ELEC-E8116 Model-based control systems
/exercises and solutions 7

1. Consider a simple integrator:

$$\dot{x}(t) = u(t)$$

Find an optimal control law that minimises a cost-function

$$J = \int_0^1 (x^2(t) + u^2(t))\,dt$$

Further, consider the case, when the optimization horizon is infinite.

Solution: We can always define

$$J_1 = \tfrac{1}{2}J = \tfrac{1}{2}\int_0^1 (x^2(t) + u^2(t))\,dt$$

without changing the solution (only the cost changes). Of course, the original cost could also be written as

$$J = \frac{1}{2}\int_0^1 (2x^2(t) + 2u^2(t))\,dt$$

and continue from there. However, the first alternative is now used:

The Riccati equation:

$$-\dot{\boldsymbol{S}}(t) = \boldsymbol{A}^T\boldsymbol{S}(t) + \boldsymbol{S}(t)\boldsymbol{A}^T - \boldsymbol{S}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{S}(t) + \boldsymbol{Q}, \quad \boldsymbol{S}(1) = 0$$

and the optimal control law is:

$$\boldsymbol{u}^*(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{S}(t)\boldsymbol{x}^*(t)$$

Now for the given process we have:

$$A = 0; B = 1; R = 1; Q = 1$$

and

$$\dot{S}(t) = S^2(t) - 1,$$

which is a differential equation that should be solved with respect to time, Hence, by separating the variables

$$\frac{dS}{dt} = S^2 - 1 \Leftrightarrow \int \frac{1}{S^2-1} dS = \int dt$$

$$\Rightarrow \int \frac{1}{S^2 - 1} dS = t + C_1$$

$$\Rightarrow \int \frac{1}{S - 1} dS - \int \frac{1}{S + 1} dS = 2t + 2C_1$$

$$\Rightarrow ln\left(\left|\frac{S - 1}{S + 1}\right|\right) = 2t + 2C_1$$

$$\Rightarrow \left|\frac{S - 1}{S + 1}\right| = |e^{2C_1}e^{2t}| \Leftrightarrow \frac{S - 1}{S + 1} = \pm e^{2C_1}e^{2t} = Ce^{2t}$$

$\Rightarrow S(t) = \frac{1+Ce^{2t}}{1-Ce^{2t}}$

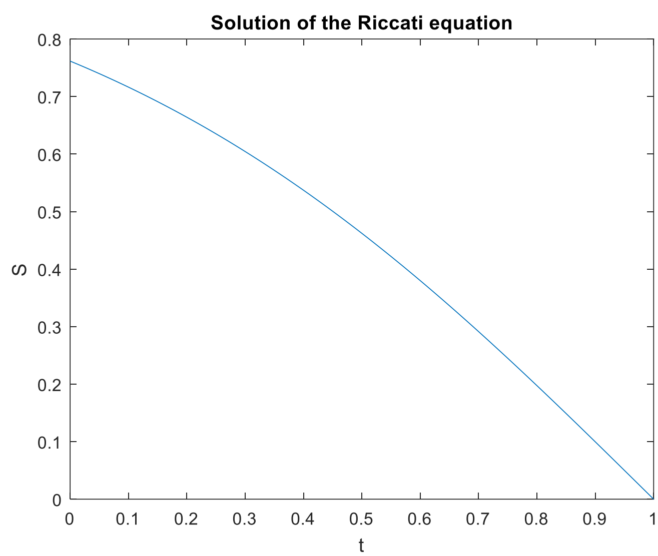
Now solve the unknown parameter$C$by using the fact $S(t_f) = S(1) = 0$, giving:

$$1 + Ce^2 = 0 \Leftrightarrow C = -e^{-2}$$

and

$$S(t) = \frac{1-e^{2(t-1)}}{1+e^{2(t-1)}}$$

Now the optimal control law is: $u * (t) = -S(t)x(t)$



If the optimization horizon were infinite, the solution of the Riccati equation would be

$S(t) = \frac{1-e^{2\left(t-t_f\right)}}{1+e^{2\left(t-t_f\right)}} = \frac{e^{-2t}-e^{-2t_f}}{e^{-2t}+e^{-2t_f}} \rightarrow 1$ as $t_f \rightarrow \infty$. The same constant solution could

have been obtained directly from $\dot{S}(t) = S^2(t) - 1$by setting the derivative zero and taking the positive (positive definite) root of S.

