Aalto University
School of Electrical
Engineering

ELEC-E8740 — Nonlinear Continuous-Time Models and Discrete-Time Dynamic Models

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Intended Learning Outcomes

After this lecture, you will be able to:

- construct nonlinear continuous-time state-space models,
- distinguish continuous-time and discrete-time models,
- construct discrete-time linear and non-linear state-space models.

Recap

- Higher order ODEs and SDEs can be transformed to a first-order vector-valued equation system
- The deterministic linear state-space model is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u \mathbf{u}(t)$$
$$\mathbf{y}_n = \mathbf{G}\mathbf{x}_n + \mathbf{r}_n$$

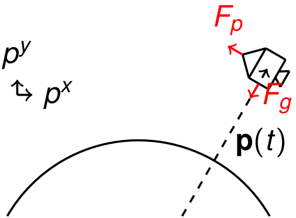
- The stochastic linear state-space model with stochastic input process $\mathbf{w}(t)$ is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_w \mathbf{w}(t)$$
$$\mathbf{y}_n = \mathbf{G}\mathbf{x}_n + \mathbf{r}_n$$

- The 2D Wiener velocity model is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

Example: Dynamic Model for a Spacecraft (1/2)



- Gravitational acceleration:

$$g \approx g_0 \left(\frac{r_e}{|\mathbf{p}(t)|} \right)^2 ,$$

Example: Dynamic Model for a Spacecraft (2/2)

- Gravitational pull: $\mathbf{F}_g = -m g_0 r_e^2 \frac{\mathbf{p}(t)}{|\mathbf{p}(t)|^3}$
- Propulsion: $\mathbf{F}_p = F_p \frac{1}{|\mathbf{p}(t)|} \begin{bmatrix} -p^y(t) \\ p^x(t) \end{bmatrix}$
- Differential equation:

$$m\mathbf{a}(t) = -m g_0 r_e^2 \frac{\mathbf{p}(t)}{|\mathbf{p}(t)|^3} + \frac{1}{|\mathbf{p}(t)|} \begin{bmatrix} -p^y(t) \\ p^x(t) \end{bmatrix} u(t).$$

- State vector:

$$\mathbf{x}(t) = \begin{bmatrix} p^x(t) & p^y(t) & v^x(t) & v^y(t) \end{bmatrix}^\mathsf{T}.$$

Can not be written as $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u \mathbf{u}(t)$.

Nonlinear Differential Equation Systems

- Nonlinear ordinary differential equation system (b_{ij} may depend on $x_n(t)$):

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(x_1(t), x_2(t), \ldots, x_{d_x}(t)) + b_{11}\, u_1(t) + \ldots b_{1 d_u}\, u_{d_u}(t) \\
\dot{x}_2(t) &= f_2(x_1(t), x_2(t), \ldots, x_{d_x}(t)) + b_{21}\, u_1(t) + \ldots b_{2 d_u}\, u_{d_u}(t) \\
&\vdots \\
\dot{x}_{d_x}(t) &= f_{d_x}(x_1(t), x_2(t), \ldots, x_{d_x}(t)) + b_{d_x 1}\, u_1(t) + \ldots b_{d_x d_u}\, u_{d_u}(t)
\end{aligned}
$$

- State vector: $\mathbf{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & \ldots & x_{d_x}(t) \end{bmatrix}^\mathsf{T}$
- In vector form:

$$
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{d_x}(t) \end{bmatrix} = \begin{bmatrix} f_1(\mathbf{x}(t)) \\ f_2(\mathbf{x}(t)) \\ \vdots \\ f_{d_x}(\mathbf{x}(t)) \end{bmatrix} + \begin{bmatrix} b_{11}(\mathbf{x}(t)) & \ldots & b_{1 d_u}(\mathbf{x}(t)) \\ b_{21}(\mathbf{x}(t)) & & \vdots \\ \vdots & \ddots & \\ b_{d_x 1}(\mathbf{x}(t)) & \ldots & b_{d_x d_u}(\mathbf{x}(t)) \end{bmatrix} \mathbf{u}(t).
$$

