Aalto University
School of Electrical
Engineering

ELEC-E8101: Digital and Optimal Control

Lecture 9
Disturbances

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Slides based on ELEC-E8101 material by Themistoklis Charalambous

In the previous lecture…

- We revisited effect of feedback on input-output dynamics

$$R(z) \longrightarrow \boxed{H(z) = \frac{P(z)K(z)}{1 + P(z)K(z)}} \longrightarrow Y(z)$$

- And error dynamics

$$E(z) = \frac{1}{1 + P(z)K(z)} R(z)$$

In the previous lecture…

- Discretized PID controllers

$$\begin{cases} P(t) = K_p e(t) \\ I(t) = K_i \int_{-\infty}^{t} e(\tau) d\tau \\ D(t) = K_d \frac{de(t)}{dt} \end{cases} \Rightarrow \begin{cases} P(kh) = K_p e(kh) \\ I(kh) = K_i \sum_{n=-\infty}^{k-1} e(nh)h = K_i h \sum_{n=-\infty}^{k-1} e(nh) \\ D(kh) = K_d \frac{e(kh)-e(kh-h)}{h} = \frac{K_d}{h} \Delta e(kh) \end{cases}$$

getting

$$u(kh) = K_p e(kh) + K_i h \sum_{n=-\infty}^{k-1} e(nh) + \frac{K_d}{h} \Delta e(kh)$$

and in *z*-domain

$$U(z) = \underbrace{\left(K_p + \frac{K_i h}{z-1} + \frac{K_d}{h} \frac{z-1}{z} \right)}_{H_{\text{PID}}} E(z)$$

On this lecture

We will talk about

- Deterministic and stochastic disturbances
- Models for stochastic disturbances/noise

We will also revisit

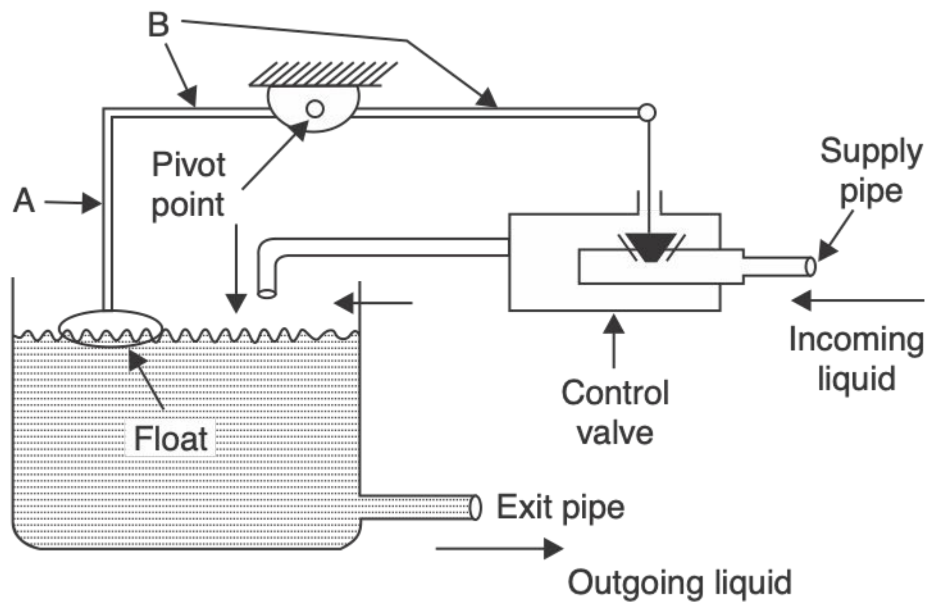
- State observers

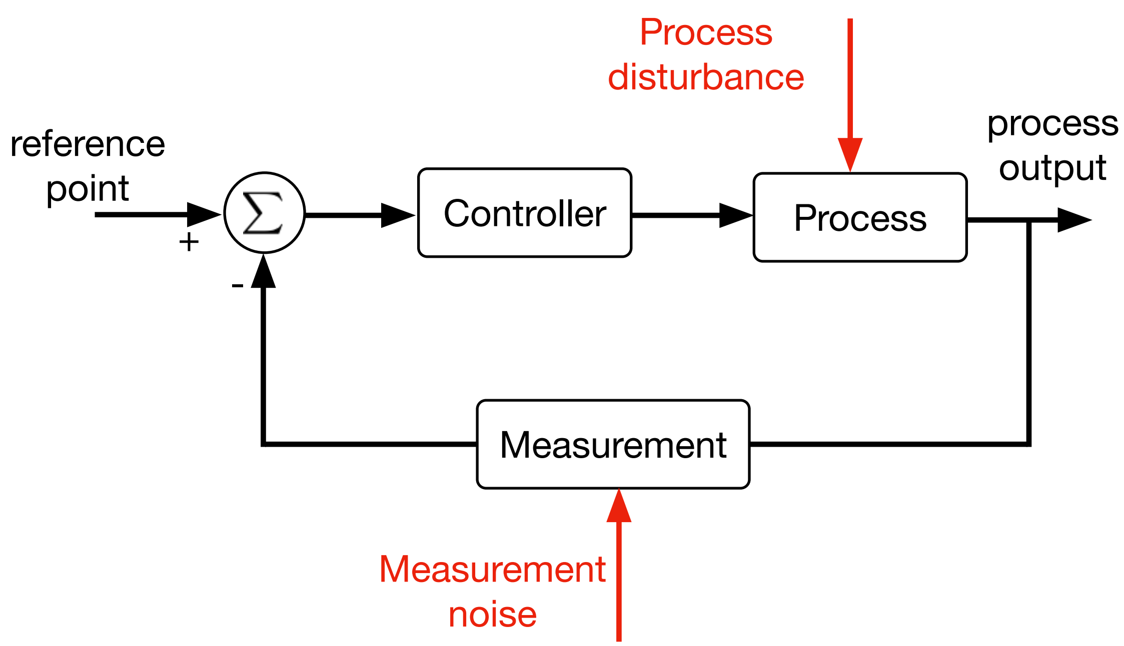
By the end of *this* lecture, you should be able to:

- Explain different types of disturbances.
- Understand the characteristics and effect of noise in dynamical processes
- Compute the mean and covariance matrices of dynamical processes

Disturbances

Closed-loop system

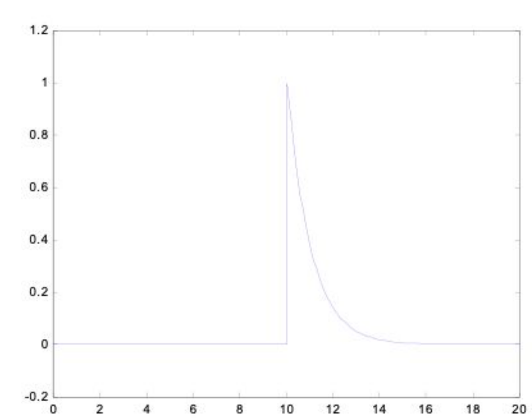


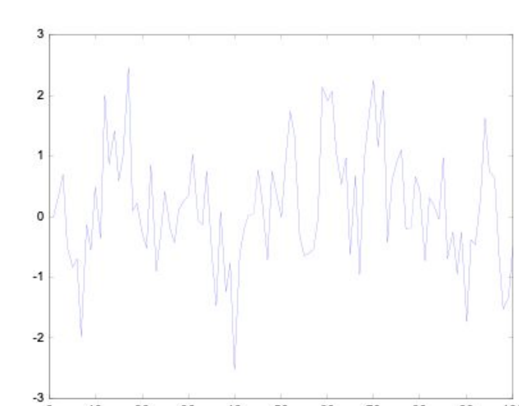


- Existence of disturbances is one of the key reasons why a control system is needed

- Based on where they appear in the process, they are mainly classified into:
  - **process disturbances** (due to modeling or external factors)
  - **measurement disturbances** (i.e., we can only obtain noisy measurements of the system output)

- Based on the type of disturbance, they are classified into:
  - **deterministic** (e.g., impulse, step, ramp)
  - **stochastic**

