Aalto University
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MS-E2114 Investment Science
Lecture 11: Options pricing in continuous time

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Overview

Stochastic processes

Black-Scholes equation

Application of Black-Scholes equation

Synthetic and exotic options

Previous lecture

- In the two previous lectures, we priced options in the binomial lattice
  - The asset price dynamics were modelled using the binomial lattice formulas in Lecture 9
  - Lecture 10 presented recursive formulas for calculating the arbitrage-free price of a derivative
- In this lecture, we determine the prices of options using a continuous-time model for the price of the underlying asset
  - The price dynamics is modelled with stochastic processes
  - Arbitrage-free prices of derivatives are expressed with stochastic differential equations

Overview

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Random walk

- Let Δt be the length of the time period in years
- Consider the stochastic process

$$\left.\begin{array}{rl} z(t_{k+1}) & = z(t_k) + \varepsilon(t_k)\sqrt{\Delta t} \\ t_{k+1} & = t_k + \Delta t \end{array}\right. , \quad k = 0, 1, \ldots, N,$$

  where $\varepsilon(t_i)$ and $\varepsilon(t_j)$ $(i \neq j)$ are independent and $\varepsilon(t_i) \sim \mathcal{N}(0, 1)$
  - Independent $\Rightarrow \mathrm{Cov}[\varepsilon(t_i), \varepsilon(t_j)] = 0$ when $i \neq j$
  - $\varepsilon(t_i)$ are normally distributed with mean 0 and variance 1
- $\varepsilon(t_i)$ is scaled with $\sqrt{\Delta t}$ in order to
  1. Make variance of the process depend linearly on Δt
  2. Scale variance of the process so that when $t_k - t_j = 1$, then $\mathrm{Var}[z(t_k) - z(t_j)] = 1$
- This process is a **random walk**

Random walk

- By using the definition of the random walk recursively, we get for any $j < k$ that

$$z(t_k) = z(t_j) + \sum_{i=j}^{k-1} \varepsilon(t_i)\sqrt{\Delta t}$$

- Thus, by rearranging terms, for any $j < k$, it holds that

$$\Delta z_{t_j \to t_k} = z(t_k) - z(t_j) = \sum_{i=j}^{k-1} \varepsilon(t_i)\sqrt{\Delta t}$$

Random walk

- As a result, $\Delta z_{t_j \to t_k} = z(t_k) - z(t_j)$ is normally distributed and

$$\mathbb{E}\left[z(t_k) - z(t_j)\right] = \sum_{i=j}^{k-1} \mathbb{E}[\varepsilon(t_i)]\sqrt{\Delta t} = 0$$

$$\mathrm{Var}\left[z(t_k) - z(t_j)\right] = \mathbb{E}\left[\left(\sum_{i=j}^{k-1} \varepsilon(t_i)\sqrt{\Delta t}\right)^2\right] = \mathbb{E}\left[\sum_{i=j}^{k-1} \varepsilon(t_i)^2 \Delta t\right]$$

$$= (k-j) \cdot 1 \cdot \Delta t = t_k - t_j$$

Wiener process / Brownian motion

- By rearranging the terms and using the definition of t_{k+1}, the random walk can be written as

$$\Delta z_{t_k} = z(t_k + \Delta t) - z(t_k) = \varepsilon(t_k)\sqrt{\Delta t}$$

- In the limit $\Delta t \to 0$, the random walk becomes the **Wiener process** $z(t)$ defined by the equation

$$dz = \varepsilon(t)\sqrt{dt},$$

where $\varepsilon(t) \sim \mathcal{N}(0, 1)$

- This process is also known as **Brownian motion**

Wiener process / Brownian motion

- Wiener process $z(t)$ is characterized by the following properties:

1. For any $s < t$, $z(t) - z(s)$ is normally distributed such that
   - $\mathbb{E}[z(t) - z(s)] = 0$
   - $\mathrm{Var}[z(t) - z(s)] = t - s$
2. For any $0 \leq t_1 < t_2 \leq t_3 < t_4$, differences $z(t_2) - z(t_1)$ and $z(t_4) - z(t_3)$ are uncorrelated
3. $z(t_0) = 0$ with probability 1

