ELEC-E8101 Digital and Optimal Control
Intermediate exam 1   19. 10. 2022   Solutions

- *Write the name of the course, your name and student number to each answer sheet.*
- *There are three (3) problems and each one must be answered.*
- *No other literature except the Table of Formulas is allowed. A calculator can be used. Only basic calculations with it are allowed (e.g. no matrix calculations, Laplace- or Z-transformations, www-connections).*
- *The table of formulas must be returned, if you have received it from the exam supervisor.*

Three (3) problems, max. 6 points each.

In solving the problems you can use the Tables as much as you wish. Note above the conditions for using the pocket calculator; it is only an aid for basic calculations**. Write your solutions in a way, where it becomes perfectly clear how you have obtained the solution.**

1. You are working in a company called *Full-service automation house*, which does automation projects for the industry. Your client asks for a short written explanation to the following concepts, which appear in your documents. Write those short descriptions. Please note that the client is somewhat aware of continuous time control theory and automation but does not know digital control. Also, the client does not want to read long explanations.

- zero-order hold (1 p.)
- pulse transfer function (1 p.)
- alias effect (1 p.)
- hidden oscillations (1 p.)
- BIBO stability (1 p.)
- final value theorem and its use in the determination of static gain (discrete time case) (1p.)

Solution

Zero order hold (ZOH) means digital to analog (D/A) conversion of a signal, in which the pulse amplitude is kept at a constant level until the next pulse enters at the next sampling instant. In discrete-time control system this is mostly used when changing the output of a digital controller to a continuous input signal to the process.

Pulse transfer function is obtained by dividing the z-transformation of the discrete time system output with the z-transformation of the input signal. It gives an algebraic relationship between the pulses relating input to output in a dynamical system. The pulse transfer function is also the z-transformation of the pulse response of the system.

The Alias effect occurs when a continuous time signal is sampled with a frequency which is too sparse with respect to the frequency content of the continuous signal. That means that information is lost in sampling. By the Shannon's sampling theorem the sampling frequency must be at least twice as big as the highest frequency of interest in the original signal. In practice, the sampling frequency must be 2-10 times higher.

Hidden oscillations mean that some oscillation in the continuous time signal does not show in the sampled signal. This can happen due to the Alias effect but also in some problematic cases related to system observability.

A system is Bounded input – Bounded output (BIBO) stable when **every** bounded input signal leads to a bounded output signal.

The final value theorem in discrete-time case says that if the limits exist then for a signal $y(k)$ and its z-transformation $Y(z)$, then

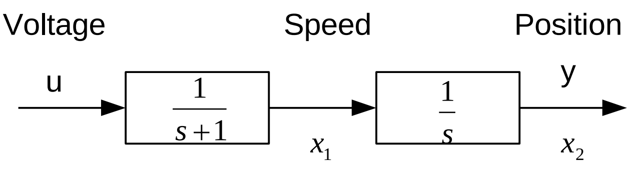
$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(1-z^{-1})Y(z) = \lim_{z\to 1}\left(\frac{z-1}{z}\right)Y(z)$$

Let $y(k)$ be the step response of a system given by the pulse transfer function $H(z)$. Then it follows

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}\left(\frac{z-1}{z}\right)H(z)\left(\frac{z}{z-1}\right) = \lim_{z\to 1} H(z) = H(1)$$

which gives a handy way to calculate the static gain for step input. The limits exist when the system H is asymptotically stable.

2. A DC/DC motor can be described by the simple model below, where *u* is the voltage input and *y* the shaft position.



a. Determine the transfer function and state-space representation of the continuous time system. Use the state variables shown in the figure. (1 p.)

b. Determine the corresponding discrete-time pulse transfer function when zero order hold is assumed and the sampling interval is *h*. (3 p.)

c. Explain verbally the correspondence $z = e^{sh}$. Verify that it holds in this case. (2 p.)