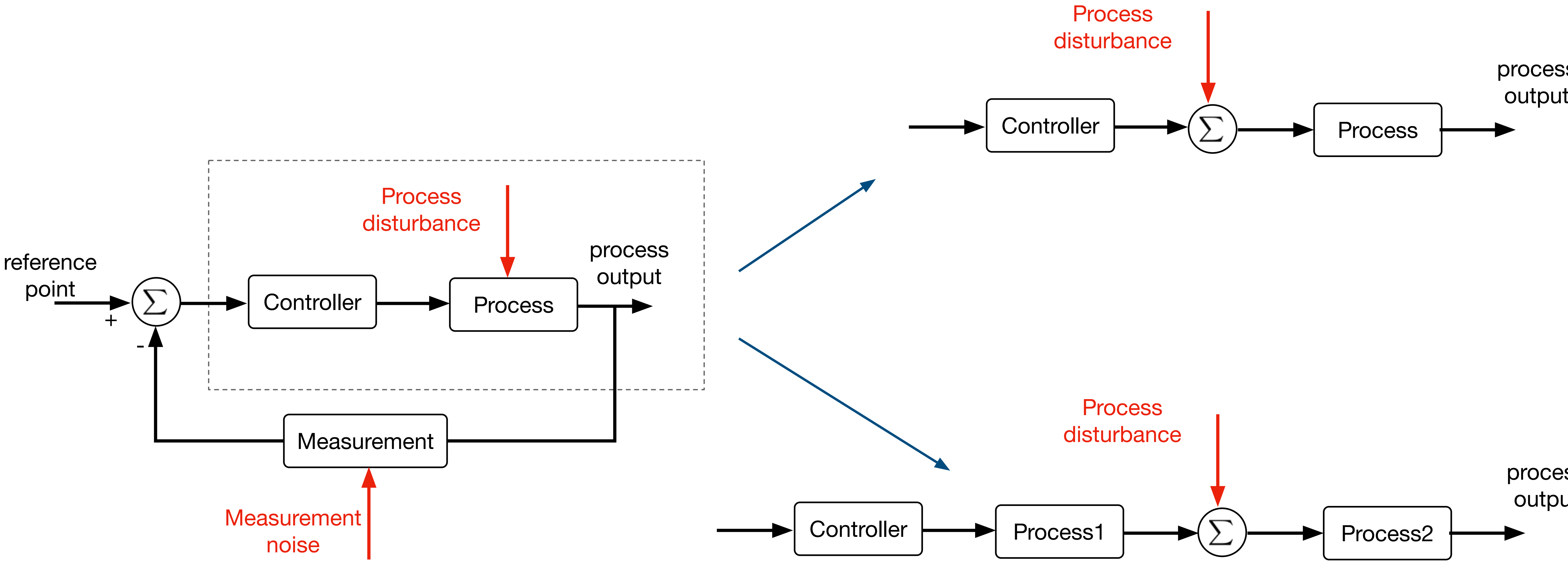




Process Disturbances

- **Typically load disturbances, typically vary slowly.**
- **May also represent modeling inaccuracies/errors.**

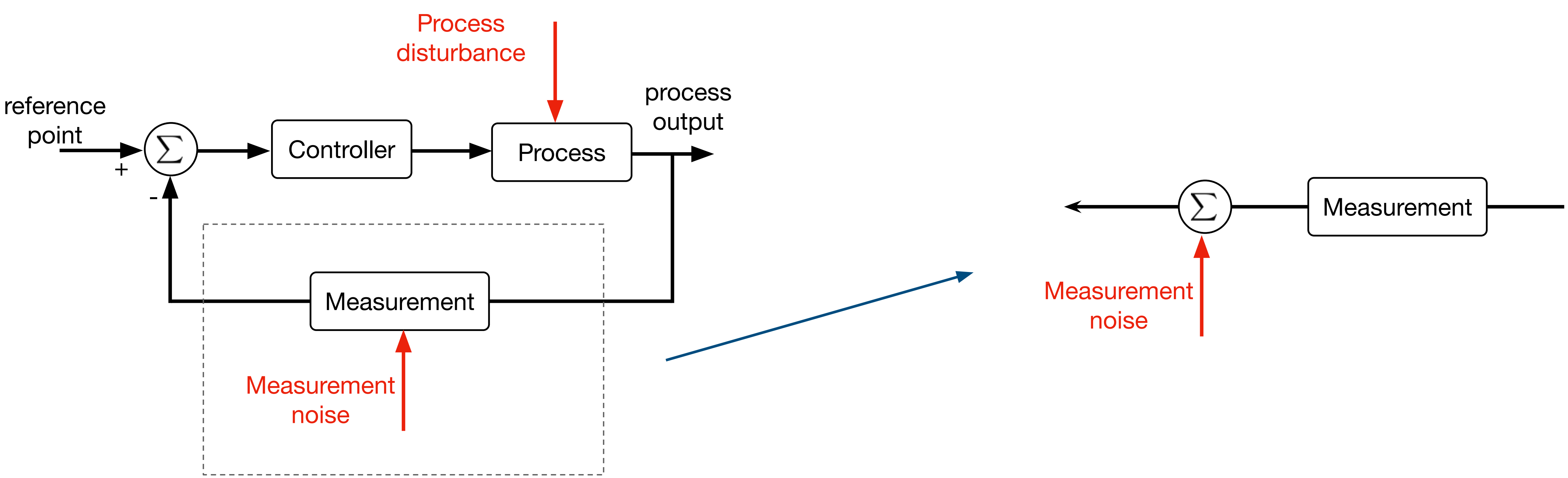
- **How process disturbances affect the process?**
  - Often modeled as additive components, affecting process input.
  - Or in the middle of a two-part process.



Measurement Disturbances

- **Often vary fast, often random in nature.**

- **How measurement disturbances affect the process?**
  - Often modeled as additive components, affecting measurement.



Sometimes word “disturbance” is reserved for process disturbances, and “noise” refers to measurement disturbance.

Reduction of effects of disturbances

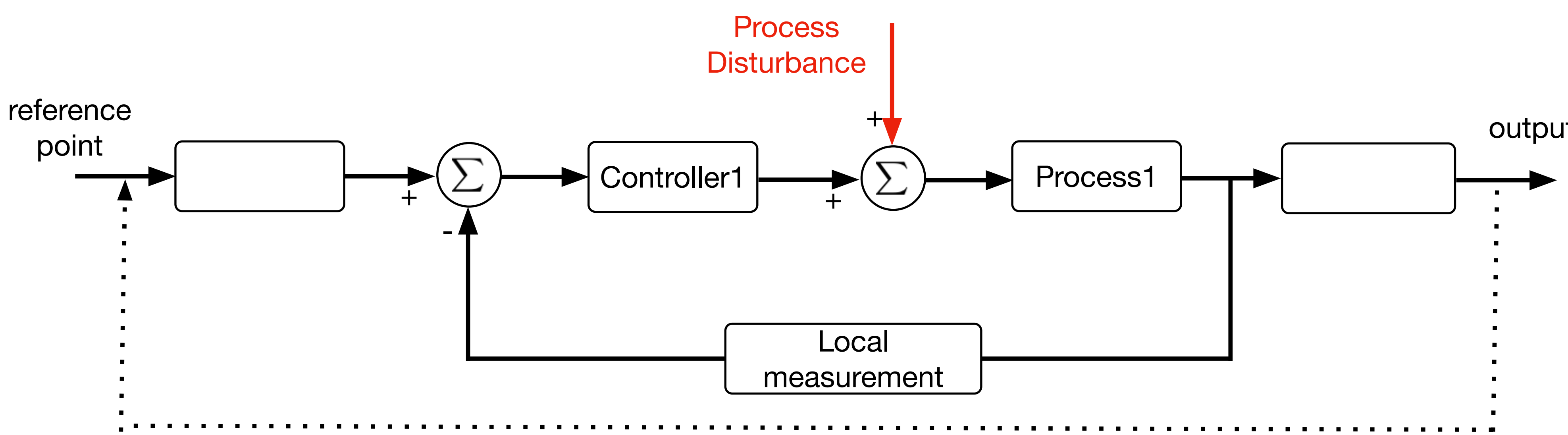
- **Reduce the source of the disturbances** (related to process and measurement design)

- Examples:
  - Buffer vessel in process industry
  - Better positioning of the measurement sensor
  - Better sampling
  - Sensor improvement (for obtaining less noise) or replacement (with sensors of less noise)
  - More sensors and use of sensor fusion

Which of these reduce process disturbances, which measurement disturbances?

