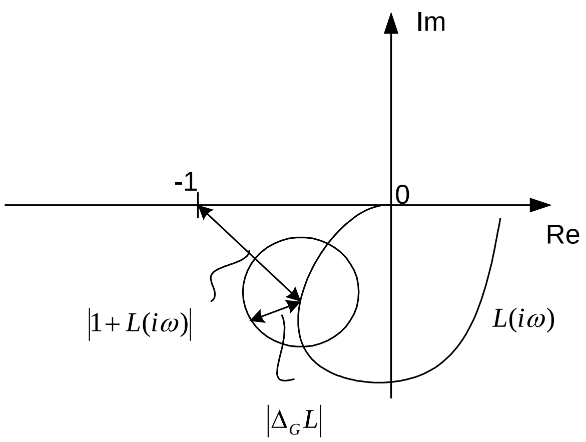
ELEC-E8116 Model-based control systems
/ exercises with solutions 6

Problem 1. Consider a SISO-system in a one-degree-of-freedom control configuration. The connection between the real and nominal system is

$$G_0(s) = G(s)(1+\Delta_G(s))$$

By using the Nyquist stability criterion derive a condition to the system to be robustly stable.

Solution. In the one-degree-of-freedom configuration $F_y = F_r$ and the loop transfer function of the nominal system is $L = GF_y$. The Nyquist curve is seen in the figure



At each point $L(i\omega)$ a circle with radius $L\Delta_G$ describes the model uncertainty such that the real curve is ceratinly inside the circle. Assuming that the nominal system is stable (no poles in RHP), the closed loop system is robustly stable exactly when the Nyquist curve does not encircle the critical point (-1,0). That can be expressed as (see figure)

$$|\Delta_G L| < |1+L|, \quad \forall\omega$$

and further

$$|T| < \frac{1}{|\Delta_G|}, \quad \forall\omega$$

which is the same as

$$\|\Delta_G T\|_\infty < 1$$

The result is the same as in the textbook formulas (6.28) and (6.29).

Problem 2. Consider the first order process

$$G_P(s) = \frac{k}{\tau s + 1} e^{-\theta s}$$

with parameter uncertainties such that $2 \le k, \theta, \tau \le 3$. The system is modelled with

$$G_0(s) = G(s)(1 + \Delta_G(s))$$

in which the nominal model is chosen to be the first order model without delay

$$G(s) = \frac{\bar{k}}{\bar{\tau} s + 1} = \frac{2.5}{2.5s + 1}$$

Discuss possible candidates for the function $\Delta_G(s)$.

Solution.
To be exact, an accurate uncertainty area in each frequency of the complex function $G_P(i\omega)$ should be determined (corresponding to the parameter variations). This is very difficult, however, and in practice an approximate solution with uncertainty circles is used; see the solution to problem 1 and the equation

