ELEC-E8101 Digital and Optimal Control
Exercise 8 - Solution
Autumn 2022

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a)

When backward difference approximation is used in the discretization of a transfer function the Laplace variable *s* is replaced with $\frac{z-1}{zh}$. Hence, the discrete PID controller is

$$\begin{aligned} H_{bw}(s) &= K\left(1+\frac{1}{T_I\frac{z-1}{zh}}+T_D\frac{z-1}{zh}\right) \\ &= K\left(\frac{\left(1+\frac{T_D}{h}+\frac{h}{T_I}\right)z^2+\left(-1-2\frac{T_D}{h}\right)z+\frac{T_D}{h}}{z^2-z}\right) \end{aligned}$$

b)

When the Tustin approximation is used in the discretization of a continuous time transfer function the Laplace variable *s* is replaced with $\frac{2}{h}\frac{z-1}{z+1}$. Hence, the discrete PID controller is:

$$\begin{aligned} H_{Tustin}(s) &= K\left(1+\frac{1}{T_I\frac{2}{h}\frac{z-1}{z+1}}+T_D\frac{2}{h}\frac{z-1}{z+1}\right) \\ &= K\left(\frac{\left(1+\frac{2T_D}{h}+\frac{h}{2T_I}\right)z^2+\left(\frac{h}{T_I}-2\frac{T_D}{h}\right)z+\left(-1+\frac{h}{2T_I}+\frac{2T_D}{h}\right)}{z^2-1}\right) \end{aligned}$$

c)
The practical PID controller can be split to proportional, integral and derivative parts as

$$\begin{aligned} G_{PID}(s) &= K\left((Y_{ref}(s)-Y(s))+\frac{1}{T_I s}(Y_{ref}(s)-Y(s))-\frac{T_D s}{1+T_D s/N}Y(s)\right) \\ &= K(G_P(s)+G_I(s)+G_D(s)) \end{aligned}$$

Considering the integral part and approximating the integral as a sum, it can be written as

$$u_I[k] = u_i[k-1]+\frac{h}{T_I}e[k]$$

$$U_I(z) = z^{-1}U_I(z)+\frac{h}{T_I}E(z)$$

Thus, its transfer function is

$$G_I(z) = \frac{U_I(z)}{E_I(z)} = \frac{hz}{T_I(z-1)}$$

For the derivative part, noting the backward discretization from (a), the derivative part is

$$G_D(z) = -\frac{T_D \frac{z-1}{zh}}{1 + T_D \frac{z-1}{zhN}} = -\frac{T_D(z-1)}{\left(h + \frac{T_D}{N}\right)z - \frac{T_D}{N}}$$

Thus, the entire controller can be written as

$$U_{practical}(z) = K\left(1 + \frac{h}{T_I}\frac{z}{z-1}\right)E(z) - K\frac{T_D(z-1)}{\left(h + \frac{T_D}{N}\right)z - \frac{T_D}{N}}Y(z)$$

What about the sampling time?

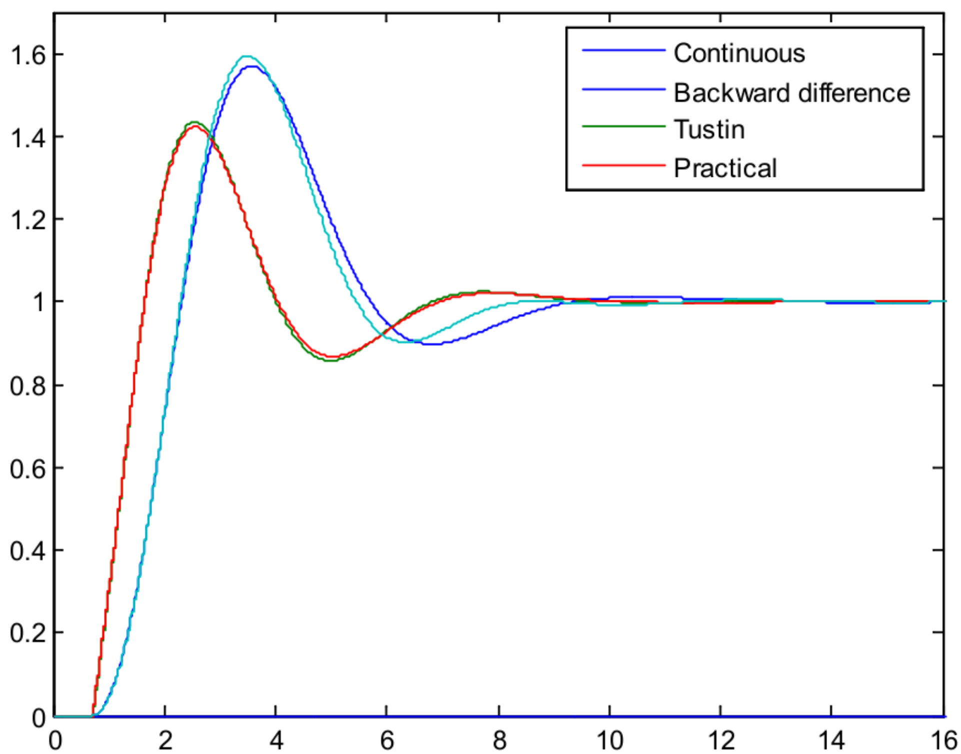
If we want that the discrete PID controller behaves like the continuous time PID controller the sampling rate should be high. Usually in commercial controller units, the sampling time is short and constant and depends on the plant dynamics (for example 200 ms for a temperature controller, 1 ms for a robot motion controller).

