Exercise 7 – Solutions

1.11.2022

#1 Incomplete information

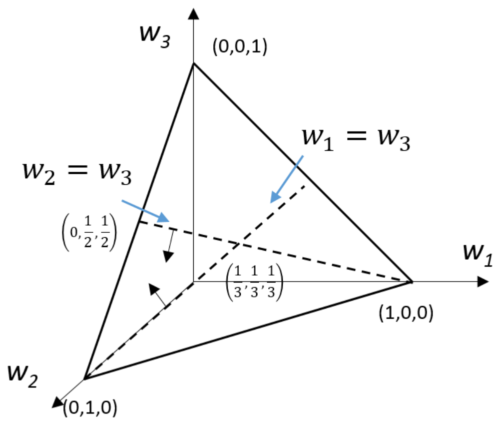
a) The preference statements imply inequalities

$$w_2\left(v_2^N(x_2^*) - v_2^N(x_2^0)\right) \geq w_3\left(v_3^N(x_3^*) - v_3^N(x_3^0)\right) \geq w_1\left(v_1^N(x_1^*) - v_1^N(x_1^0)\right) \Leftrightarrow$$
$$w_2 \geq w_3 \geq w_1.$$

b) The set of feasible weights:

$$S = \left\{w \in \mathbb{R}^3 | w_2 \geq w_3 \geq w_1, \sum_{i=1}^{3} w_i = 1, w_i \geq 0\ \forall i\right\}.$$

The extreme points of this set are (0,1,0), (0, ½, ½), (1/3, 1/3, 1/3) .



c) Table 1 shows the alternatives' overall values at the extreme points of S.

	w=(0,1,0)	w=(0,1/2,1/2)	w=(1/3,1/3,1/3)
$V(x^1)$	0.50	0.55	0.37
$V(x^2)$	0.40	0.45	0.40
$V(x^3)$	0.50	0.50	0.67

Because the minimum and maximum overall values are obtained at the extreme points, the value intervals become

$$V(x^1) \in [0.37, 0.55], \quad V(x^2) \in [0.40, 0.45], \quad V(x^3) \in [0.50, 0.67].$$

d) Alternative x^k dominates x^j, iff

$\min_w \left(V(x^k, w, v) - V(x^j, w, v)\right) \geq 0$ and

$\max_w \left(V(x^k, w, v) - V(x^j, w, v)\right) > 0.$

The alternatives' pairwise value differences at each extreme point are:

	w=(0,1,0)	w=(0,1/2,1/2)	w=(1/3,1/3,1/3)
$V(x^1) - V(x^2)$	0.10	0.10	-0.03
$V(x^2) - V(x^3)$	-0.10	-0.05	-0.27
$V(x^1) - V(x^3)$	0	0.05	-0.30

Because the minimum and maximum value differences are obtained at the extreme points, it is concluded that x^3 dominates x^2 and no other dominance relationships exist.

#2 Sensitivity analysis

a) V(A)=200>V(B)=195>V(C)=185.

b) The normalized value function $V^N(x)$ is a positive affine transformation of $V(x)$:

$$V^N(x) = A * V(x) + B = Ax_1 + Ax_2 + B \quad (1)$$

Now, the condition $V^N(0,0) = 0$ implies that $B = 0$.

Then, substituting $B = 0$ and $V^N(105,105) = 1$ to (1) implies that

$210A = 1 \Leftrightarrow A = 1/210.$

$V^N(x)$ can also be written as $V^N(x) = w_1 v_1^N(x_1) + w_2 v_2^N(x_2)$. Therefore, it applies

$$V^N(x) = w_1 v_1^N(x_1) + w_2 v_2^N(x_2) = \frac{1}{210}x_1 + \frac{1}{210}x_2,$$

based on which with i=1,2, it now holds

$$v_i^N(x_i) = \frac{1}{210 w_i} x_i. \quad (2)$$

Moreover, since necessarily now $v_i^N(0) = 0$ and $v_i^N(105) = 1$, one can solve from (2) that $w_1 = w_2 = \frac{1}{2} = 0.5$, and thereby

$$V^N(x) = w_1 v_1^N(x_1) + w_2 v_2^N(x_2) = 0.5 * \frac{x_1}{105} + 0.5 * \frac{x_2}{105}. \quad (3)$$

The weights with which B gets the same value as A are found by solving

$$\begin{cases} w_1 v_1^N(100) + w_2 v_2^N(100) = w_1 v_1^N(90) + w_2 v_2^N(105) \\ w_1 + w_2 = 1 \end{cases}, \quad (4)$$

where $v_1^N(x_1) = \frac{x_1}{105}$ and $v_2^N(x_2) = \frac{x_2}{105}$.