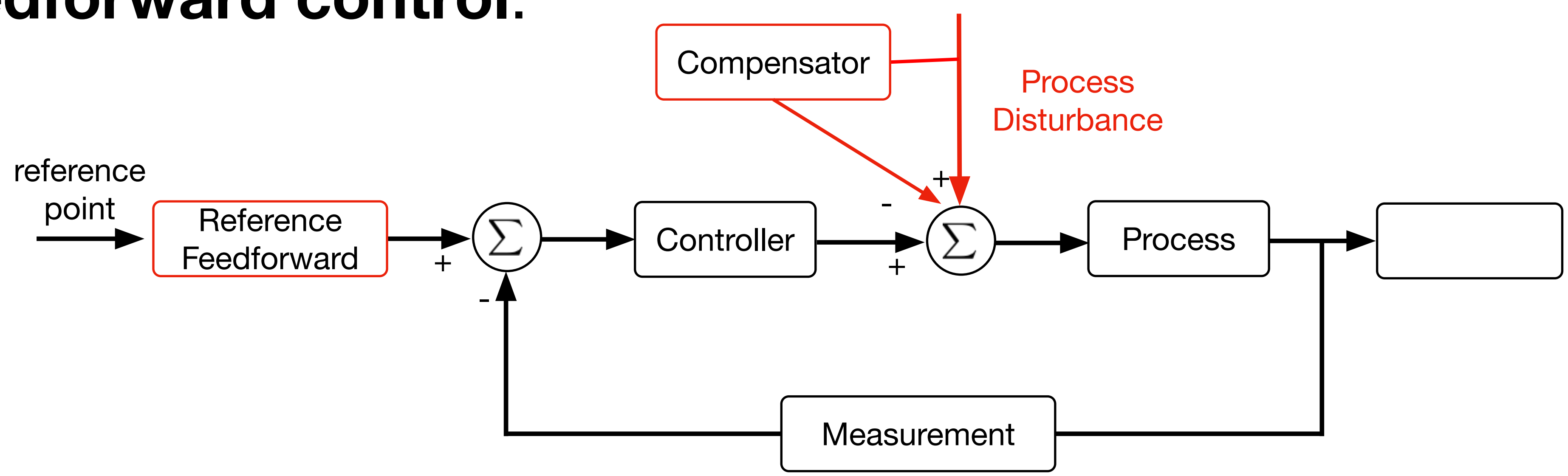
Reduction of effects of disturbances

- **By local feedback**
  - For example:
    - reduce variations in supply pressure to valves by introducing a pressure regulator
    - control current of electric motor to achieve desired torque
  - Necessary that disturbances enter the system locally in a well-defined way
  - Necessary to have access to a measured variable that is influenced by the disturbance and to have access to a control variable that enters the system in the neighborhood of the disturbance
  - Dynamics relating the measured variable to the control variable should be such that a high-gain control loop can be used - no need to have detailed characteristics about the process



Reduction of effects of disturbances

- **By feedforward control:**



- Disturbance is measured, and a control signal that attempts to counteract the disturbance is generated and applied to the process

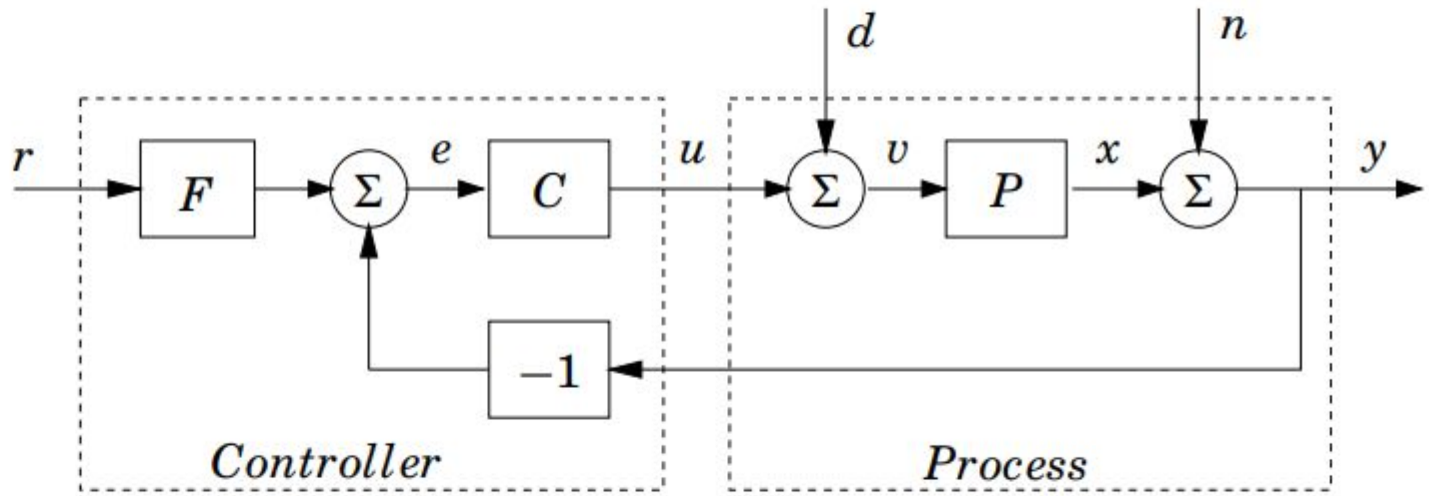
- Particularly useful for disturbances generated by changes in the command or reference signals

- Also for modellable model disturbances (e.g. gravity compensation for robot)

Reduction of effects of disturbances

- **By prediction**:
  - Extension of the feedforward principle that may be used when the disturbance cannot be measured
  - Disturbance is predicted using measurable signals, and the feedforward signal is generated from the prediction
  - Not necessary to predict the disturbance itself; sufficient to model a signal that represents the effect of the disturbance on the important process variables

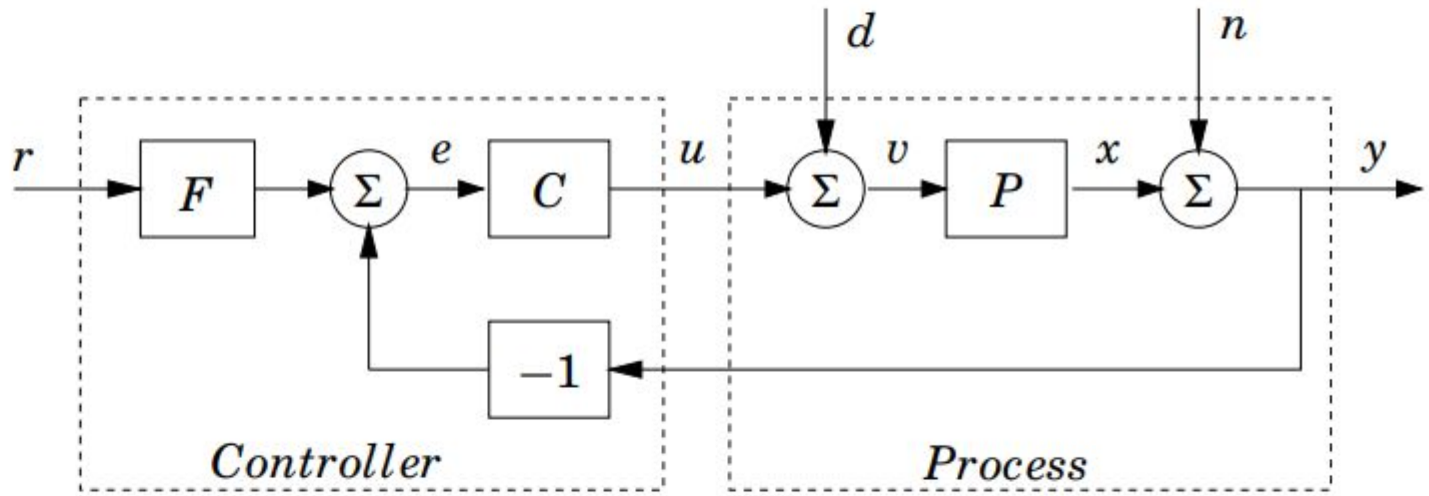
Effect of control on disturbances



- **Relations between inputs r, n, d and variables of interest x, y, u**

$$
\begin{aligned}
X &= \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R \\
Y &= \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R \\
U &= -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R
\end{aligned}
$$