Solution

a. From the figure the transfer function is obtained directly

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+1)}$$

It follows that

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2+s} \Rightarrow \ddot{y}(t) + \dot{y}(t) = u(t)$$

By using the state variables in the figure

$$x_2(t) = y(t), \quad x_1(t) = \dot{y}(t) = \dot{x}_2(t)$$

we get (note the differential equation)

$$\begin{aligned} \dot{x}_1(t) &= \ddot{y}(t) = -x_1(t) + u(t) \\ \dot{x}_2(t) &= x_1(t) \end{aligned}$$

The state-space representation is

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \end{aligned}$$

b. The easiest way is to use the Tables, ZOH equivalents. We note that the ZOH equivalent to

$\frac{a}{s(s+a)}$ is $\frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$ where the coefficients are given in the table. The pulse transfer function is

$$\frac{\left(h-1+e^{-h}\right)z+1-e^{-h}-he^{-h}}{z^2-\left(1+e^{-h}\right)z+e^{-h}}=\frac{\left(h-1+e^{-h}\right)z+1-e^{-h}-he^{-h}}{(z-1)(z-e^{-h})}$$

Another method would be to start from the continuous-time state-space representation, to discretize it (ZOH) and then to change the discrete state-space representation into pulse transfer function.

A third way would be to use the formula

$$H(z)=\frac{z-1}{z}\cdot Z\left\{L^{-1}\left\{G(s)\frac{1}{s}\right\}\Big|_{t=kh}\right\}$$

All methods lead to the same result. However, the last two need somewhat lengthy calculations.

c. The relationship $z=e^{sh}$ determines, how the poles (*s*) are mapped into discrete time model poles (z) under ZOH discretization. Here *s* = 0 and -1.

$$z=e^{0\cdot h}=1,\quad z=e^{-1\cdot h}=e^{-h}\quad \text{Ok.}$$

4. Consider the model

$$\begin{cases} x(kh+h)=\begin{bmatrix}1 & h\\ 0 & 1\end{bmatrix}x(kh)+\begin{bmatrix}\frac{1}{2}h^2\\ h\end{bmatrix}u(kh)\\ y(kh)=\begin{bmatrix}1 & 0\end{bmatrix}x(kh)\end{cases}$$

where *h* is the sampling interval.

a. Determine the pulse transfer function. Based on that, deduce what kind of process the model describes? What is the corresponding continuous-time transfer function, under the assumption of ZOH? (2 p.)

b. Is the system i. reachable, ii. observable? Explain in words what your answers mean. (2 p.)

c. Is the process asymptotically stable? Is it BIBO stable? (2 p.)

Solution

a.

$$H(z) = C\left(zI - \Phi\right)^{-1}\Gamma$$

$$zI - \Phi = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z-1 & -h \\ 0 & z-1 \end{bmatrix}$$

$$(zI - \Phi)^{-1} = \frac{1}{(z-1)^2}\begin{bmatrix} z-1 & 0 \\ h & z-1 \end{bmatrix}^T = \frac{1}{(z-1)^2}\begin{bmatrix} z-1 & h \\ 0 & z-1 \end{bmatrix}$$

$$H(z) = \begin{bmatrix} 1 & 0 \end{bmatrix}\frac{1}{(z-1)^2}\begin{bmatrix} z-1 & h \\ 0 & z-1 \end{bmatrix}\begin{bmatrix} \frac{1}{2}h^2 \\ h \end{bmatrix} = \frac{1}{(z-1)^2}\begin{bmatrix} z-1 & h \end{bmatrix}\begin{bmatrix} \frac{1}{2}h^2 \\ h \end{bmatrix}$$

$$= \frac{1}{(z-1)^2}\left[\frac{1}{2}h^2 z - \frac{1}{2}h^2 + h^2\right] = \frac{\frac{1}{2}h^2(z+1)}{(z-1)^2}$$

It is a double integrator (two poles at $z = 1$). Also, from the tables it is immediately observed that the pulse transfer function is the ZOH-equivalent to the transfer function $1/s^2$.

b. i.The reachability (controllability) matrix is

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix} = \begin{bmatrix} \frac{1}{2}h^2 & \frac{3}{2}h^2 \\ h & h \end{bmatrix}; \quad \det W_c = \frac{1}{2}h^3 - \frac{3}{2}h^3 = -h^3 \neq 0$$

The system is reachable, which means that any state from any initial state can be reached in a finite time by using a suitable control signal.

ii. The observability matrix is

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & h \end{bmatrix}; \quad \det Wo = h \neq 0$$

so the system is observable. It means that any initial state can be determined by observing the input and output signals for a finite time. (Consequently, it means that by observing the input and output signals the state can be estimated such that the estimation error approaches zero).

c. The two poles of the system are $z = 1$, which are on the unit circle. The system is not asymptotically stable and not BIBO stable. (For example, a unit step input causes the output to grow without limit).