and the solution to the internal forces

$$\sum F_Z = F_{Z1} + \frac{3}{2}\frac{M}{L} = 0 \quad \Leftrightarrow \quad F_{Z1} = -\frac{3}{2}\frac{M}{L}, \quad \leftarrow$$

$$\sum M_Y = M_{Y1} - \frac{1}{2}M = 0 \quad \Leftrightarrow \quad M_{Y1} = \frac{1}{2}M, \quad \leftarrow$$

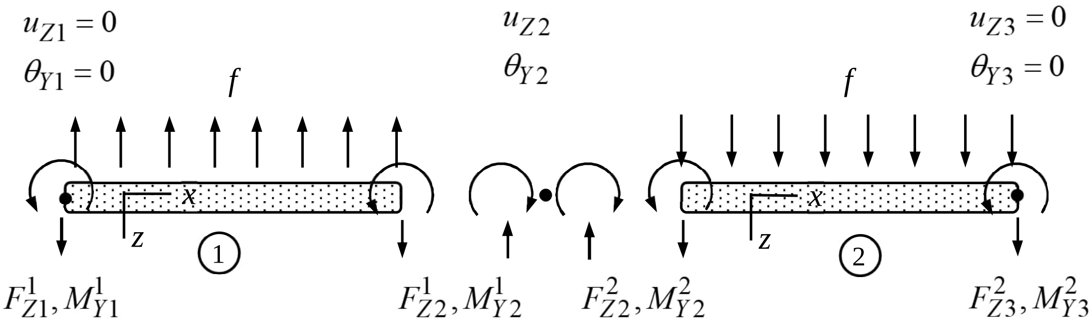
$$\sum F_Z = F_{Z2} - \frac{3}{2}\frac{M}{L} = 0 \quad \Leftrightarrow \quad F_{Z2} = \frac{3}{2}\frac{M}{L}. \quad \leftarrow$$

External load acting on the beam shown consists of piecewise constant parts having equal magnitudes but opposite signs. Determine displacement u_{Z2} and rotation θ_{Y2} of the mid-point (point 2). Young's modulus of the material and the second moments of area are E and I, respectively. Use two beam elements of equal length.



Solution

Only the displacement in the Z – direction and rotation in the Y – direction matter in the planar beam bending problem. From the figure, the non-zero displacement/rotation components are u_{Z2} and θ_{Y2} . Free body diagrams of the two bending beam elements and node 2 are (nodes 1 and 3 are constrained)



Element contributions of the two xz – plane bending beams (formulae collection) and the equilibrium equations of node 2 are (notice that the distributed force in the element contribution is the transverse component in the material system associated with beam)

Beam 1: $$\begin{Bmatrix} F_{Z1}^1 \\ M_{Y1}^1 \\ F_{Z2}^1 \\ M_{Y2}^1 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}, \quad (f_z = -f)$$

Beam 2: $$\begin{Bmatrix} F_{Z2}^2 \\ M_{Y2}^2 \\ F_{Z3}^2 \\ M_{Y3}^2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ 0 \\ 0 \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}, \quad (f_z = f)$$

Node 2: $-F_{Z2}^1 - F_{Z2}^2 = 0$ and $-M_{Y2}^1 - M_{Y2}^2 = 0$.

Elimination of the internal forces from the two equilibrium equations of node 2 using the element contributions gives the forms

Node 2: $$-[\frac{EI}{L^3}(12u_{Z2}+6L\theta_{Y2})+6\frac{fL}{12}]-[\frac{EI}{L^3}(12u_{Z2}-6L\theta_{Y2})-6\frac{fL}{12}]=0 \quad \text{and}$$

$$-[\frac{EI}{L^3}(6Lu_{Z2}+4L^2\theta_{Y2})+L\frac{fL}{12}]-[\frac{EI}{L^3}(-6Lu_{Z2}+4L^2\theta_{Y2})+L\frac{fL}{12}]=0\,.$$

Matrix representation of the two equilibrium equations, containing u_{Z2} and θ_{Y2} as the unknowns, is

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}-fL^2\begin{Bmatrix} 0 \\ -1/6 \end{Bmatrix}=0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}=\frac{fL^5}{EI}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}^{-1}\begin{Bmatrix} 0 \\ -1/6 \end{Bmatrix}=\frac{fL^5}{EI}\begin{bmatrix} 1/24 & 0 \\ 0 & 1/(8L^2) \end{bmatrix}\begin{Bmatrix} 0 \\ -1/6 \end{Bmatrix}=\frac{1}{48}\frac{fL^3}{EI}\begin{Bmatrix} 0 \\ -1 \end{Bmatrix} \quad \Leftrightarrow$$