Effect of control on disturbances



- **For pure feedback control (Gang of Four)**

N,R to X
R to Y
D to U

$$\frac{PC}{1+PC}$$

D to X,Y

$$\frac{P}{1+PC}$$

$$\frac{C}{1+PC}$$

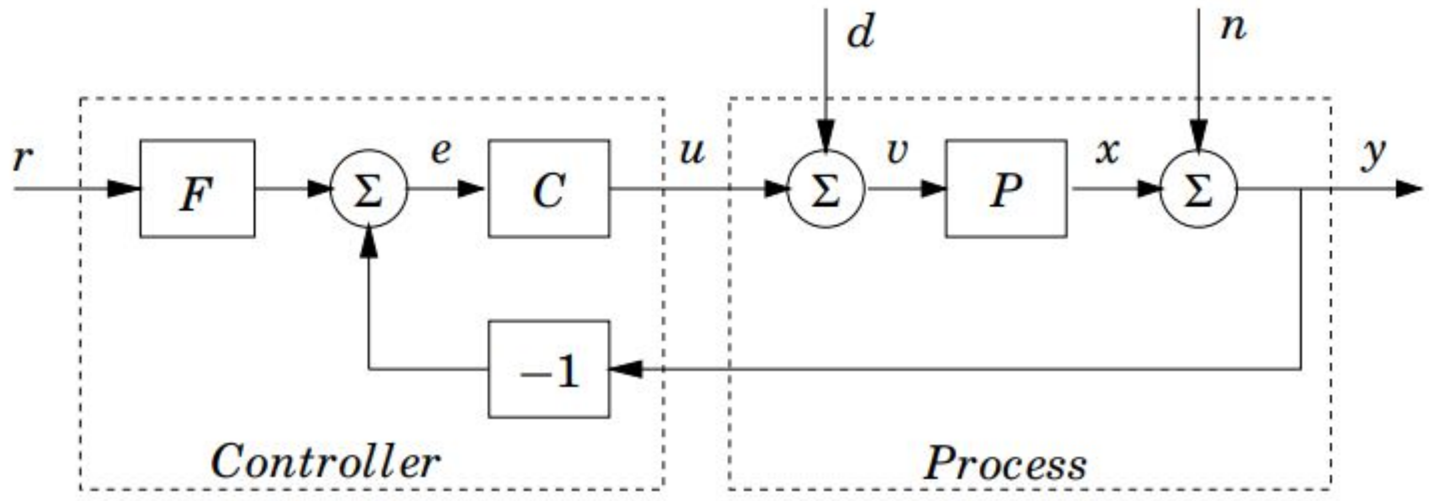
N,R to U

$$\frac{1}{1+PC}$$

N to Y

How do we want these to behave?

Effect of control on disturbances



- **For pure feedback control (Gang of Four)**

Compromise! $\boxed{N,R \text{ to } X}$

R to Y

D to U

D to X,Y

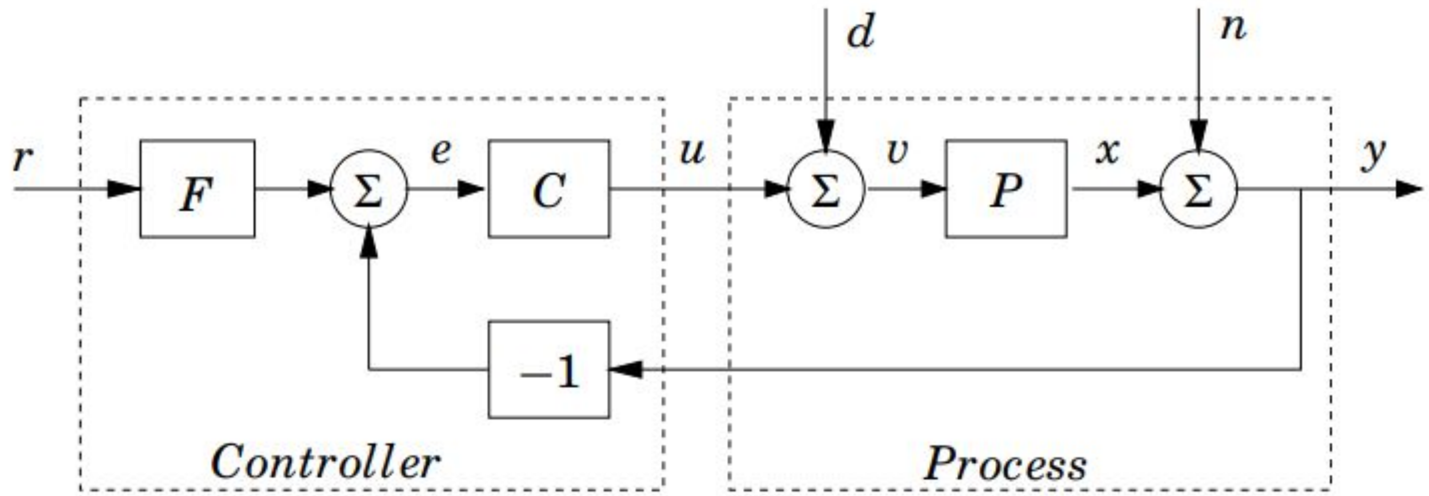
$$\frac{PC}{1+PC} \qquad \frac{P}{1+PC}$$