The solution is $w_1 = \frac{1}{3}, w_2 = \frac{2}{3}$. Now B is the most preferred alternative, if $w_2 \geq 2/3$.

Similarly, the weights with which C gets the same value as A are found by solving

$$\begin{cases} w_1 v_1^N(100) + w_2 v_2^N(100) = w_1 v_1^N(105) + w_2 v_2^N(80) \\ w_1 + w_2 = 1 \end{cases} \quad (5)$$

The solution is $w_1 = \frac{4}{5}, w_2 = \frac{1}{5}$. Thus, C is the most preferred one, if $w_1 \geq 4/5$.

B is the closest competitor, because (1/3,2/3) is closer to (0.5,0.5) than (0.8, 0.2):

$$\|(1/3, 2/3) - (0.5, 0.5)\|_2 = \sqrt{(1/6)^2 + (1/6)^2} = \frac{\sqrt{2}}{6} <$$
$$\frac{3\sqrt{2}}{10} = \sqrt{2(3/10)^2} = \|(4/5, 1/5) - (0.5, 0.5)\|_2$$

c) By performing corresponding calculations as in Equations (1) - (2) in part b), one now obtains that

$$A = \frac{1}{1155}, \qquad B = 0, \qquad v_i^N = \frac{1}{1155 w_i} x_i \ \forall i$$

Since necessarily now $v_1^N(0) = 0$ and $v_1^N(1050) = 1$ and $v_2^N(0) = 0$ and $v_2^N(105) = 1$ we get (like in part b) that

$$w_1 = \frac{10}{11}, \qquad w_2 = \frac{1}{11}, \qquad v_1^N(x_1) = \frac{x_1}{1050}, \qquad v_2^N(x_2) = \frac{x_2}{105}.$$

The weights where B and C get the same score as A are calculated again using Equations (4) and (5). For B the weights are now $w_1 = \frac{5}{6}, w_2 = \frac{1}{6}$. And for C they are $w_1 = \frac{40}{41}, w_2 = \frac{1}{41}$.

C is the closest competitor, because it maximizes *V'* with $w_1 \geq 40/41$ while B maximizes *V'* with $w_2 \geq 1/6$ and

$$\|(40/41, 1/41) - (10/11, 1/11)\|_2 = \sqrt{2(30/451)^2} \approx 0.0941 <$$
$$0.1071 \approx \sqrt{2(5/66)^2} = \|(5/6, 1/6) - (10/11, 1/11)\|_2$$

Extra Remarks:

In part a), the decreasing value order of the alternatives is A, B, C. This is reflected in Figure 1(a), where the coordinate points of the alternatives are located on different contour lines of the normalized additive value function.

In part b), the closest competitor is B, i.e., a smaller Euclidean change in the attribute weights is needed for B to become as good as A, compared to the change needed for C to become as good as A. This is visually reflected in Figure 1: The slope of the contour lines of the normalized additive value function in Figure 1(b) differs less from the slope of the contour lines in Figure 1(a) compared to the case in Figure 1(c).

In part c), the scale of the first attribute is increased from [0, 105] to [0, 1050]. This causes the weight of the first attribute to increase from 1/2 to 10/11. Now, C becomes the closest competitor to A. This is visually reflected in Figure 2: Compared to the slope of the contour lines in Figure 2(b), the slope of the contour lines in Figure 2(c) is closer to the slope of the contour lines in Figure 2(a).