$$G_0(s) = G(s)(1 + \Delta_G(s))$$

The largest relative error must be determined

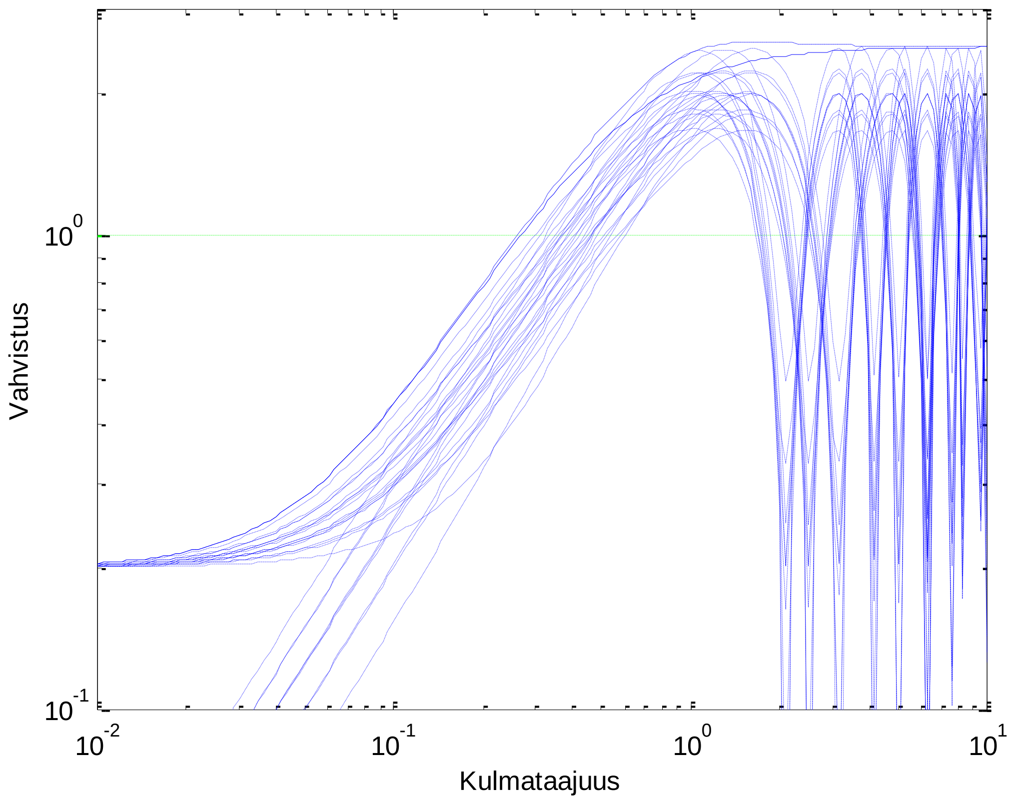
$$l_I(\omega) = \max_{G_P \in \Pi} \left| \frac{\mathrm{G_P}(i\omega) - G(i\omega)}{G(i\omega)} \right|$$

(Π means all possible models when the parameters vary within the given intervals).

Then we can take

$$|\Delta_G(i\omega)| \ge l_I(\omega), \quad \forall \omega$$

To calculate $l_I(\omega)$ choose the values 2, 2.5 and 3 for each variable (k, θ, τ). That does not necessarily describe the worst possible situation, but it is a step to the right direction. For the functions $l_I(\omega)$ we obtain $3^3 = 27$ curves, which are shown in the figure



The curve $l_I(\omega)$ must at each freqiency be larger than the dotted curves. It is seen that the value of $l_I(\omega)$ in small frequencies is 0.2 and 2.5 in large frequencies. For a candidate of Δ_G try a first order model, which corresponds to this behaviour

$$\Delta_{G1}(s) = \frac{Ts + 0.2}{(T/2.5)s + 1}, \quad T = 4$$

From the solid line it is seen that this is pretty good except near the frequency $\omega = 1$ where $\Delta_{G1}(s)$ is a little too small to cover all uncertainties. Increase the magnitude a bit near that particular frequency

$$\Delta_{G2}(s) = \Delta_{G1}(s)\frac{s^2 + 1.6s + 1}{s^2 + 1.4s + 1}$$

which is good (dash-dotted line in the figure).

Problem 3. Consider the process described in Exercise 5, Problem 1 with the exception that the process model is uncertain. The true system is

$$G_0(s) = G(s)(1 + \Delta_G(s))$$

in which the relative uncertainty has been modeled as

$$\Delta_G(s) = \frac{10s + 0.33}{(10/5.25)s + 1}$$

Is the controlled (closed loop) system robustly stable?

Solution.

The process model and controller were

$$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)} \qquad K(s) = 1.136(1 + \frac{1}{12.7s})$$

As noticed in problem 27 the determination of the relative error Δ_G is difficult. Often a simple error model is used, e.g. the first order transfer function

$$\Delta(s) = \frac{\tau s + r_0}{(\tau / r_\infty)s + 1}$$

where r_0 is the relative error of the stationary state, $1/\tau$ is approximately that angular frequency, in which the relative error reaches the 100% level, and r_∞ is the relative error in high frequencies (typically $r_\infty \geq 2$).