$$\frac{C}{1+PC} \qquad \frac{1}{1+PC}$$

N,R to U

N to Y

Effect of control on disturbances

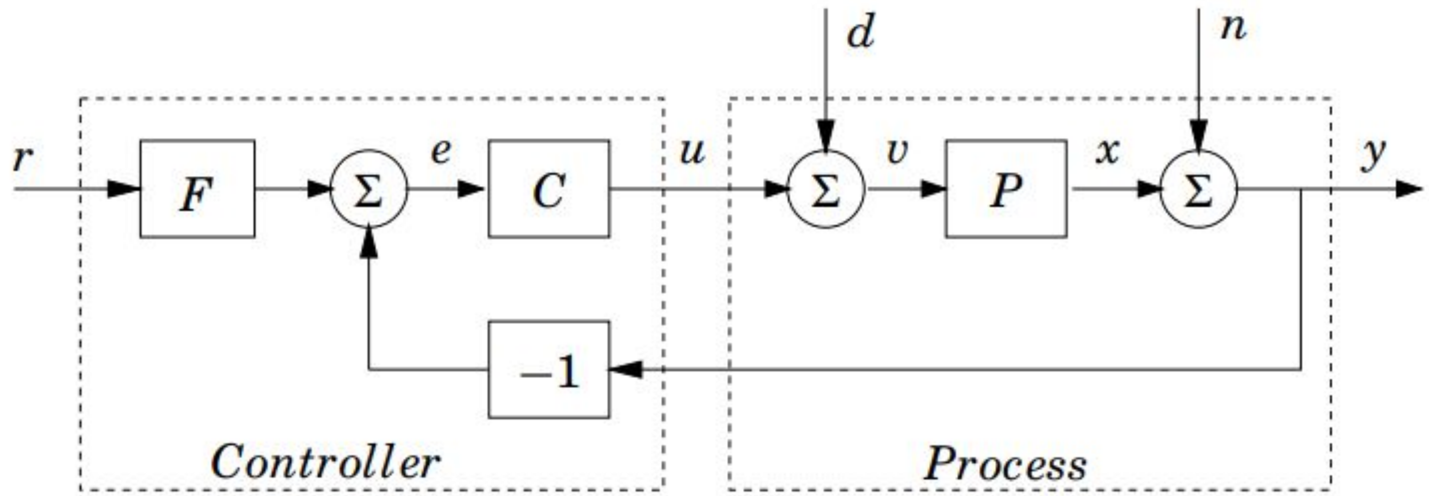


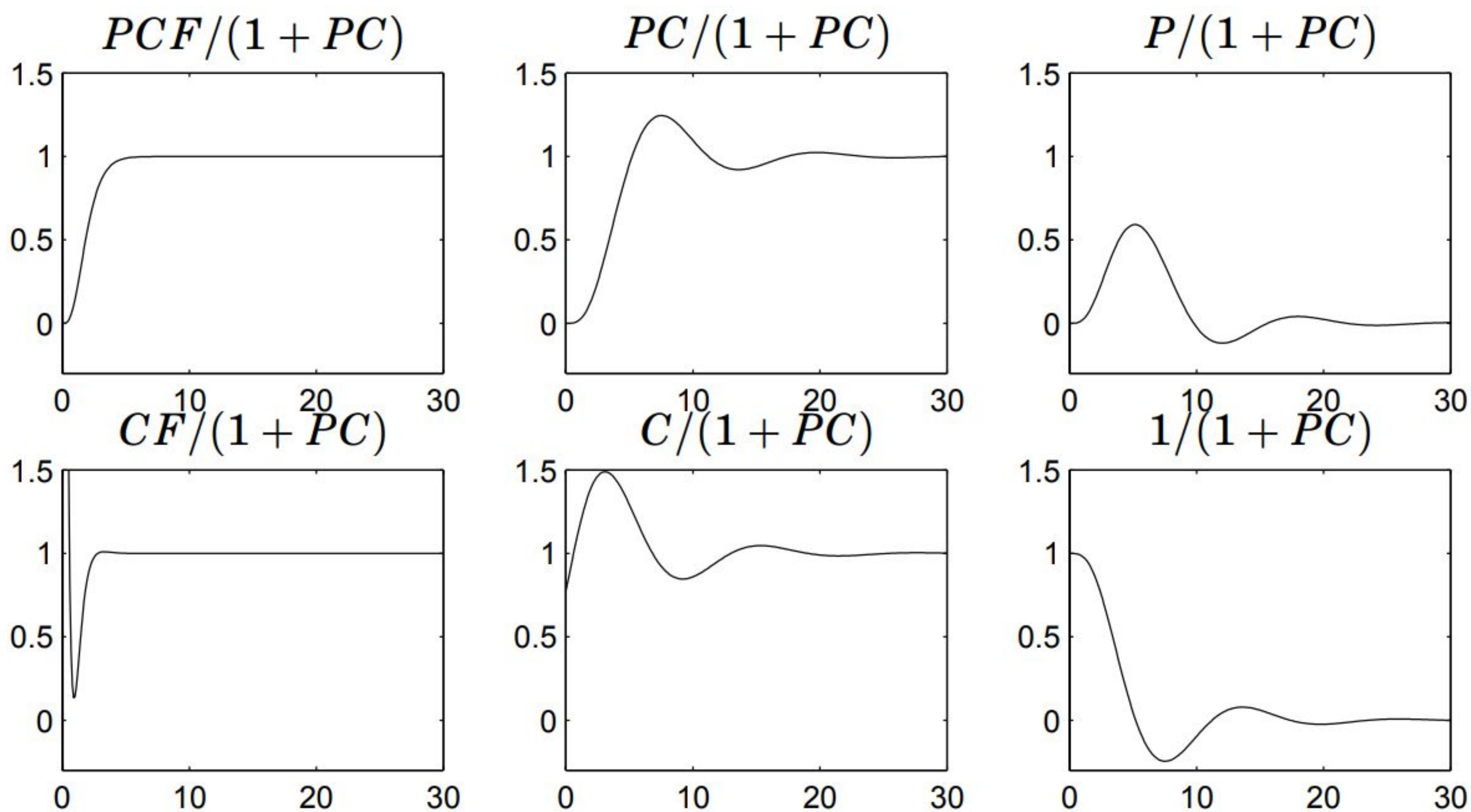
- **Feedback + feedforward (Gang of Six)**

R to X,Y	*N to X* *D to U*	*D to X,Y*
$\dfrac{PCF}{1+PC}$	$\dfrac{PC}{1+PC}$	$\dfrac{P}{1+PC}$
$\dfrac{CF}{1+PC}$	$\dfrac{C}{1+PC}$	$\dfrac{1}{1+PC}$
R to U	*N to U*	*N to Y*

Feedback can be designed to deal with disturbances, feedforward response to reference changes!

Example





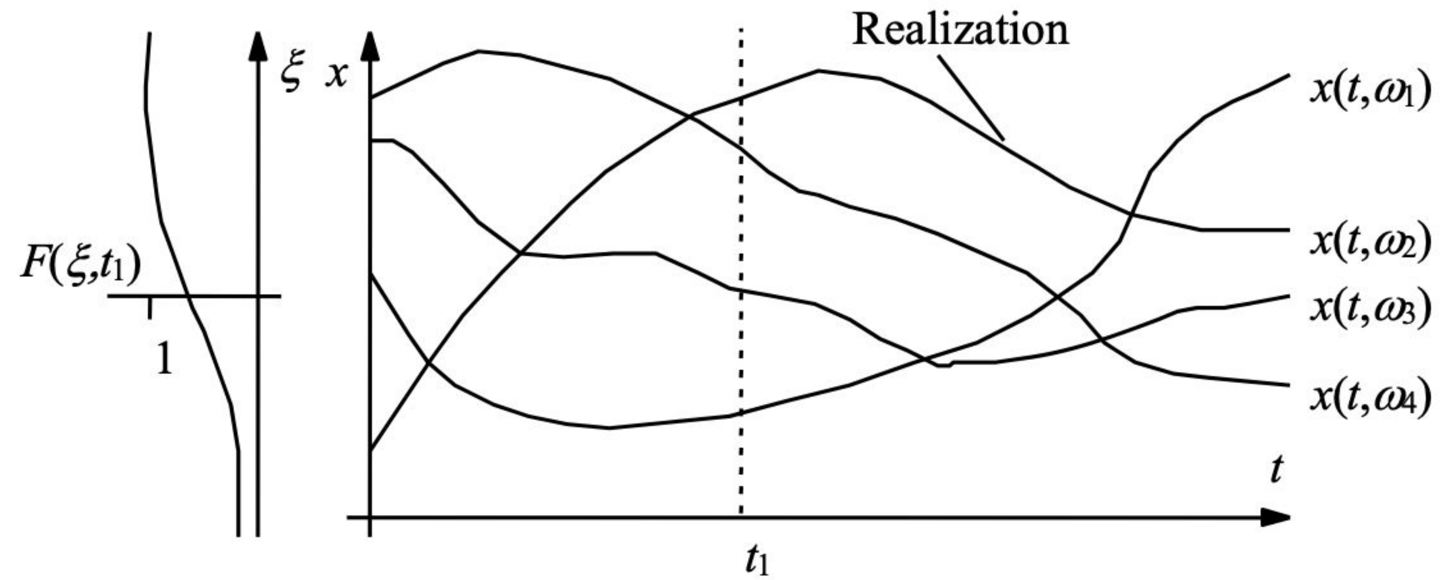
Now for something different: Modeling random disturbances

Stochastic models of disturbances

- Natural to use stochastic (random) concepts to describe disturbances
  - possible to describe a wide class of disturbances → permits good formulation of prediction problems

- A stochastic process (random process, random function) can be regarded as a family of stochastic variables $\{x[k], k \in T\}$. In this context, T is the time index

- A stochastic process may be considered as a function of 2 variables,
  - If variable ω is fixed, $x[\bullet,\omega]$ is called realization
  - If variable k is fixed, $x[k,\bullet]$ is a random variable



Concepts of stochastic processes

- The distribution function of a stochastic process is defined as follows (P denotes probability)

$$F(\xi_1, \xi_2, \ldots, \xi_n; k_1, k_2, \ldots, k_n) \triangleq P\{x[k_1] < \xi_1, x[k_2] < \xi_2, \ldots, x[k_n] < \xi_n\}$$

