ELEC-E8116 Model-based control systems
Intermediate exam 1. 20. 10. 2022 / Solutions

- *Write the name of the course, your name and student number to each answer sheet.*
- *There are three (3) problems and each one must be answered.*
- *No literature is allowed. A calculator can be used as a calculating aid. However, it must not be used for advanced calculations, e.g. matrix calculus, Laplace transformations, connection to the Internet etc.*

1. a. Explain briefly the following concepts

- Conservative control law (1 p.)
- Robust stability (1 p.)
- "Push through"-rule (1 p.)

1. b. Explain shortly the concept "Fundamental restrictions in control". Name and discuss the main items there. (2 p.)

Solution:

a. – Conservative control law means that the controller might be Ok "on paper" (stability, some norm conditions etc.), but it lacks performance (it is "lazy"). For example, a controller that gives only a zero output signal might give a stable closed loop, but is definitely conservative. I practice, conservatism is a bad problem in mathematically oriented control theory (e.g. in robust control), because "on paper" it is not easy to detect. Therefore it must always be investigated.
-The system is robustly stable, if it remains stable in spite on process model uncertainties (usually restricted by a given bound.) In practice that means that the controller has been designed so that it can tolerate some degree of model uncertainty.
-"Push-Through"-rule is a formula in matrix calculus, which is often handy in matrix expression manipulations. Let *A* and *B* be *mxn* and *nxm* dimensional matrices, respectively, Assume that the inverse matrices below are non-singular. Then it holds

$$\mathrm{A}\left(\mathrm{I}+\mathrm{BA}\right)^{-1}=\left(\mathrm{I}+\mathrm{AB}\right)^{-1}\mathrm{A}$$

where the *I*'s are identity matrices of suitable dimensions.

b. Fundamental restrictions in control mean such performance limitations in (closed loop) control, which cannot be removed by any means. In other words, there does not exist any controller or controller tuning that would exceed some performance limit. Mostly, causes of such limitations are given by

- unstable systems
- systems with delay
- non-minimum phase systems
- limitations in control signal range
- system inverse (e.g. disturbance rejection actually means finding the process inverse or an approximation to it in control design)

The fundamental restriction is the correspondence between the sensitivity functions.

$$S(j\omega)+T(j\omega)=I$$

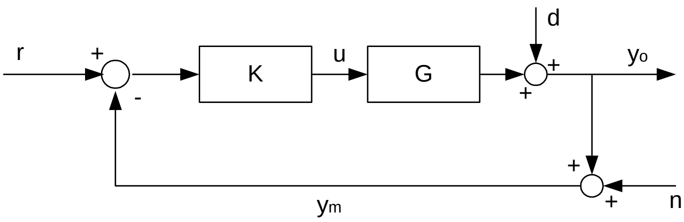
Both sensitivities are determined by the loop transfer function *L*, which in turn is given by the product of process and controller transfer functions.

Removing an unstable process pole or nonminimum phase zero by a direct cancellation by the controller leads to an internally unstable closed loop system (i.e. some finite input signal causes some output to grow without limit).

Other fundamental restrictions are e.g. ability to mitigate disturbance by a bounded control signal, the Bode equation, which couples the magnitude and phase frequency functions, the waterbed formula, the conditions for bandwidth (closed loop performance) limited by unstable poles and nonminimun phase zeros.

For detailed results, see lecture slides, Chapter 4.

2. Consider a multivariable control configuration.



Write the equations describing the system and identify

a. the closed loop transfer function (1 p.)
b. the sensitivity function (1 p.)
c. the complementary sensitivity function. Show that $S + T = I$ and explain the result. (1 p.)
d. show and discuss the meaning of the identity (MIMO case)

$$u=G^{-1}\left[G_c r-(1-S)d\right] \qquad \text{(2 p.)}$$

Solution:

a,b.

$$\begin{aligned} y_0 &= d+Gu=d+GK\left[r-\left(y_0+n\right)\right]=d+GKr-GK\left(y_0+n\right) \\ &\Rightarrow \left(I+GK\right)y_0=d+GKr-GKn \\ &\Rightarrow y_0=\left(I+GK\right)^{-1}d+\left(I+GK\right)^{-1}GKr-\left(I+GK\right)^{-1}GKn \\ &\qquad =Sd+G_c r-G_c n \end{aligned}$$

S is the sensitivity function, G_c is the closed loop transfer function.

c. The complementary sensitivity function is

$$\mathrm{T} = \left(\mathrm{I}+\mathrm{GK}\right)^{-1}\mathrm{GK} = \mathrm{GK}\left(\mathrm{I}+\mathrm{GK}\right)^{-1}$$

$$\mathrm{S}+\mathrm{T} = \left(\mathrm{I}+\mathrm{GK}\right)^{-1} + \left(\mathrm{I}+\mathrm{GK}\right)^{-1}\mathrm{GK} = \left(\mathrm{I}+\mathrm{GK}\right)^{-1}\left[\mathrm{I}+\mathrm{GK}\right] = \mathrm{I}$$