White noise

- Recall that $dz = \varepsilon(t)\sqrt{dt}$
- Thus, the derivative of $z(t)$ is $dz/dt = \varepsilon(t)/\sqrt{dt}$
- The derivative dz/dt is a normally distributed random variable with variance of $1/dt$, which is infinite
- $z(t)$ is not differentiable anywhere, because dz/dt is not a real number (instead, it is a random variable with infinite variance)
- This can be also verified as follows:

$$\mathbb{E}\left[\left(\frac{z(s)-z(t)}{s-t}\right)^2\right] = \frac{s-t}{(s-t)^2} = \frac{1}{s-t} \to \infty$$

  when $s \to t$
- The term dz/dt is called **white noise**

Generalized Wiener process

- **The generalized Wiener process** (or alternatively, Brownian motion with drift) is an extension of the Wiener process which has a term for deterministic shift

$$dx(t) = adt + bdz,$$

where $x(t)$ is a stochastic process, a and b are constants, dt is differential in time, and z is a Wiener process

- Integration yields

$$x(t) = x(0) + at + bz(t)$$

Itô process

- **Itô process** is an extension of the generalized Wiener process such that the deterministic and stochastic shifts are functions on x and t, defined through

$$dx(t) = a(x,t)dt + b(x,t)dz,$$

where $a(x,t)$ and $b(x,t)$ are integrable functions

Itô's lemma

Theorem

*(**Itô's lemma**) Suppose that the random process x is defined by the Itô process*

$$dx(t) = a(x,t)dt + b(x,t)dz, \tag{1}$$

where $z(t)$ is a standard Wiener process. If the process $y(t)$ is defined by $y(t) = F(x,t)$, then $y(t)$ satisfies the Itô equation

$$dy(t) = \left(\frac{\partial F}{\partial x}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}b^2\right)dt + \frac{\partial F}{\partial x}bdz,$$

where z is the Wiener process in Equation (1).

Simplified proof from Wikipedia

Informal derivation [edit]

A formal proof of the lemma relies on taking the limit of a sequence of random variables. This approach is not presented here since it involves a number of technical details. Instead, we give a sketch of how one can derive Itô's lemma by expanding a Taylor series and applying the rules of stochastic calculus.

Assume X_t is an Itô drift-diffusion process that satisfies the stochastic differential equation

$$dX_t = \mu_t\, dt + \sigma_t\, dB_t,$$

where B_t is a Wiener process. If $f(t,x)$ is a twice-differentiable scalar function, its expansion in a Taylor series is

$$df = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial x}\, dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\, dx^2 + \cdots.$$

Substituting X_t for x and therefore $\mu_t\, dt + \sigma_t\, dB_t$ for dx gives

$$df = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial x}(\mu_t\, dt + \sigma_t\, dB_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(\mu_t^2\, dt^2 + 2\mu_t\sigma_t\, dt\, dB_t + \sigma_t^2\, dB_t^2\right) + \cdots.$$

In the limit $dt \to 0$, the terms dt^2 and $dt\, dB_t$ tend to zero faster than dB^2, which is $O(dt)$. Setting the dt^2 and $dt\, dB_t$ terms to zero, substituting dt for dB^2 (due to the quadratic variance of a Wiener process), and collecting the dt and dB terms, we obtain

$$df = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x}\, dB_t$$

as required.

Itô's lemma

Proof: By definition, we have:

$$dy = dF = \lim_{\Delta t \to 0} \Delta y$$

Given the initial values of t and $x(t)$ (and thus $y(t) = F(x, t)$) and a finite time difference Δt, we can approximate the resulting change in $y(t)$ as follows:

$$\Delta z \approx \varepsilon(t)\sqrt{\Delta t}$$

$$\Delta x \approx a(x, t)\Delta t + b(x, t)\Delta z$$

$$\Delta y \approx \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\Delta x)^2$$

where the term with $(\Delta x)^2$ is needed because Δz depends on $\sqrt{\Delta t}$. The approximation becomes increasingly accurate as Δt gets smaller.