- The expected (or mean) value of a stochastic process x is defined as

$$m[k] \triangleq E\{x[k]\} = \int_{-\infty}^{+\infty} \xi dF(\xi; k)$$

- The variance is defined as

$$\sigma_x^2[k] \equiv \mathrm{var}\{x[k]\} \triangleq E\{(x[k] - m[k])^2\} = E\Big\{\big(x[k] - E\{x[k]\}\big)^2\Big\}$$

$$= \int_{-\infty}^{+\infty} \big(\xi - m[k]\}\big)^2 dF(\xi; k)$$

Concepts of stochastic processes

- For computing the mean and variance, sometimes the derivative of F, called the density function, is used instead, where

$$\int_{-\infty}^{+\infty} p(x)dx = 1$$

- The expected (or mean) value of a stochastic process x is simplified to

$$m[k] \triangleq E\{x[k]\} = \int_{-\infty}^{+\infty} xp(x)dx$$

- The variance is simplified to

$$\sigma_x^2 \equiv \mathrm{var}\{x\} = E\Big\{\big(x - E\{x\}\big)^2\Big\} = \int_{-\infty}^{+\infty} \big(x - E\{x\}\big)^2 p(x)dx$$

Some useful properties

- Suppose a is constant, x and y are stochastic variables. Then

$$
\begin{aligned}
E\{a\} &= a \\
E\{ax\} &= aE\{x\} \\
E\{x+y\} &= E\{x\} + E\{y\}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{var}\{a\} &= 0 \\
\mathrm{var}\{ax\} &= a^2 \mathrm{var}\{x\}
\end{aligned}
$$

- If x and y are independent random variables, then

$$E\{xy\} = E\{x\}E\{y\}$$

$$\mathrm{var}\{x+y\} = \mathrm{var}\{x\} + \mathrm{var}\{y\}$$

For refreshing memory from basic probability course

Concepts of stochastic processes

- The definitions of mean and variance are extended to vector functions; the variance is extended to *covariance*

- The expected (or mean) value of a stochastic process $\mathbf{x}$ is given by

$$\mathbf{m}[k] \triangleq E\{\mathbf{x}[k]\}$$

- The variance is given by

$$\begin{aligned}\mathrm{var}\{\mathbf{x}[k]\} &\triangleq E\{(\mathbf{x}[k]-\mathbf{m}[k])(\mathbf{x}[k]-\mathbf{m}[k])^T\}\\ &= E\Big\{\big(\mathbf{x}[k]-E\{\mathbf{x}[k]\}\big)\big(\mathbf{x}[k]-E\{\mathbf{x}[k]\}\big)^T\Big\}\end{aligned}$$

- The covariance function is given by

$$\begin{aligned}\mathbf{r}_{\mathbf{xx}}(s,k) \equiv \mathrm{cov}\{\mathbf{x}[s],\mathbf{x}[k]\} &\triangleq E\{(\mathbf{x}[s]-\mathbf{m}[s])(\mathbf{x}[k]-\mathbf{m}[k])^T\}\\ &= E\Big\{\big(\mathbf{x}[s]-E\{\mathbf{x}[s]\}\big)\big(\mathbf{x}[k]-E\{\mathbf{x}[k]\}\big)^T\Big\}\\ &= \int\int(\xi_1-\mathbf{m}[s])(\xi_2-\mathbf{m}[k])^T dF(\xi_1,\xi_2;s,k)\end{aligned}$$

Stationarity and covariance

- A stochastic process is called stationary if the finite-dimensional distribution of $\{x[k_1], x[k_2], \ldots, x[k_n]\}$ is identical to that of $\{x[k_1+k], x[k_2+k], \ldots, x[k_n+k]\}$ for all $k, n, k_1, k_2, \ldots, k_n$.

- The process is weakly stationary, if the two first moments of the distributions (mean and covariance) are the same for all values of k. The value of the covariance function then depends only on the time difference $\tau = s - k$, i.e.,

$$\mathbf{r}_{\mathbf{xx}}(s,k) = \mathbf{r}_{\mathbf{xx}}(s-k) = \mathbf{r}_{\mathbf{xx}}(\tau) = \operatorname{cov}\Big\{\mathbf{x}[k+\tau], \mathbf{x}[k]\Big\}$$

- From the definition it is immediate that variance is auto-covariance when $\tau = 0$

$$\mathbf{r}_{\mathbf{xx}}(0) = \operatorname{cov}\Big\{\mathbf{x}[k], \mathbf{x}[k]\Big\} = \operatorname{var}\Big\{\mathbf{x}[k]\Big\}$$

- The auto-covariance function describes how the signal correlates with itself at different time instants (one of the tools used to find patterns in the data):
  - A big positive covariance value means strong correlation
  - Zero means no correlation
  - A negative value means negative correlation.

Auto-covariance, correlation & cross-covariance

- **What is the largest value of auto-covariance?**

$$|\mathbf{r}_{\mathbf{xx}}(\tau)| \leq \mathbf{r}_{\mathbf{xx}}(0)$$

- The auto-covariance matrix is symmetric, i.e., $\mathbf{r}_{\mathbf{xx}}(\tau) = \mathbf{r}_{\mathbf{xx}}(-\tau)$

- The correlation function is the normed covariance function, i.e., $\rho_{\mathbf{x}}(\tau) = \dfrac{\mathbf{r}_{\mathbf{xx}}(\tau)}{\mathbf{r}_{\mathbf{xx}}(0)}$
  - Intuitive appeal as a measure of dependence
  - Useful for confirming (or disproving) linear relationships
  - Useful for analyzing and interpreting regression models

- Cross-covariance: $\mathbf{r}_{\mathbf{xy}}(\tau) = \mathrm{cov}\{\mathbf{x}[k+\tau], \mathbf{y}[k]\}$

- The cross-covariance function describes how a signal correlates with another signal. By using cross-covariance it is possible to investigate, e.g., which signals in a large system correlate with each other

Estimations from data

- Estimates of the proposed concepts can be calculated directly from data. For example:

Expected value: $$E\{\mathbf{x}\} \approx \frac{1}{N}\sum_{k=1}^{N}\mathbf{x}[k]$$

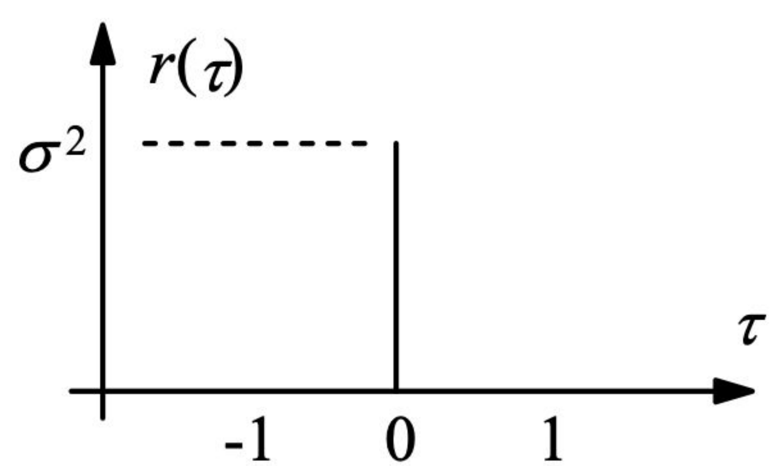
Cross-covariance: $$\mathbf{r}_{\mathbf{xy}}(\tau) \approx \frac{1}{N}\sum_{k=1}^{N}(\mathbf{x}[k+\tau] - \mathbf{m}_{\mathbf{x}}[k+\tau])(\mathbf{y}[k] - \mathbf{m}_{\mathbf{y}}[k])^T$$