In the problem the relative error of the process model has been assumed to be 0.33 in small frequencies, about. 1 in frequency 0.1 *rad/s* and 5.25 in high frequencies.

The system is robustly stable, if it holds for all frequencies (textbook formula (6.29))

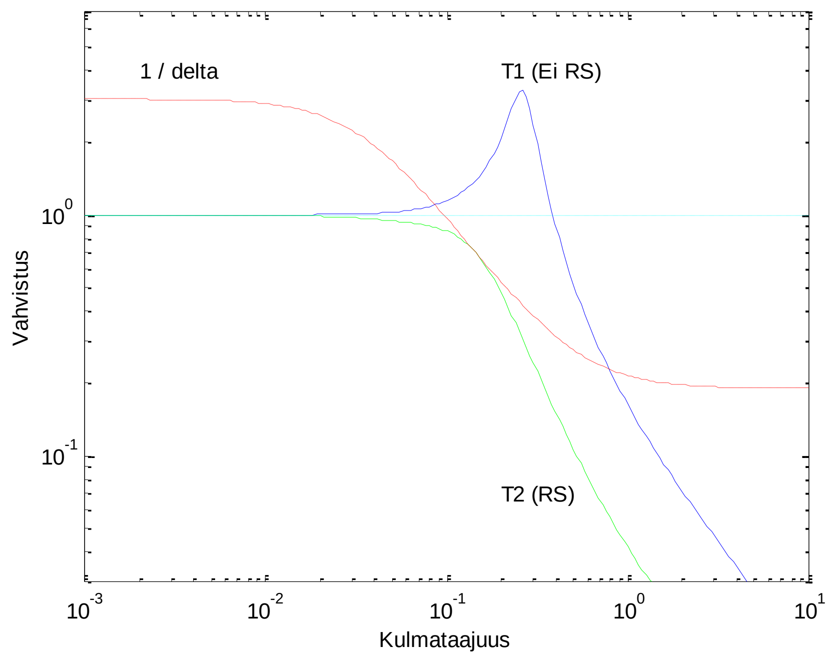
$$|T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|}$$

in which the complementary sensitivity function is

$$T = \frac{GK}{1+GK}$$

From the figure it is seen that the system is not robustly stable, because the complementary sensitivity function $T1$ exceeds $1/\Delta_G$ in frequencies above 0.1.

By decreasing the gain of the PI-controller from 1.13 to 0.31 (trial and error result) a robustly stable closed-loop system is obtained (curve $T2$).



Problem 4. Let a closed-loop SISO-system be stable. Prove that the maximum delay that can be added to the process without causing closed-loop instability is

$$\theta_{\max} = PM / \omega_c$$

where *PM* is the phase margin of the (original) system and ω_c is the gain crossover frequency.

Solution.
Let *G* be the original transfer function. In the gain crossover frequency it holds

$$|G(i\omega_c)| = 1 \quad \text{and} \quad \arg G(i\omega_c) = -\pi + PM$$

When pure delay is added to the process

$$\left|G(i\omega_c)e^{-i\theta\omega_c}\right| = |G(i\omega_c)| = 1$$

which means that the gain crossover frequency remains the same. For the phase

$$\arg G(i\omega_c)e^{-i\theta\omega_c} = \arg G(i\omega_c) - \theta\omega_c = -\pi + PM - \theta\omega_c$$

At the stability limit

$$\arg G(i\omega_c)e^{-i\theta_{\max}\omega_c} = -\pi + PM - \theta_{\max}\omega_c = -\pi$$

from which

$$\theta_{\max} = \frac{PM}{\omega_c}$$

Problem 5. Let the weight of the sensitivity function be given as

$$\frac{1}{W_s} = A\frac{\dfrac{s}{A\omega_0}+1}{\dfrac{s}{B\omega_0}+1}, \quad 0 < A << 1,\ B >> 1$$

Sketch a schema for the magnitude plot of the frequency response and investigate its characteristics. What is the slope in the increasing part of the curve? What is the magnitude at frequency ω_0?