The result is fundamental. Note that the sensitivity functions are functions of (angular) frequency, so the formula applies at each frequency. The result actually explains, why control design always meets contradictions: to keep *S* "small" in some frequency range, leads to *T* "large" in that range.

d. Writing the formulas in the topology shown in the figure gives

$$\begin{aligned}
&\mathrm{y}_0 = \mathrm{Gu}+\mathrm{d}\\
&\Rightarrow \mathrm{u} = \mathrm{G}^{-1}\left(\mathrm{y}_0-\mathrm{d}\right) = \mathrm{G}^{-1}\left[\mathrm{Sd}+\mathrm{G_c r}-\mathrm{G_c n}-\mathrm{d}\right]\\
&\qquad = \mathrm{G}^{-1}\left[\mathrm{G_c r}-(\mathrm{I}-\mathrm{S})\mathrm{d}-\mathrm{G_c n}\right]\\
&\qquad = \mathrm{G}^{-1}\left[\mathrm{G_c r}-(\mathrm{I}-\mathrm{S})\mathrm{d}\right]
\end{aligned}$$

where the noise term *n* has been assumed zero in the last phase. The result means that perfect disturbance rejection

$$\mathrm{u} = \mathrm{G}^{-1}(\mathrm{r}-\mathrm{d})$$

is achieved by a controller with $\mathrm{S}=0,\ \mathrm{T}\approx\mathrm{G_c}=\mathrm{I}$, even if the disturbance *d* were not measurable.

3. Find the poles, zeros and a minimal realization to the system

$$G(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{2s-3}{(s+1)(s+2)} \\ \dfrac{s-2}{s+1} & \dfrac{s}{s+2} \end{bmatrix} \qquad \text{(1+1+3 p.)}$$

Hint to the last part: You should know the number of states in the minimal realization. Write the terms in the transfer function matrix by means of weighted sums of terms 1/(*s*+1), 1/(*s*+2) and control inputs. Then choose the state variables.

Solution:

The minors are

$$\frac{2}{s+1}, \frac{2s-3}{(s+1)(s+2)}, \frac{s-2}{s+1}, \frac{s}{s+2},$$
$$\frac{2s}{(s+1)(s+2)} - \frac{(s-2)(2s-3)}{(s+1)^2(s+2)} = \ldots = \frac{3(3s-2)}{(s+1)^2(s+2)}$$

The least common denominator is the pole polynomial

$$p(s) = \left(s+1\right)^2 (s+2)$$

so that the system has three poles; a double pole at -1 and a pole at -2. The *minimal realization* has three states.

In the largest minor above (one largest minor in this case) the pole polynomial is already in the denominator. The greatest common divisor of largest minors is the zero polynomial. In this case trivially

$$z(s) = 3(3s-2)$$

so that the system has one (RHP) zero at $z_1 = 2/3$.

Note that Matlab gives the correct poles and zeros in the minimal realization only, when the *minreal* command is used in *pole* and *tzero* commands.

To construct a minimal realization write first

$$Y_1(s) = \frac{2}{s+1}U_1(s) + \frac{2s-3}{\left(s+1\right)\left(s+2\right)}U_2(s) = \ldots = \frac{2}{s+1}U_1(s) - \frac{5}{s+1}U_2(s) + \frac{7}{s+2}U_2(s)$$
$$Y_2(s) = \frac{s-2}{s+1}U_1(s) + \frac{s}{\left(s+2\right)}U_2(s) = \ldots = U_1(s) - \frac{3}{s+1}U_1(s) + U_2(s) - \frac{2}{s+2}U_2(s)$$

By inspection choose the three state variables, which give the state equations

$$X_1(s) = \frac{1}{s+1}U_1(s) \Rightarrow \dot{x}_1(t) + x_1(t) = u_1(t)$$
$$X_2(s) = \frac{1}{s+1}U_2(s) \Rightarrow \dot{x}_2(t) + x_2(t) = u_2(t)$$
$$X_3(s) = \frac{1}{s+2}U_2(s) \Rightarrow \dot{x}_3(t) + 2x_3(t) = u_2(t)$$

The state-space representation becomes

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & -5 & 7 \\ -3 & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} u(t)$$

Note that

$$Y_1 = 2X_1 - 5X_2 + 7X_3$$
$$Y_2 = U_1 - 3X_1 + U_2 - 2X_3$$

Note that Matlab does not give the same minimal realization. By different choice of state variables there exist an infinite number of (minimal) realizations to the same transfer function.