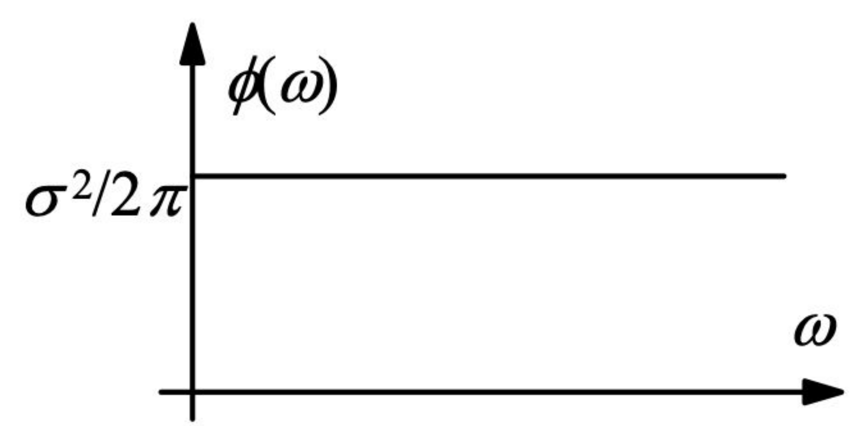
White noise in discrete-time systems

- At each time instant the signal value is a random variable without any correlation to any other signal (or to itself) at any time instant

Auto-covariance: $$r(\tau) = \begin{cases} \sigma^2, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}$$

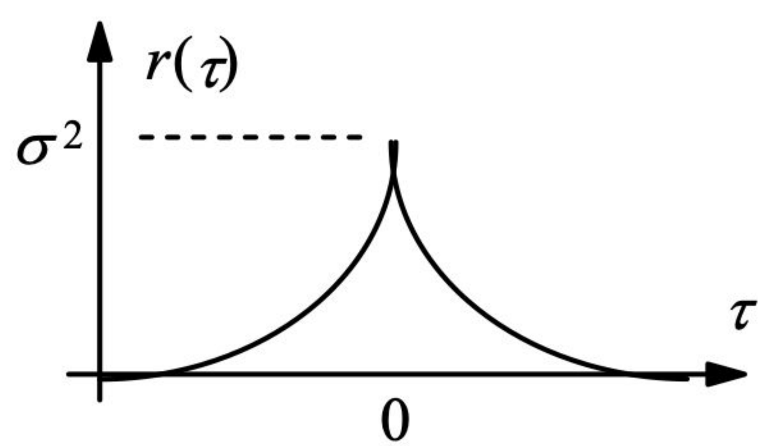
Auto-spectral density: $$\phi(\omega) = \frac{\sigma^2}{2\pi}$$

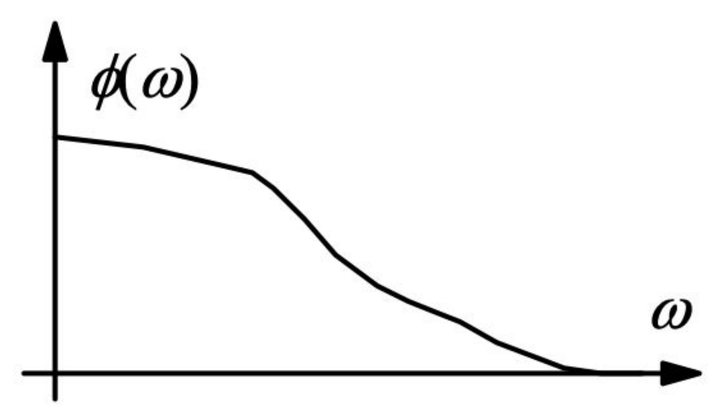




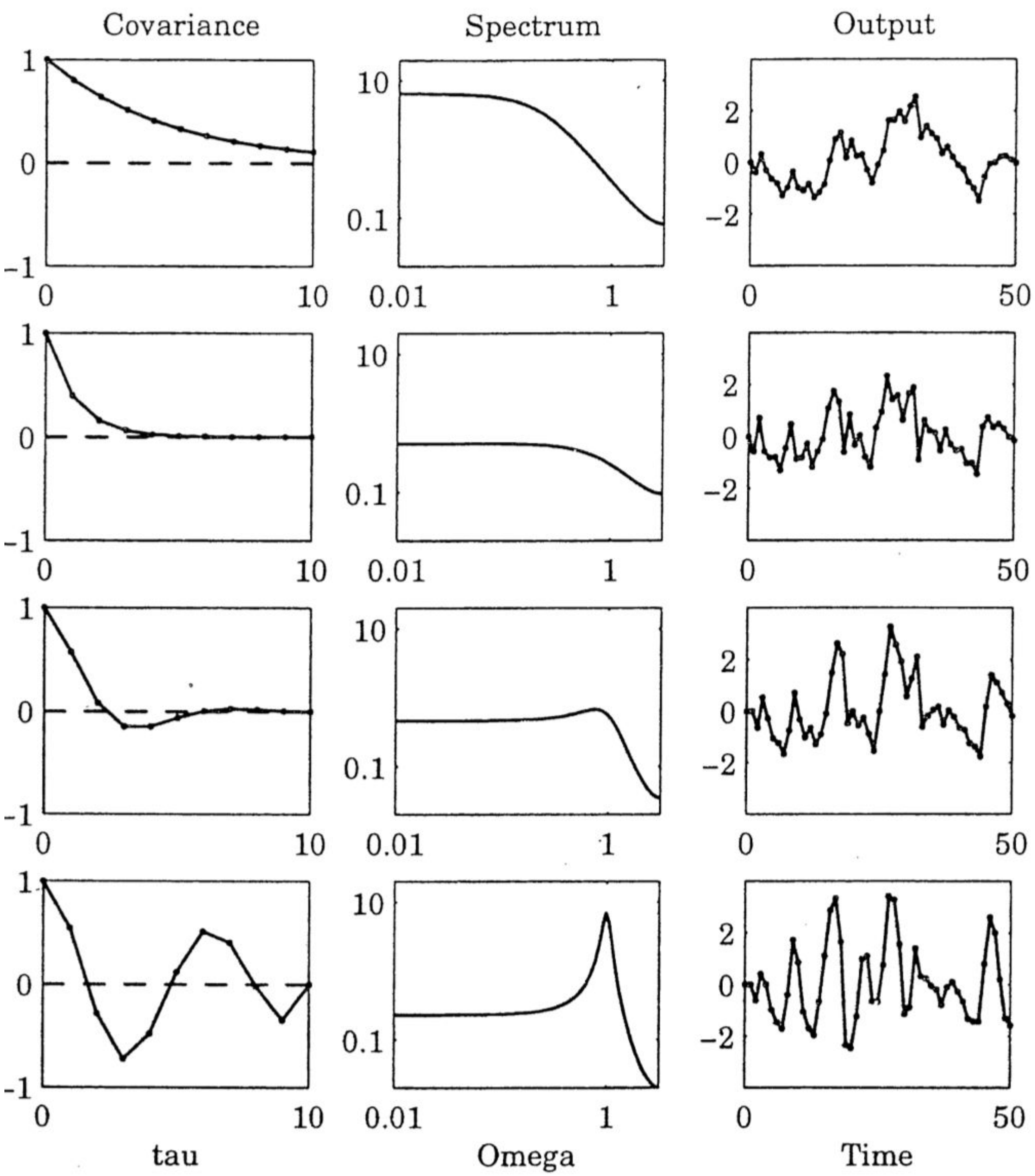
Colored noise in discrete-time systems

- All needed stochastic signals can be generated by filtering white noise, so that:
  - the covariance “spreads” (a sample correlates with previous values and values to come), and
  - in the spectrum certain frequencies are weighted more (the signal is more powerful at certain frequencies)





Example of processes



- Covariance functions, spectral densities, and sample functions for some stationary random processes. All processes have unit variance.
- In practical work, it is useful to have a good understanding of how signal properties are related to the spectrum

Stochastic difference equations

- Consider the representation

$$\mathbf{x}[k+1] = \Phi\mathbf{x}[k] + \mathbf{v}[k]$$

where is an independent zero-mean random variable with covariance (correlates neither with nor with itself at any time instant); is therefore white noise

- Suppose that the initial state has the mean $\mathbf{m}_0$ and covariance . Consider the behavior of as a function of time: $\mathbf{m}[k] = E\{\mathbf{x}[k]\}$

- Take expectations from both sides

$$E\{\mathbf{x}[k+1]\} = E\{\Phi\mathbf{x}[k] + \mathbf{v}[k]\} = E\{\Phi\mathbf{x}[k]\} + E\{\mathbf{v}[k]\} = \Phi \underbrace{E\{\mathbf{x}[k]\}}_{=\mathbf{m}[k]} + \underbrace{E\{\mathbf{v}[k]\}}_{=0}$$

$$\Rightarrow \mathbf{m}[k+1] = \Phi\mathbf{m}[k], \quad \mathbf{m}[0] = \mathbf{m}_0$$

- The mean value behaves exactly according to system dynamics!