Generate a second order model, where the slope is twice as large as in the previous case. Investigate again the characteristics. What is the magnitude at frequency ω_0?

Solution:

$$\frac{1}{W_s(j\omega)} = A\frac{\dfrac{j\omega}{A\omega_0}+1}{\dfrac{j\omega}{B\omega_0}+1} = A\frac{\dfrac{1}{A\omega_0}+\dfrac{1}{j\omega}}{\dfrac{1}{B\omega_0}+\dfrac{1}{j\omega}} \quad \text{Clearly } \frac{1}{W_s(j0)} = A, \quad \frac{1}{W_s(j\infty)} = B$$

For $\omega \to \omega_0$ $$\left|\frac{1}{W_s(j\omega)}\right| = A\sqrt{\frac{1+\left(\dfrac{\omega}{A\omega_0}\right)^2}{1+\left(\dfrac{\omega}{B\omega_0}\right)^2}} = \underbrace{A\sqrt{\frac{1+\left(\dfrac{1}{A}\right)^2}{1+\left(\dfrac{1}{B}\right)^2}}}_{\omega=\omega_0} = \sqrt{\frac{1+A^2}{1+\dfrac{1}{B^2}}} \approx 1,$$ because B is "large" and A is "small".

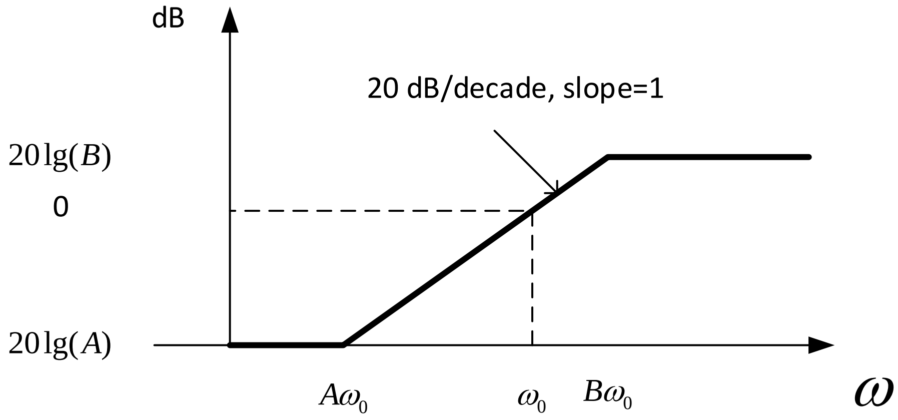
The Bode diagram (amplitude) is shown below:

Note that for the absolute value of the term $1+j\omega T$ in the frequency response it holds

$$\sqrt{1+(\omega T)^2} \underset{\omega=1/T}{=} \sqrt{2} \approx 3\text{dB}$$ which can be approximated as 0 dB. For higher frequencies

$$\sqrt{1+(\omega T)^2} \approx \sqrt{(\omega T)^2} = \omega T \Rightarrow 20\lg(\omega T) = 20\lg(\omega) + 20\lg(T)$$

increases 20dB/decade (slope = 1) from zero decibels at $\omega = 1/T$.



Note that in the lecture slides an example of *Mixed Sensitivity Design* was shown with the desired sensitivity weight

$\frac{1}{W_s(s)} = \frac{s + \omega_B^* A}{\frac{s}{M} + \omega_B^*}$. This is the same parameterization as in the problem, by $M = B, \omega_B^* = \omega_0$.