Note that, although the change in the boundaries of the attribute scales can change the closest competitor, it doesn't change the value order of the alternatives:

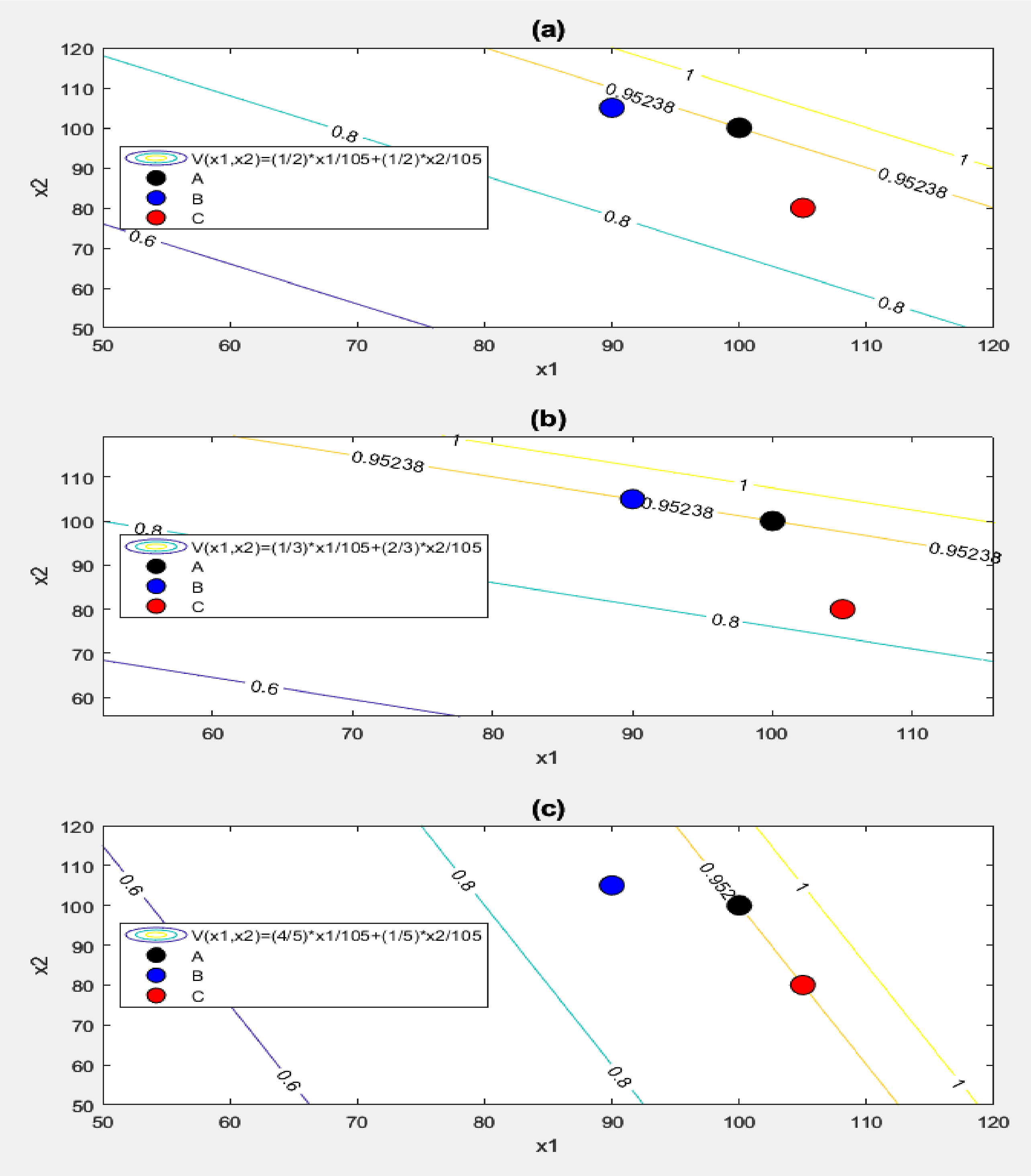
$$V^1(x1, x2) = \frac{1}{2} * \frac{x1}{105} + \frac{1}{2} * \frac{x2}{105}, \qquad V^1(A) = \frac{100}{105}, V^1(B) = \frac{97.5}{105}, V^1(C) = \frac{92.5}{105}$$

$$V^2(x1, x2) = \frac{10}{11} * \frac{x1}{1050} + \frac{1}{11} * \frac{x2}{105},$$

$$V^2(A) = \frac{2000}{11 * 1050}, V^2(B) = \frac{1950}{11 * 1050}, V^2(C) = \frac{1850}{11 * 1050}$$

Now, $V^2(x1, x2) = \frac{20}{110} * V^1(x1, x2)$.