Itô's lemma

Proof (cont'd): Substituting Δx into Δy, we get:

$$\Delta y \approx \frac{\partial F}{\partial x}(a(x,t)\Delta t + b(x,t)\Delta z) + \frac{\partial F}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(a(x,t)\Delta t + b(x,t)\Delta z)^2$$

Now we have for the last term:

$$\begin{aligned}
&(a(x,t)\Delta t + b(x,t)\Delta z)^2 = \\
&a(x,t)^2\Delta t^2 + 2a(x,t)\Delta t b(x,t)\Delta z + b(x,t)^2\Delta z^2 = \\
&a(x,t)^2\Delta t^2 + 2a(x,t)\Delta t b(x,t)\varepsilon(t)\sqrt{\Delta t} + b(x,t)^2(\varepsilon(t)\sqrt{\Delta t})^2 = \\
&a(x,t)^2\Delta t^2 + 2a(x,t)b(x,t)\varepsilon(t)\Delta t^{3/2} + b(x,t)^2\varepsilon(t)^2\Delta t
\end{aligned}$$

We keep only the term for Δt, as others will be infinitely smaller than this term as Δt goes to zero.

Itô's lemma

Proof (cont'd): By substituting $(a(x,t)\Delta t + b(x,t)\Delta z)^2$ with $b(x,t)^2\varepsilon(t)^2\Delta t$, we now have:

$$\begin{aligned}\Delta y \approx{}& \frac{\partial F}{\partial x}(a(x,t)\Delta t + b(x,t)\Delta z)\\ &+ \frac{\partial F}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} b(x,t)^2\varepsilon(t)^2\Delta t\end{aligned}$$

Since $\varepsilon(t)$ is a standard normal distribution, $\varepsilon(t)^2$ follows the χ^2-distribution with one degree of freedom[1]. This distribution has an expected value of 1 and variance of 2.

[1] $\chi^2(k)$ distribution with k degrees of freedom is the sum of k independent random variables which follow the standard normal distribution

Itô's lemma

Proof (cont'd): Let us denote $E = \varepsilon(t)^2 - 1$ so that $\mathbb{E}[E] = 0$ and we get:

$$\begin{aligned}\Delta y \approx & \frac{\partial F}{\partial x}(a(x,t)\Delta t + b(x,t)\Delta z) \\ & + \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} b(x,t)^2\right)\Delta t \\ & + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} b(x,t)^2 E\Delta t\end{aligned}$$

We can note now that $E\Delta t$ has a variance of $2\Delta t^2$ while Δz has a variance of Δt. Thus, the variance of $E\Delta t$ is infinitely smaller than the variance of Δz as Δt goes to zero, and hence the term can be neglected, as it also has been set to have zero expected value.

Itô's lemma

Proof (cont'd): By rearranging the terms we get:

$$\Delta y \approx \left(\frac{\partial F}{\partial x} a(x,t) + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} b(x,t)^2 \right) \Delta t + \frac{\partial F}{\partial x} b(x,t) \Delta z$$

By taking the limit value $dy = \lim_{\Delta t \to 0} \Delta y$, we get Itô's lemma:

$$dy = \left(\frac{\partial F}{\partial x} a(x,t) + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} b(x,t)^2 \right) dt + \frac{\partial F}{\partial x} b(x,t) dz \quad \square$$