The second order model is

$$\frac{1}{W_s} = A \frac{(\frac{j\omega}{A^{1/2}\omega_0} + 1)^2}{(\frac{j\omega}{B^{1/2}\omega_0} + 1)^2}$$

Similar calculus as above shows that the amplitude curve is as in the above figure but with the angular frequencies $\left(A^{1/2}\omega_0, \omega_0, B^{1/2}\omega_0\right)$ instead of $\left(A\omega_0, \omega_0, B\omega_0\right)$. The curve increases 40 dB/decade, slope is 2. Note that this is again the same as

$$\frac{1}{W_s(s)} = \frac{(s + \omega_B^* A^{1/2})^2}{(\frac{s}{M^{1/2}} + \omega_B^*)^2}.$$

Problem 6. Consider the angular frequencies $\omega_B, \omega_c, \omega_{BT}$ which are used to define the bandwidth of a controlled system. State the definitions. Prove that when the phase margin is less than 90 degrees ($PM < \pi/2$) it holds $\omega_B < \omega_c < \omega_{BT}$. Interpretations?

Solution: Definitions:

ω_B : where S crosses $1/\sqrt{2} \approx -3\text{dB}$ from below.

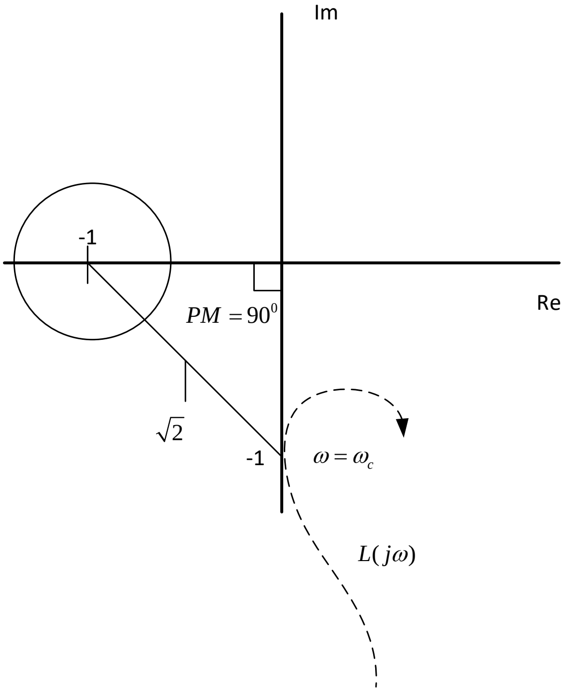
ω_c : where L crosses 1 = 0 dB (gain crossover (angular) frequency)

ω_{BT} : where T crosses $1/\sqrt{2} \approx -3\text{dB}$ from above.

At the gain crossover frequency it holds

$$|L(j\omega_c)| = 1 \Rightarrow |T(j\omega_c)| = \left|\frac{L(j\omega_c)}{1+L(j\omega_c)}\right| = \frac{|L(j\omega_c)|}{|1+L(j\omega_c)|} = \frac{1}{|1+L(j\omega_c)|} = \left|\frac{1}{1+L(j\omega_c)}\right| = |S(j\omega_c|$$

(Note that $L(j\omega_c)$ is a complex number and so $|1+L(j\omega_c)| \neq 1+|L(j\omega_c)|$.
$|1+x+jy| = \sqrt{(1+x)^2+y^2} \neq 1+\sqrt{x^2+y^2}$, except in some rare exceptional cases (when?)).



The figure shows the Nyquist diagram of L where the phase margin PM = 90 degrees. In the gain crossover frequency then
$|S(j\omega_c| = |T(j\omega_c| = 1/\sqrt{2} \approx -3\text{ dB}$ (The distance from the point (-1,0) is inversely proportional to the absolute value of S. See lecture slides, Chapter 3).

So, at ω_c all the bandwidths would coincide.

But when PM < 90 degrees $|S(j\omega_c| = |T(j\omega_c| > 1/\sqrt{2}$, which implies directly that

S approaches from below $\Rightarrow \omega_B < \omega_c$
T approaches from above $\Rightarrow \omega_{BT} > \omega_c$.