The following rules of thumb can be used when choosing the sampling time for a PID controller:

$$hN/T_D \approx 0.2 \ldots 0.6 \quad N \approx 10 \quad h/L \approx 0.01 \ldots 0.06$$

where *L* is the delay. Thus, for this system, the sampling time can be $h \approx 0.007 \ldots 0.042$.

Simulating the system for *h=0.014* gives



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We first find the pulse transfer function of G(s) with the ZOH:

$$\begin{aligned} G(z) &= \frac{z-1}{z} Z\left\{\frac{G(s)}{s}\right\} \\ &= \frac{z-1}{z} Z\left\{\frac{0.1}{s^2(s+0.1)}\right\} \\ &= \frac{z-1}{z} \frac{\left((0.1-1+e^{-0.1})z + (0.1-1-e^{-0.1})\right)z}{0.1(z-e^{-0.1})(z-1)^2} \\ &= \frac{0.0484(z+0.9672)}{(z-1)(z-0.9048)} \end{aligned}$$

Noting from the task description that

$$u[k] = -0.5u[k-1] + 13(e[k] - 0.88e[k-1]),$$

the controller transfer function is

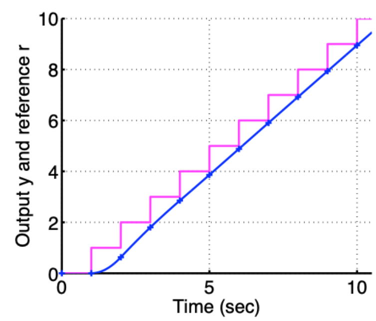
$$D(z) = \frac{U(z)}{E(z)} = 13\frac{1-0.88z^{-1}}{1+0.5z^{-1}} = 13\frac{z-0.88}{z+0.5}$$

Now, check the steady state error for unit ramp. Ramp can be written

$$R(z) = \frac{z}{(z-1)^2}$$

Therefore the steady state error is (*math done either by hand or by computer*)

$$\begin{aligned} e_{ss} &= \lim_{z\to 1}(z-1)E(z) \\ &= \lim_{z\to 1}\left[(z-1)\frac{z}{(z-1)^2}\frac{1}{1+D(z)G(z)}\right] = \ldots = 0.96 \end{aligned}$$



Thus, as required

$$e_{ss} < 1$$

Checking the specifications for overshoot and settling time,

$$M_P < 16\% \Rightarrow \zeta > 0.5$$

$$t_s < 10s \Rightarrow |z| < 0.01^{0.1} = 0.63,$$

find first closed loop poles of the system as roots of $1 + D(z)G(z) = 0$:

$$1 + 13\frac{z-0.88}{z+0.5}\frac{0.0484(z+0.9672)}{(z-1)(z-0.9048)} = 0$$

gives

$$z = \{0.88, \; -0.050 \pm 0.304j\}$$

However, the pole at z=0.88 is canceled by the zero. Therefore,

$$z = -0.050 \pm 0.304j = re^{\pm j\pi\theta} = 0.31e^{\pm 1.73j\pi} \Rightarrow \zeta = 0.56$$

(note that $z = e^{-\zeta\omega_0 T}e^{-j\sqrt{1-\zeta^2}\omega_0 T}$)

Thus, all specs are satisfied, also visible in the plot