Itô's lemma example: Geometric Brownian motion

- For example, consider the **geometric Brownian motion**, which is an Itô process, defined through

$$dx(t) = axdt + bxdz$$

- Geometric Brownian motion is obtained if one assumes that the natural logarithm of the price process is a *generalized Wiener process*
- The multiplicative model

$$\ln S(k+1) - \ln S(k) = w(k), \quad w(k) \sim \mathcal{N}(\nu, \sigma^2)$$

  has the continuous time counterpart

$$d \ln S(t) = \nu dt + \sigma dz$$

- That is, $\ln S(t)$ is a generalized Wiener process

Itô's lemma example: Geometric Brownian motion

- Now, define $x(t) = \ln S(t)$ to get

$$dx(t) = d \ln S(t) = \nu dt + \sigma dz$$

- By Itô's lemma, the random process $y(t) = F(x, t)$ $= S(t) = e^{\ln S(t)} = e^x$ is an Itô process that satisfies

$$\begin{aligned} dy &= \left(\frac{\partial e^x}{\partial x} \nu + \frac{\partial e^x}{\partial t} + \frac{1}{2} \frac{\partial^2 e^x}{\partial x^2} \sigma^2 \right) dt + \frac{\partial e^x}{\partial x} \sigma dz \\ &= \left(\nu + \frac{1}{2} \sigma^2 \right) e^x dt + e^x \sigma dz \\ \Rightarrow \frac{dS(t)}{S(t)} &= \left(\nu + \frac{1}{2} \sigma^2 \right) dt + \sigma dz \end{aligned}$$

Itô's lemma example: Geometric Brownian motion

- Thus, we have

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dz$$
$$\Leftrightarrow dS(t) = \mu S(t)dt + \sigma S(t)dz$$

  where $\mu = \nu + \frac{1}{2}\sigma^2$
- ν and σ are the expectation and volatility of $\ln S(t)$, which is a generalized Wiener process

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Black-Scholes equation

- We have priced options in binomial lattices
- Analogous results can be derived by using stochastic differential equations
- Assume that
  1. Price of underlying asset S follows the geometric Brownian motion

     $$dS = \mu S dt + \sigma S dz, \tag{2}$$

     where z is a Wiener process
  2. Value of the risk-free asset B satisfies the differential equation

     $$dB = rBdt$$

  3. $f(S, t)$ is the value of a derivative security of the underlying asset S at time t

Black-Scholes equation

Theorem

*(**Black-Scholes equation**) Suppose that the price S of a security is governed by the geometric Brownian motion* (2) *and the interest rate is r. A derivative of this security has a price $f(S,t)$, which satisfies the partial differential equation*

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf.$$

Black-Scholes equation

Proof: Apply Itô's lemma to $f(S, t)$:

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S}\sigma S dz$$

Form a *replicating portfolio G* from the underlying asset and the risk-free asset, i.e., invest x_t in the underlying asset and y_t in the risk-free asset. The value differential of this portfolio is

$$dG = x_t dS + y_t dB = x_t(\mu S dt + \sigma S dz) + y_t rB dt$$
$$\Rightarrow dG = (x_t \mu S + y_t rB)dt + x_t \sigma S dz$$

Black-Scholes equation

To construct a replicating portfolio, the amounts x_t and y_t must be selected so that

1. the coefficient multiplying dz is the same in the derivative and the replicating portfolio, and
2. the price of the replicating portfolio G must be the price of the derivative $f(S, t)$,

Finally, the no-arbitrage principle requires that

3. the coefficient multiplying dt is the same in the derivative and the replicating portfolio

as otherwise there would be arbitrage opportunities.

Black-Scholes equation

Thus, we first have

$$x_t \sigma S = \frac{\partial f}{\partial S} \sigma S \Rightarrow x_t = \frac{\partial f}{\partial S}$$

and then secondly,

$$\begin{aligned} G = x_t S + y_t B = \frac{\partial f}{\partial S} S + y_t B = f(S,t) \\ \Rightarrow y_t = \frac{1}{B}\left[f(S,t) - S\frac{\partial f}{\partial S}\right] \end{aligned}$$

Black-Scholes equation

Finally, the coefficients multiplying dt must be the same, i.e.,

$$\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = x_t \mu S + y_t r B$$

We substitute x_t and y_t to the above to obtain

$$\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = \frac{\partial f}{\partial S}\mu S + \frac{1}{B}\left[f(S,t) - S\frac{\partial f}{\partial S}\right] rB$$

$$\Rightarrow \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf. \qquad \square$$