Figure 1



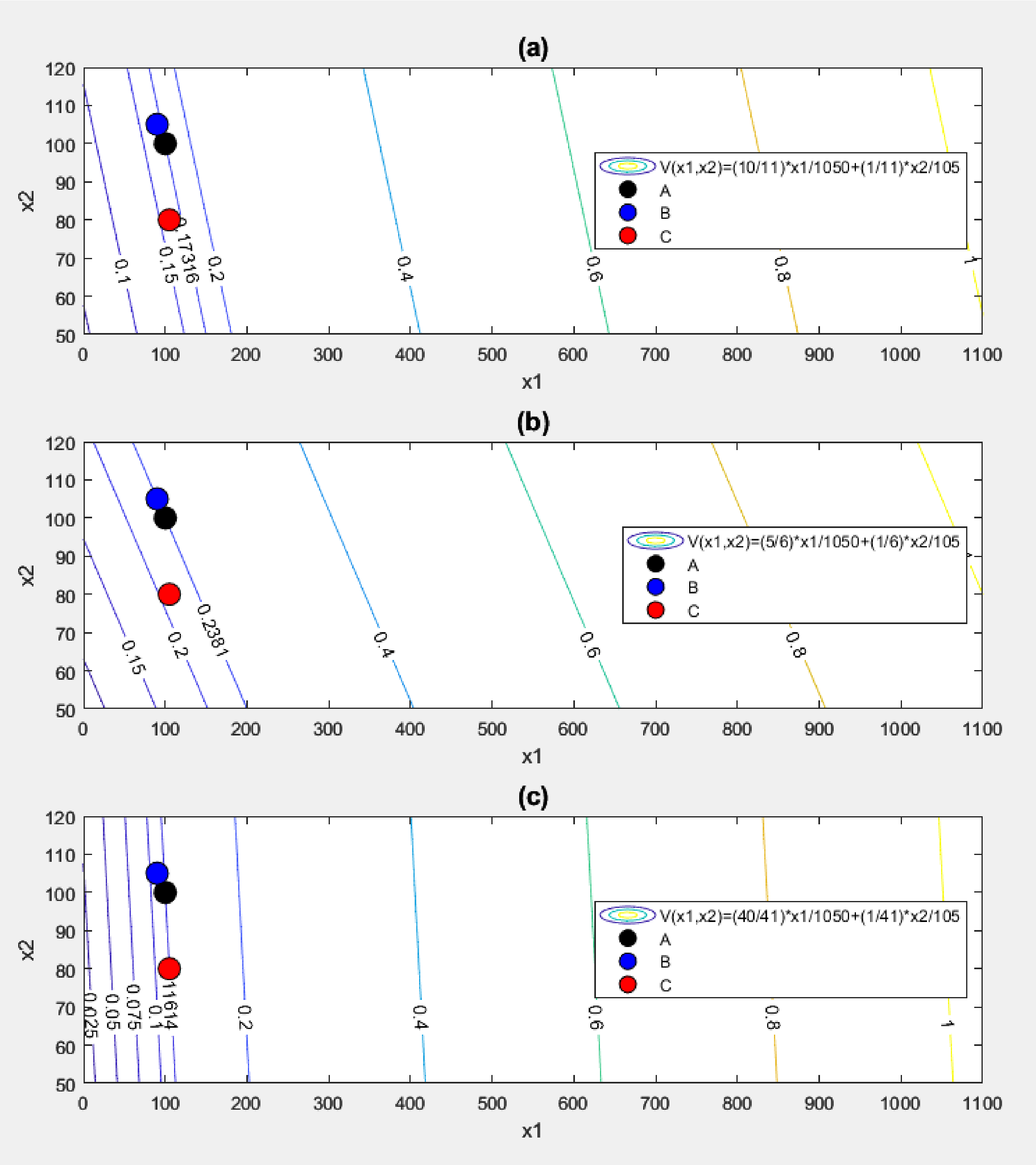

(a)
V(x1,x2)=(10/11)*x1/1050+(1/11)*x2/105
A
B
C
x1
x2
(b)
V(x1,x2)=(5/6)*x1/1050+(1/6)*x2/105
A
B
C
x1
x2
(c)
V(x1,x2)=(40/41)*x1/1050+(1/41)*x2/105
A
B
C
x1
x2

Figure 2