Nonlinear Continuous-Time State-Space Models

- Deterministic nonlinear dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_u(\mathbf{x}(t))\mathbf{u}(t)$$

- Stochastic nonlinear dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$

- Nonlinear measurement model:

$$\mathbf{y}_n = \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n$$

- Stochastic nonlinear state-space model:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t) \\ \mathbf{y}_n &= \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n\end{aligned}$$

Example: Dynamic Model for a Spacecraft (2)

- Differential equation:

$$m\mathbf{a}(t) = -m g_0 r_e^2 \frac{\mathbf{p}(t)}{|\mathbf{p}(t)|^3} + \frac{1}{|\mathbf{p}(t)|} \begin{bmatrix} -p^y(t) \\ p^x(t) \end{bmatrix} w(t).$$

- State vector:

$$\mathbf{x}(t) = \begin{bmatrix} p^x(t) & p^y(t) & v^x(t) & v^y(t) \end{bmatrix}^\mathsf{T}.$$

- Vector form:

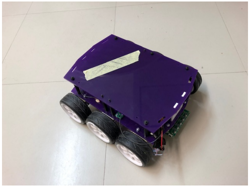
$$\begin{aligned}
\begin{bmatrix} v^x(t) \\ v^y(t) \\ a^x(t) \\ a^y(t) \end{bmatrix} &= \begin{bmatrix} v^x(t) \\ v^y(t) \\ -g_0 r_e^2 \frac{p^x(t)}{|\mathbf{p}(t)|^3} \\ -g_0 r_e^2 \frac{p^y(t)}{|\mathbf{p}(t)|^3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{p^y(t)}{m|\mathbf{p}(t)|} \\ \frac{p^x}{m|\mathbf{p}(t)|} \end{bmatrix} w(t) \\
&= \begin{bmatrix} f_1(\mathbf{x}(t)) \\ f_2(\mathbf{x}(t)) \\ f_3(\mathbf{x}(t)) \\ f_4(\mathbf{x}(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{p^y(t)}{m|\mathbf{p}(t)|} \\ \frac{p^x(t)}{m|\mathbf{p}(t)|} \end{bmatrix} w(t),
\end{aligned}$$

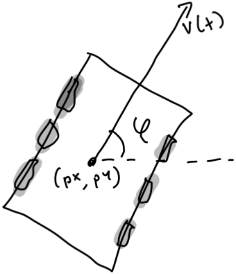
Example: Robot Navigation in 2D (1/4)

- Quasi-constant turn model:

$$\begin{aligned}\dot{p}^x(t) &= v(t)\cos(\varphi(t))\\ \dot{p}^y(t) &= v(t)\sin(\varphi(t))\\ \dot{v}(t) &= w_1(t)\\ \dot{\varphi}(t) &= w_2(t)\end{aligned}$$

- The state is $\mathbf{x}(t) = \begin{bmatrix} p^x(t) & p^y(t) & v(t) & \varphi(t)\end{bmatrix}^\mathsf{T}$.
- Position measurement: picks $p^x(t)$ and $p^y(t)$
- Speed measurements (odometry): $v(t)$
- Magnetometer (compass): $\varphi(t)$.



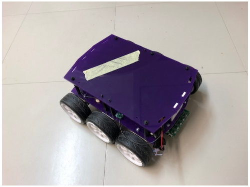


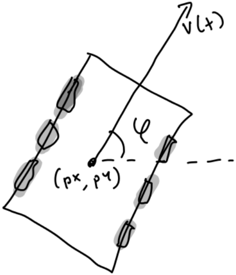
Example: Robot Navigation in 2D (2/4)

- Gyroscope measures $\dot{\varphi}(t)$.
- Accelerometer measures $\dot{v}(t)$.
  - Word of warning: accelerometers are usually not accurate enough for this.
- Putting these into the equations we get the model