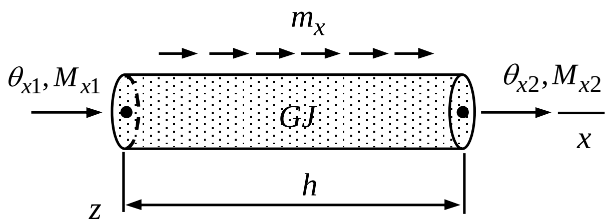
$u_{Z2}=0$ and $\theta_{Y2}=-\frac{1}{48}\frac{fL^3}{EI}$. ←

The boundary value problem defining the element contribution of a torsion bar consist of

$$GJ\frac{d^2\phi}{dx^2}+m_x=0 \quad x\in]0,h[\,,$$

$$\phi(0)=\theta_{x1} \quad \text{and} \quad \phi(h)=\theta_{x2}\,,$$

$$GJ\frac{d\phi}{dx}(0)=-M_{x1} \quad \text{and} \quad GJ\frac{d\phi}{dx}(h)=M_{x2}\,,$$



in which the shear modulus G, cross-sectional area of the bar A, and external distributed moment per unit length m_x are constants. Derive the element contribution of a torsion bar element with the aid of the boundary value problem.

Solution

The equations defining the element contribution of a torsion bar consist of the equilibrium equation, and boundary conditions for rotations and moments at the nodes. As the number of boundary conditions is four, existence of the solution is possible only under certain condition on ”data” GJ, m_x, h, θ_{x1}, θ_{x2}, M_{x1}, M_{x2}. The condition for the data is the torsion bar element contribution.

First, integration of the equilibrium twice is used to find the generic solution (any method to find the solution goes)

$$\phi=a+bx-\frac{m_x}{2GJ}x^2=\{1 \quad x\}\begin{Bmatrix}a\\b\end{Bmatrix}-\frac{m_x}{2GJ}x^2\,.$$

After that, the rotation boundary conditions are used to express the integration constants a and b in terms of the nodal rotations

$$\begin{Bmatrix}\theta_{x1}\\\theta_{x2}\end{Bmatrix}=\begin{Bmatrix}\phi(0)\\\phi(h)\end{Bmatrix}=\begin{bmatrix}1&0\\1&h\end{bmatrix}\begin{Bmatrix}a\\b\end{Bmatrix}-\frac{m_x}{2GJ}\begin{Bmatrix}0\\h^2\end{Bmatrix} \quad\Leftrightarrow\quad \begin{Bmatrix}a\\b\end{Bmatrix}=\frac{1}{h}\begin{bmatrix}h&0\\-1&1\end{bmatrix}(\begin{Bmatrix}\theta_{x1}\\\theta_{x2}\end{Bmatrix}+\frac{m_x}{2GJ}\begin{Bmatrix}0\\h^2\end{Bmatrix})$$

to get

$$\phi=\{1 \quad x\}\frac{1}{h}\begin{bmatrix}h&0\\-1&1\end{bmatrix}(\begin{Bmatrix}\theta_{x1}\\\theta_{x2}\end{Bmatrix}+\frac{m_x}{2GJ}\begin{Bmatrix}0\\h^2\end{Bmatrix})-\frac{m_x}{2GJ}x^2\,.$$

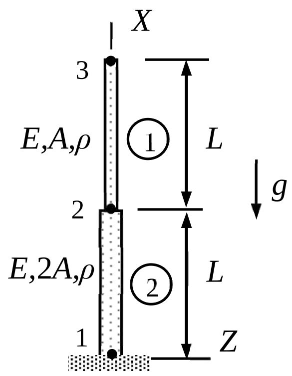
Finally, the moment boundary conditions and the rotation solution give

$$M_{x1}=-GJ\frac{d\phi}{dx}(0)=\frac{GJ}{h}(\theta_{x1}-\theta_{x2})-\frac{m_xh}{2}\,,$$

$$M_{x2}=GJ\frac{d\phi}{dx}(h)=\frac{GJ}{h}(\theta_{x2}-\theta_{x1})-\frac{m_xh}{2}$$

or in a more concise form

$$\begin{Bmatrix} M_{x1} \\ M_{x2} \end{Bmatrix} = \frac{GJ}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$



Consider the structure of the figure loaded by its own weight. Determine the displacement u_{X3} of the free end by using bar elements (1) and (2). The cross-sectional area of the bar (2) is twice that of bar (1). Acceleration by gravity g and material properties E and ρ are constants.