Stochastic difference equations

- As for the covariance function, use a new variable: $\tilde{\mathbf{x}}[k] = \mathbf{x}[k] - \mathbf{m}[k]$

- For the state covariance :

$$P[k] = \mathrm{cov}\{\mathbf{x}[k], \mathbf{x}[k]\} = E\{\tilde{\mathbf{x}}[k]\tilde{\mathbf{x}}^T[k]\}$$

- We want to see how the state covariance evolves over time. Towards this end:

$$\begin{aligned}
\tilde{\mathbf{x}}[k+1]\tilde{\mathbf{x}}^T[k+1] &= \big(\Phi\tilde{\mathbf{x}}[k] + \mathbf{v}[k]\big)\big(\Phi\tilde{\mathbf{x}}[k] + \mathbf{v}[k]\big)^T \\
&= \big(\Phi\tilde{\mathbf{x}}[k] + \mathbf{v}[k]\big)\big(\tilde{\mathbf{x}}^T[k]\Phi^T + \mathbf{v}^T[k]\big) \\
&= \Phi\tilde{\mathbf{x}}[k]\tilde{\mathbf{x}}^T[k]\Phi^T + \Phi\tilde{\mathbf{x}}[k]\mathbf{v}^T[k] + \mathbf{v}[k]\tilde{\mathbf{x}}^T[k]\Phi^T + \mathbf{v}[k]\mathbf{v}^T[k]
\end{aligned}$$

- Take expectations in both sides:

$$\begin{aligned}
E\{\tilde{\mathbf{x}}[k+1]\tilde{\mathbf{x}}^T[k+1]\} = E\{\Phi\tilde{\mathbf{x}}[k]\tilde{\mathbf{x}}^T[k]\Phi^T\} + \underbrace{E\{\Phi\tilde{\mathbf{x}}[k]\mathbf{v}^T[k]\}}_{=0} + \underbrace{E\{\mathbf{v}[k]\tilde{\mathbf{x}}^T[k]\Phi^T\}}_{=0} \\
+ \underbrace{E\{\mathbf{v}[k]\mathbf{v}^T[k]\}}_{=R_1}
\end{aligned}$$

Stochastic difference equations

- Therefore, we obtain a dynamic equation for the covariance:

$$\begin{aligned} E\{\tilde{\mathbf{x}}[k+1]\tilde{\mathbf{x}}^T[k+1]\} &= E\{\Phi\tilde{\mathbf{x}}[k]\tilde{\mathbf{x}}^T[k]\Phi^T\} + R_1 \\ &= \Phi E\{\tilde{\mathbf{x}}[k]\tilde{\mathbf{x}}^T[k]\}\Phi^T + R_1 \end{aligned}$$

$$\Rightarrow P[k+1] = \Phi P[k]\Phi^T + R_1, \quad P[0] = R_0$$

- Consider the state auto-covariance for different values of k. For example, if :

$$\begin{aligned} \mathbf{r}_{\mathbf{xx}}(k+1,k) &= E\{\tilde{\mathbf{x}}[k+1]\tilde{\mathbf{x}}^T[k]\} = E\{(\Phi\tilde{\mathbf{x}}[k] + \mathbf{v}[k])\tilde{\mathbf{x}}^T[k]\} \\ &= \Phi E\{\tilde{\mathbf{x}}[k]\tilde{\mathbf{x}}^T[k]\} + E\{\mathbf{v}[k]\tilde{\mathbf{x}}^T[k]\} = \Phi P[k] + \mathbf{0} = \Phi P[k] \end{aligned}$$

- By repeating for any value of

$$\mathbf{r}_{\mathbf{xx}}(k+\tau,k) = \Phi^\tau P[k], \quad \tau \geq 0$$

Stochastic difference equations

- If the observation equation is

$$\mathbf{y}[k] = C\mathbf{x}[k]$$

it follows

$$\begin{cases} \mathbf{m}_{\mathbf{y}}[k] = C\mathbf{m}_{\mathbf{x}}[k] \\ \mathbf{r}_{\mathbf{yy}}(k+\tau, k) = C\mathbf{r}_{\mathbf{xx}}(k+\tau, k)C^T \\ \mathbf{r}_{\mathbf{yx}}(k+\tau, k) = C\mathbf{r}_{\mathbf{xx}}(k+\tau, k) \end{cases}$$

- What if the observation is given by

$$\mathbf{y}[k] = C\mathbf{x}[k] + \mathbf{w}[k]$$

where $\mathbf{w}[k]$ is white noise?

Example

- Consider the stochastic process:

$$x[k+1] = ax[k] + v[k]$$
$$y[k] = x[k] + e[k]$$

where $v[k]$ and $e[k]$ are white noise with covariance r_1 and r_2, respectively. The initial state at time instant k_0 has mean value m_0 and auto-covariance r_0. Find how the mean, covariances evolve over time.

Solution:

- For the mean value: $m[k+1] = am[k], \quad m[k_0] = m_0$

$$\Rightarrow m[k] = a^{k-k_0} m[k_0]$$

- For the covariance: $P[k+1] = a^2 P[k] + r_1, \quad P[k_0] = r_0$

$$\Rightarrow P[k] = a^{2(k-k_0)} r_0 + \frac{1 - a^{2(k-k_0)}}{1 - a^2} r_1$$

Example

- Hence, the auto-covariance is given by

$$r_{xx}(l,k) = \begin{cases} a^{l-k}P[k], & l \geq k \\ a^{k-l}P[k], & l < k \end{cases}$$

- Assume that the process is stable ($|a| < 1$) and it has been running a long period of time ($k \to \infty$). Then, it follows

$$m[k] \to 0, \qquad P[k] = \frac{1}{1-a^2}r_1, \qquad r_{xx}(l,k) = r_{xx}(\tau) = \frac{a^{|\tau|}}{1-a^2}r_1$$

The output covariance then becomes

$$r_{yy}(\tau) = \begin{cases} \frac{1}{1-a^2}r_1 + r_2, & \tau = 0 \\ \frac{a^{|\tau|}}{1-a^2}r_1, & \tau \neq 0 \end{cases}$$

Recall: state observers

- Recall the approach using the observer/estimator

$$\begin{aligned}\hat{\mathbf{x}}[k+1] &= \Phi\hat{\mathbf{x}}[k] + \Gamma u[k] + K\tilde{y}[k] \\ &= \Phi\hat{\mathbf{x}}[k] + \Gamma u[k] + K(y[k] - \hat{y}[k]) \\ &= \Phi\hat{\mathbf{x}}[k] + \Gamma u[k] + K(y[k] - C\hat{\mathbf{x}}[k]) \\ &= (\Phi - KC)\hat{\mathbf{x}}[k] + \Gamma u[k] + Ky[k]\end{aligned}$$

and the performance was studied by comparing the estimate with the real state:

$$\begin{aligned}\tilde{\mathbf{x}}[k+1] &= \mathbf{x}[k+1] - \hat{\mathbf{x}}[k+1] \\ &= \big(\Phi\mathbf{x}[k] + \Gamma u[k]\big) - \big((\Phi - KC)\hat{\mathbf{x}}[k] + \Gamma u[k] + KC\mathbf{x}[k]\big) \\ &= (\Phi - KC)(\mathbf{x}[k] - \hat{\mathbf{x}}[k]) \\ &= \underbrace{(\Phi - KC)}_{\Phi_{\text{o}}}\tilde{\mathbf{x}}[k] = \Phi_{\text{o}}\tilde{\mathbf{x}}[k]\end{aligned}$$

- Matrix K was chosen such that the eigenvalues of Φ_{o} are at desired places in the complex plane.

Learning outcomes

By the end of *this* lecture, you should be able to:

- Explain different types of disturbances.
- Understand the characteristics and effect of noise in dynamical processes
- Compute the mean and covariance matrices of dynamical processes