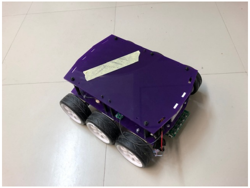
$$
\begin{aligned}
\dot{p}^x(t) &= v(t)\cos(\varphi(t)) \\
\dot{p}^y(t) &= v(t)\sin(\varphi(t)) \\
\dot{v}(t) &= a_{\text{acc}}(t) + w_1(t) \\
\dot{\varphi}(t) &= \omega_{\text{gyro}}(t) + w_2(t).
\end{aligned}
$$

- The state is still $\mathbf{x}(t) = \begin{bmatrix} p^x(t) & p^y(t) & v(t) & \varphi(t) \end{bmatrix}^\mathsf{T}$.





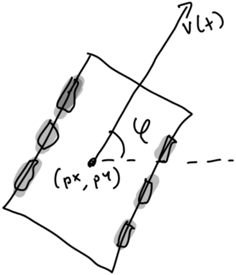
Example: Robot Navigation in 2D (3/4)



- Often we have the speed $v(t)$ directly available (e.g., from wheels)
- Then we can reduce the model to

$$\begin{aligned}
\dot{p}^x(t) &= v(t)\cos(\varphi(t)) \\
\dot{p}^y(t) &= v(t)\sin(\varphi(t)) \\
\dot{\varphi}(t) &= \omega_{\text{gyro}}(t) + w(t).
\end{aligned}$$

- The state is now $\mathbf{x}(t) = \begin{bmatrix} p^x(t) & p^y(t) & \varphi(t) \end{bmatrix}^\mathsf{T}$.
- This is a typical model used in 2D tracking.

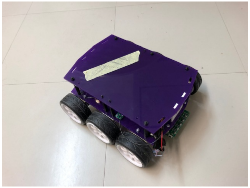


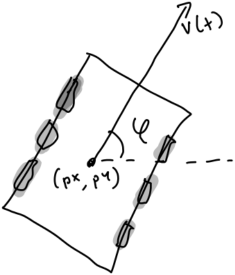
Example: Robot Navigation in 2D (4/4)

- Finally, the speed measurement is often not accurate.
- Thus it is beneficial to include additional noises to the dynamic model:

$$
\begin{aligned}
\dot{p}^x(t) &= v(t)\cos(\varphi(t)) + w_1(t) \\
\dot{p}^y(t) &= v(t)\sin(\varphi(t)) + w_2(t) \\
\dot{\varphi}(t) &= \omega_{\text{gyro}}(t) + w_3(t).
\end{aligned}
$$

- The state is stil $\mathbf{x}(t) = \begin{bmatrix} p^x(t) & p^y(t) & \varphi(t) \end{bmatrix}^\mathsf{T}$.
- This model would be a good candidate for the dynamic model in the project work.





Discrete-Time Processes and Difference Equations

- Some processes are only defined at discrete time points $t_1, t_2, \ldots$
- The discrete-time equivalent of differential equations are difference equations
- The difference of two discrete points in time takes the role of the derivative

Vector Form of Difference Equation Systems

- Equation system of d_x linear difference equations:

$$
\begin{aligned}
x_{1,n} &= a_{11} x_{1,n-1} + \cdots + a_{1 d_x} x_{d_x,n-1} + b_{11} u_{1,n} + \cdots + b_{1 d_u} u_{d_u,n} \\
x_{2,n} &= a_{21} x_{1,n-1} + \cdots + a_{2 d_x} x_{d_x,n-1} + b_{21} u_{1,n} + \cdots + b_{2 d_u} u_{d_u,n} \\
&\vdots \\
x_{d_x,n} &= a_{d_x 1} x_{1,n-1} + \cdots + a_{d_x d_x} x_{d_x,n-1} + b_{d_x 1} u_{1,n} + \cdots + b_{d_x d_u} u_{d_u,n}
\end{aligned}
$$