2. Consider the 1. order process$G(s) = \frac{1}{s-a}$, which has a realization

$$\dot{x}(t) = ax(t) + u(t)$$
$$y(t) = x(t)$$

so that the state is the measured variable. It is desired to find the control, which minimizes the criterion

$$J = \frac{1}{2}\int_0^\infty (x^2 + Ru^2)\, dt \qquad (R > 0)$$

Calculate the control and investigate the properties of the resulting closed-loop system.

Solution:
The algebraic Riccati equation is

$$aX + Xa - XR^{-1}X + 1 = 0 \quad \Rightarrow \quad X^2 - 2aRX - R = 0$$

The solution must be positive semidefinite $X \geq 0$ so that

$$X = aR + \sqrt{(aR)^2 + R}$$

The optimal control law is thus

$$u = -K_r x \qquad \text{in which} \qquad K_r = X/R = a + \sqrt{a^2 + 1/R}$$

The closed-loop system is

$$\dot{x} = ax + u = -\sqrt{a^2 + 1/R}\; x$$

which has a pole at

$$s = -\sqrt{a^2 + 1/R} < 0$$

The root locus of this pole starts from $\boldsymbol{s = -|a|}$ when$\boldsymbol{R} = \infty$ (control has an infinite weight) and moves towards -∞, when R approaches zero. Note that the root locus is exactly the same in the stable (a < 0) process case as well as in the unstable (a > 0) case.

It is easily seen that for a small R the gain crossover frequency of the open loop transfer function

$$\boldsymbol{L = GK_r = K_r/(s - a)}$$

is approximately

$$\boldsymbol{\omega_c \approx \sqrt{1/R}}$$

and the gain drops 20 dB / decade in high frequencies, which is a general property of LQ-controllers. Moreover, the Nyquist curve does not in any frequency go inside the unit circle with the center at (-1,0). This means that

$$\boldsymbol{|S(i\omega)| = 1/|1 + L(i\omega)| \leq 1}$$

for all frequencies. (Explanation: setting $L = x + iy$ gives

$$|S| = \frac{1}{|1 + x + iy|} = \frac{1}{\sqrt{(1 + x)^2 + y^2}}$$

so that $|S| = 1$ in the circumference of the unit circle centered at (-1,0). Inside the circle $|S| > 1$ and outside $|S| < 1$.)

This property is clear for the stable process ($a < 0$), because $K_r > 0$ and the phase of $L(i\omega)$ changes from zero degrees (at the angular frequency 0) to –90 degrees (at the infinite angular frequency). It is remarkable that the property holds also in the case of unstable processes ($a > 0$), even though the phase of $L(i\omega)$ varies between $\mathbf{-180°, -90°}$

3. In solving the discrete-time LQ problem an essential step is to find a “first control step” by minimizing the cont

$$J_{N-1} = \frac{1}{2}x_{N-1}^T Q x_{N-1} + \frac{1}{2}u_{N-1}^T R u_{N-1} + \frac{1}{2}(Ax_{N-1} + Bu_{N-1})^T S_N (Ax_{N-1} + Bu_{N-1})$$

Do it.

Solution:

Note: Q, R, S are symmetric $Q = Q^T$ etc.

$$\frac{\partial}{\partial x}(Ax) = A, \quad \frac{\partial}{\partial x}(x^T A x) = x^T(A + A^T) \underbrace{=}_{A\text{symmetric}} 2x^T A$$

$$J_{N-1} = \frac{1}{2}x_{N-1}^T Q x_{N-1} + \frac{1}{2}u_{N-1}^T R u_{N-1} + \frac{1}{2}(Ax_{N-1} + Bu_{N-1})^T S_N (Ax_{N-1} + Bu_{N-1})$$

$$J = \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u + \frac{1}{2}x^T A^T S A x + \frac{1}{2}x^T A^T S B u + \underbrace{\frac{1}{2}u^T B^T S A x}_{\text{scalar, can be transposed}} + \frac{1}{2}\underbrace{u^T B^T S B u}_{\text{symmetric}}$$

To solve the extreme value the derivative with respect to *u* must be zero.