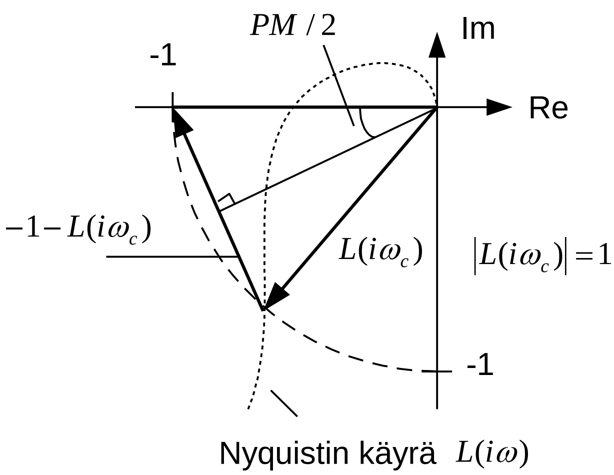
We can conclude that roughly all the frequencies described can be used to discuss bandwidth, describing the behaviour of the closed-loop system.

Problem 7. Consider a SISO-system. The maximum values of the sensitivity and complementary functions are denoted M_S and M_T , respectively. Let the gain and phase margins of a closed-loop system be GM (gain margin) and PM (phase margin). Prove that

$$GM \geq \frac{M_S}{M_S - 1} \qquad PM \geq 2\arcsin\left(\frac{1}{2M_S}\right) \geq \frac{1}{M_S} \text{ [rad]}$$

$$GM \geq 1 + \frac{1}{M_T} \qquad PM \geq 2\arcsin\left(\frac{1}{2M_T}\right) \geq \frac{1}{M_T} \text{ [rad]}$$

Solution:
Start from the figure below, where the Nyquist diagram of the loop transfer function (L) has been presented.



Denote the phase crossover frequency by ω_{180} (then the phase of L is –180 degrees). By the definition of the gain margin

$$GM = \frac{1}{|L(i\omega_{180})|} \quad \Rightarrow \quad L(i\omega_{180}) = \frac{-1}{GM}$$

We obtain

$$T(i\omega_{180}) = \frac{L(i\omega_{180})}{1+L(i\omega_{180})} = \frac{-1}{GM-1}$$

$$S(i\omega_{180}) = \frac{1}{1+L(i\omega_{180})} = \frac{1}{1-\frac{1}{GM}}$$

Now use the abbreviations $M_T = \max_{\omega} |T(i\omega)|$, $M_S = \max_{\omega} |S(i\omega)|$

and it follows that

$$M_T \geq \frac{1}{|GM-1|}; \qquad M_S \geq \frac{1}{\left|1-\frac{1}{GM}\right|}$$

and the gain margin inequalities given in the problem follow easily. Let us calculate the first as an example.

$$M_T \geq \frac{1}{|GM-1|} \Rightarrow |GM-1| \geq \frac{1}{M_T} \Rightarrow GM-1 \geq \frac{1}{M_T} \Rightarrow GM \geq 1+\frac{1}{M_T}$$

The inequality related to M_S is derived correspondingly.

Considering the phase margin note that

$$|S(i\omega_c)| = \frac{1}{|1+L(i\omega_c)|} = \frac{1}{|-1-L(i\omega_c)|}$$

in which ω_c is the gain crossover frequency (the gain of L is one in this frequency). From the figure it can be seen that

$$|S(i\omega_c)| = |T(i\omega_c)| = \frac{1}{2\sin(PM/2)}$$

and the inequalities related to phase margin follow directly. (In the last form the following fact, obtained for example by the Taylor approximation, is used: when x is positive, $\arcsin(x) > x$.)

The results show for example that if $M_T = 2$, then $GM \geq 1.5$, $PM \geq 29^\circ$.

Sometimes the maximum values (∞ - norms) M_S and M_T are used as alternatives to gain and phase margins. For example, demanding that $M_S < 2$, the often used ”rules of thumb” $GM > 2$, $PM > 30^\circ$ follow.