- Vector form:

$$
\begin{bmatrix} x_{1,n} \\ \vdots \\ x_{d_x,n} \end{bmatrix} = \begin{bmatrix} a_{11} & \ldots & a_{1 d_x} \\ \vdots & \ddots & \vdots \\ a_{d_x 1} & \ldots & a_{d_x d_x} \end{bmatrix} \begin{bmatrix} x_{1,n-1} \\ \vdots \\ x_{d_x,n-1} \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1 d_u} \\ \vdots & \ddots & \vdots \\ b_{d_x 1} & \ldots & b_{d_x d_u} \end{bmatrix} \begin{bmatrix} u_{1,n} \\ \vdots \\ u_{d_x,n} \end{bmatrix}
$$

- Compact notation:

$$
\mathbf{x}_n = \mathbf{F} \mathbf{x}_{n-1} + \mathbf{B}_u \mathbf{u}_n
$$

Deterministic Discrete-Time State-Space Model

- Linear discrete-time dynamic model:

$$\mathbf{x}_n = \mathbf{F}\mathbf{x}_{n-1} + \mathbf{B}_u \mathbf{u}_n$$

- Deterministic, linear discrete-time state-space model:

$$\begin{aligned}\mathbf{x}_n &= \mathbf{F}\mathbf{x}_{n-1} + \mathbf{B}_u \mathbf{u}_n \\ \mathbf{y}_n &= \mathbf{G}\mathbf{x}_n + \mathbf{r}_n.\end{aligned}$$

with $\mathrm{E}\{\mathbf{r}_n\} = 0$, $\mathrm{Cov}\{\mathbf{r}_n\} = \mathbf{R}_n$, $\mathrm{Cov}\{\mathbf{r}_n, \mathbf{r}_m\} = 0$ $(n \neq m)$

Conversion of Lth Order Difference Equation (1/2)

- Lth order difference equation (with single input u_n):

$$z_n = c_1 z_{n-1} + c_2 z_{n-2} + \cdots + c_L z_{n-L} + d_1 u_n$$

- It is easier to choose $\mathbf{x}_{n-1}$ on the RHS (c.f. continuous case)
- A possible choice:

$$x_{1,n-1} = z_{n-1},\ x_{2,n-1} = z_{n-2},\ \ldots,\ x_{d_x,n-1} = z_{n-L}.$$

Conversion of Lth Order Difference Equation (2/2)

- Difference equation system:

$$
\begin{aligned}
x_{1,n} &= c_1 x_{1,n-1} + c_2 x_{2,n-1} + \cdots + c_L x_{d_x,n-1} + d_1 u_n \\
x_{2,n} &= z_{n-1} = x_{1,n-1} \\
&\vdots \\
x_{d_x,n} &= z_{n-L+1} = x_{d_x+1,n-1}
\end{aligned}
$$

- Vector form:

$$
\begin{bmatrix} x_{1,n} \\ x_{2,n} \\ \vdots \\ x_{d_x,n} \end{bmatrix}
=
\begin{bmatrix} c_1 & c_2 & \dots & c_L \\ 1 & 0 & & \vdots \\ \vdots & \ddots & & \\ 0 & \dots & 1 & 0 \end{bmatrix}
\begin{bmatrix} x_{1,n-1} \\ x_{2,n-1} \\ \vdots \\ x_{d_x,n-1} \end{bmatrix}
+
\begin{bmatrix} d_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_n,
$$

Stochastic Linear State-Space Model (1/2)

- Dynamics are not entirely deterministic and inputs may not always be known
- Let the process noise $\mathbf{q}_n$ (random variable) take the place of the input $\mathbf{u}_n$ (or in addition to $\mathbf{u}_n$)
- Stochastic linear discrete-time dynamic model:

$$\mathbf{x}_n = \mathbf{F}\mathbf{x}_{n-1} + \mathbf{B}_q\mathbf{q}_n$$

Stochastic Linear State-Space Model (2/2)