$$\begin{aligned}\frac{\partial J}{\partial u} &= u^T R + \frac{1}{2}x^T A^T S B + \frac{1}{2}x^T A^T S B + u^T B^T S B \\ &= u^T R + x^T A^T S B + u^T B^T S B = 0\end{aligned}$$

Taking the transpose does not change the equation

$$\begin{aligned}&Ru + B^T S A x + B^T S B u = 0 \\ &\Rightarrow\ (R + B^T S B)u = -B^T S A x \\ &\Rightarrow\ u* = -(R + B^T S B)^{-1} B^T S A x\end{aligned}$$

Note that the inverse exists, because *S* is positive semidefinite and *R* is positive definite. Also, the *Hessian*

$$\frac{\partial^2 J}{\partial u^2} = \frac{\partial}{\partial u}(u^T R + u^T B^T S B)^T = \frac{\partial}{\partial u}(Ru + B^T S B u) = R + B^T S B > 0 \quad \text{(pos. def.)}$$

shows that the extreme value is a minimum.

4. The discrete time LQ problem and its solution can be given as

$$\begin{aligned}&\boldsymbol{x_{k+1} = A_k x_k + B_k u_k}, \quad \boldsymbol{k > i} \\ &\boldsymbol{J_i = \frac{1}{2}x_N^T S_N x_N + \frac{1}{2}\sum_{k=i}^{N-1}\left(x_k^T Q_k x_k + u_k^T R_k u_k\right)} \\ &\boldsymbol{S_N \geq 0}, \quad \boldsymbol{Q_k \geq 0}, \quad \boldsymbol{R_k > 0}\end{aligned}$$

(final state free)

$$\begin{aligned}
S_k &= (A_k - B_k K_k)^T S_{k+1}(A_k - B_k K_k) + K_k^T R_k K_k + Q_k \\
K_k &= (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k, \quad k < N \\
u_k^* &= -K_k x_k, \quad k < N \\
J_i^* &= \frac{1}{2} x_i^T S_i x_i
\end{aligned}$$

Show that the Riccati equation can also be written in the form

$$\boldsymbol{S_k = A_k^T \left[S_{k+1} - S_{k+1} B_k \left(B_k^T S_{k+1} B_k + R_k\right)^{-1} B_k^T S_{k+1}\right] A_k + Q_k, k < N, S_N} \textbf{ given}$$

(The "Joseph-stabilized form" of the Riccati equation)

Solution: Start from the equations

$$S_k = (A_k - B_k K_k)^T S_{k+1}(A_k - B_k K_k) + K_k^T R_k K_k + Q_k \qquad (1)$$

$$K_k = ({B_k}^T S_{k+1} B_k + R_k)^{-1} {B_k}^T S_{k+1} A_k$$

and try to reach

$$\boldsymbol{S_k = A_k^T \left[S_{k+1} - S_{k+1} B_k \left(B_k^T S_{k+1} B_k + R_k\right)^{-1} B_k^T S_{k+1}\right] A_k + Q_k} \qquad (2)$$

First note in equation (1) that when Q and R have been chosen to be symmetric and S_N is symmetric, then S_i is symmetric for all i (verification by taking the transpose of S_k in equation (1); remember the calculation rules of transposition).

Start from (1) and use the short notation$S_{k+1} = S$, $K_k = K$ etc.

$$\begin{aligned}
& A^T SA - A^T SBK - K^T B^T SA + K^T B^T SBK + K^T RK + Q \\
&= A^T SA - A^T SBK - K^T B^T SA + K^T [B^T SB + R] K + Q \\
&= A^T SA - A^T SB(B^T SB + R)^{-1} B^T SA - A^T SB(B^T SB + R)^{-1} B^T SA \\
&\quad + A^T SB(B^T SB + R)^{-1} \underbrace{(B^T SB + R)(B^T SB + R)^{-1}}_{I} B^T SA + Q \\
&= A^T \{S - SB(B^T SB + R)^{-1} B^T S - SB(B^T SB + R)^{-1} B^T S + SB(B^T SB + R)^{-1} B^T S\} A \\
&\qquad + Q \\
&= A^T \{S - SB(B^T SB + R)^{-1} B^T S\} A + Q
\end{aligned}$$

which is the same as (2).

Note that especially in the calculation of the transpose of K the fact that Q, R and S are symmetric, has been utilized.