Solution

Let the material coordinate systems of both bars coincide with the structural system. The element and node tables are

	model	properties	geometry
1	BAR	{{E}, {A}, {-A g ρ, 0, 0}}	Line[{2, 3}]
2	BAR	{{E}, {2 A}, {-2 A g ρ, 0, 0}}	Line[{1, 2}]

	{X,Y,Z}	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, 0, 0}
3	{2 L, 0, 0}	{uX[3], 0, 0}	{0, 0, 0}

According to the recipe for assembly (build of the system equations), element contributions are first written in terms of the displacement and force components in the structural coordinate system (notice that gravity is acting in the direction of the negative x-axis):

$$\text{Bar 1}: \begin{Bmatrix} F_{X2}^1 \\ F_{X3}^1 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} + \frac{gA\rho L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\text{Bar 2}: \begin{Bmatrix} F_{X1}^2 \\ F_{X2}^2 \end{Bmatrix} = \frac{E2A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} + \frac{2gA\rho L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

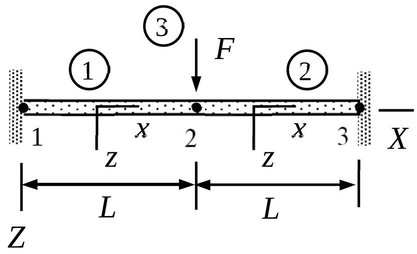
For equilibrium of nodes, sums of the internal forces and moments acting on the nodes need to vanish. In build of the system equations (minimal equation set) it is enough to consider the non-constrained directions of displacements at nodes 2 and 3:

$$\begin{Bmatrix} F_{X2}^1 + F_{X2}^2 \\ F_{X3}^1 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} + \frac{gA\rho L}{2}\begin{Bmatrix} 3 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} = -\frac{1}{2}\frac{gA\rho L^2}{EA}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{Bmatrix} 3 \\ 1 \end{Bmatrix} = -\frac{1}{4}\frac{gA\rho L^2}{EA}\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{Bmatrix} 3 \\ 1 \end{Bmatrix} = -\frac{1}{2}\frac{gA\rho L^2}{EA}\begin{Bmatrix} 2 \\ 3 \end{Bmatrix} \quad \Leftrightarrow$$

$$u_{X2} = -\frac{g\rho L^2}{E} \text{ and } u_{X3} = -\frac{3}{2}\frac{g\rho L^2}{E}.$$

Determine the displacement u_{Z2} at node 2 of the beam structure shown. Use two beam elements of equal length. Assume that rotation $\theta_{Y2}=0$. Point force of magnitude F is acting on node 2. Young's modulus of the material E and the second moment of area I are constants.



Solution

In hand calculations, explicit forms of the node and element tables are not needed. In simple cases, the relationship between the displacement, rotation, force, and moment components of the material coordinate and structural coordinate systems can also be deduced from the figure. The beam and point force element contributions are

$$\begin{Bmatrix} F_{z1} \\ M_{y1} \\ F_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix} \text{ and } \begin{Bmatrix} F_{X1} \\ F_{Y1} \\ F_{Z1} \end{Bmatrix} = -\begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix}.$$

Element contributions need to be written in terms of the displacement and rotation components of the structural coordinate system. The structure has just one-degree of freedom u_{Z2}.

Beam 1: $$\begin{Bmatrix} F_{Z1}^1 \\ M_{Y1}^1 \\ F_{Z2}^1 \\ M_{Y2}^1 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ 0 \end{Bmatrix},$$

Beam 2: $$\begin{Bmatrix} F_{Z2}^2 \\ M_{Y2}^2 \\ F_{Z3}^2 \\ M_{Y3}^2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ 0 \\ 0 \\ 0 \end{Bmatrix},$$ Force 3: $$\begin{Bmatrix} F_{X2}^3 \\ F_{Y2}^3 \\ F_{Z2}^3 \end{Bmatrix} = -\begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix}.$$

Assembly means writing the equilibrium equations of the nodes. In practice, the equations giving the displacements and rotations are obtained by summing the internal forces in directions where displacement and rotation components are not constrained. If point forces are considered as one node element, the sum is over the elements connected to a node.

$$\sum F_{Z2}^i = F_{Z2}^1 + F_{Z2}^2 + F_{Z2}^3 = 24\frac{EI}{L^3}u_{Z2} - F = 0 \quad \Leftrightarrow \quad u_{Z2} = \frac{1}{24}\frac{FL^3}{EI}.$$