- Stochastic linear discrete-time dynamic model:

$$\mathbf{x}_n = \mathbf{F}\mathbf{x}_{n-1} + \mathbf{B}_q \mathbf{q}_n$$

- The process noise follows

$$\mathbf{q}_n \sim p(\mathbf{q}_n)$$

  with $\mathrm{E}\{\mathbf{q}_n\} = 0$, $\mathrm{Cov}\{\mathbf{q}_n\} = \mathbf{Q}_n$, and $\mathrm{Cov}\{\mathbf{q}_m, \mathbf{q}_n\} = 0$ $(m \neq n)$

- Stochastic linear discrete-time state-space model:

$$\begin{aligned}
\mathbf{x}_n &= \mathbf{F}\mathbf{x}_{n-1} + \mathbf{B}_q \mathbf{q}_n \\
\mathbf{y}_n &= \mathbf{G}\mathbf{x}_n + \mathbf{r}_n
\end{aligned}$$

Nonlinear Discrete-Time Dynamic Model

- Difference equations may also be nonlinear
- Nonlinear difference equation system (with process noise inputs):

$$
\begin{aligned}
x_{1,n} &= f_1(x_{1,n-1}, x_{2,n-1}, \ldots, x_{d_x,n-1}) + b_{11} q_{1,n} + \cdots + b_{1 d_q} q_{d_q,n} \\
x_{2,n} &= f_2(x_{1,n-1}, x_{2,n-1}, \ldots, x_{d_x,n-1}) + b_{21} q_{1,n} + \cdots + b_{2 d_q} q_{d_q,n} \\
&\vdots \\
x_{d_x,n} &= f_{d_x}(x_{1,n-1}, x_{2,n-1}, \ldots, x_{d_x,n-1}) + b_{d_x 1} q_{1,n} + \cdots + b_{d_x d_q} q_{d_q,n}
\end{aligned}
$$

- Vector form:

$$
\begin{bmatrix} x_{1,n} \\ \vdots \\ x_{d_x,n} \end{bmatrix} = \begin{bmatrix} f_1(x_{1,n-1}, x_{2,n-1}, \ldots, x_{d_x,n-1}) \\ \vdots \\ f_{d_x}(x_{1,n-1}, x_{2,n-1}, \ldots, x_{d_x,n-1}) \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1 d_u} \\ \vdots & \ddots & \vdots \\ b_{d_x 1} & \ldots & b_{d_x d_u} \end{bmatrix} \begin{bmatrix} q_{1,n} \\ \vdots \\ q_{d_u,n} \end{bmatrix}
$$

Nonlinear Discrete-Time State-Space Model

- Compact notation of the dynamic model:

$$\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{B}_q \mathbf{q}_n$$

- Nonlinear discrete-time state-space model:

$$\begin{aligned} \mathbf{x}_n &= \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{B}_q \mathbf{q}_n \\ \mathbf{y}_n &= \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n \end{aligned}$$

  where:
  - $\mathbf{q}_n \sim p(\mathbf{q}_n)$, $\mathrm{E}\{\mathbf{q}_n\} = 0$, $\mathrm{Cov}\{\mathbf{q}_n\} = \mathbf{Q}_n$
  - $\mathbf{r}_n \sim p(\mathbf{r}_n)$, $\mathrm{E}\{\mathbf{r}_n\} = 0$, $\mathrm{Cov}\{\mathbf{r}_n\} = \mathbf{R}_n$

Summary

- Nonlinear continuous-time state-space model:

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t) \\
\mathbf{y}_n &= \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n
\end{aligned}
$$

- Linear discrete-time state-space model:

$$
\begin{aligned}
\mathbf{x}_n &= \mathbf{F}\mathbf{x}_{n-1} + \mathbf{B}_q\mathbf{q}_n \\
\mathbf{y}_n &= \mathbf{G}\mathbf{x}_n + \mathbf{r}_n
\end{aligned}
$$

- Nonlinear discrete-time state-space model:

$$
\begin{aligned}
\mathbf{x}_n &= \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{B}_q(\mathbf{x}_{n-1})\mathbf{q}_n \\
\mathbf{y}_n &= \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n
\end{aligned}
$$