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Application of Black-Scholes equation

- In general, the Black-Scholes equation does not have a closed form solution
- Certain special cases satisfy the equation

1. A derivative whose value is the same as that of the underlying asset (i.e., $f(S,t) = S$)

$$\frac{\partial f}{\partial t} = 0 \wedge \frac{\partial f}{\partial S} = 1 \wedge \frac{\partial^2 f}{\partial S^2} = 0$$
$$\Rightarrow 0 + 1rS + 0 = rS$$

2. Risk-free asset as a derivative instrument (i.e., $f(S,t) = e^{rt}$)

$$\frac{\partial f}{\partial t} = re^{rt} \wedge \frac{\partial f}{\partial S} = 0 \wedge \frac{\partial^2 f}{\partial S^2} = 0$$
$$\Rightarrow re^{rt} + 0 + 0 = re^{rt}$$

Application of Black-Scholes equation

- How to use Black-Scholes equation?

1. Pick or guess $f(S, t)$: If it does not satisfy the BS-equation, there are arbitrage opportunities
   $\Rightarrow$ Not a possible price process under the no-arbitrage principle
2. Give boundary conditions (e.g., value of option on expiry) and solve the resulting partial differential equation
   - For example, the boundary conditions of a European call option are

$$C(0, t) = 0, \quad C(S, T) = \max(S - K, 0)$$

   - An American call can be exercised before expiry
     $\Rightarrow$ The value of the option satisfies

$$C(S, t) \geq \max(S - K, 0)$$

Application of Black-Scholes equation

- Example: Consider an American call with unlimited time to expiry (perpetual call)
  - Boundary conditions

$$
\begin{aligned}
C(S,t) &\geq \max\{S-K,0\} \\
C(S,t) &\leq S
\end{aligned}
$$

  - Solution $C(S,t) = S$ satisfies these conditions
  - Interpretation: The price of the underlying asset will in the long run increase so much that the strike price of the option becomes irrelevant

  $\Rightarrow$ The option and the stock have the same value
- The Black-Scholes equation has an analytical solution for a European call on a non-dividend-paying stock

Call option formula

Theorem

*(**Black-Scholes call option formula**) Consider a European call option with strike price K and expiration time T. If the underlying stock pays no dividends during the time $[0, T]$ and if interest is compounded continuously at a constant rate r, the Black-Scholes solution is $f(S,t) = C(S,t)$, defined by*

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \quad \textit{where}$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

and where $N(x)$ denotes the standard cumulative normal probability distribution $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy$

Call option formula example

- Let us revisit the example in Lecture 10 in which the stock price is 80 € and volatility 0.40
- Consider a European call which expires in four months with the strike price 85 €
- What is the price of the option, when the risk-free rate is 8% and no dividends are paid?

Call option formula example

- We have the following parameters for the call option formula:

$$S = 80, \quad K = 85 \quad r = 0.08, \quad \sigma = 0.40$$

$$\begin{aligned} d_1 =& \frac{\ln(80/85) + (0.08 + 0.40^2/2)(4/12)}{0.40\sqrt{4/12}} = 0.0316 \\ d_2 =& d_1 - 0.40\sqrt{4/12} = -0.2625 \end{aligned}$$

$$\begin{aligned} N(d_1) =& 0.4874 \\ N(d_2) =& 0.3965 \end{aligned}$$

$$C(S, t) = 80 \cdot 0.4874 - 85 \cdot e^{-0.08(4/12)} \cdot 0.3965 = 6.18$$

- The value of the call is 6.18 €, which is slightly less than the price we obtained from the binomial lattice (6.40 €)

Put option formula

- Pricing formula for a European put option $P(S, t)$ on a non-dividend-paying stock can be obtained from the put-call parity (see Lecture 9):

$$C(S,t) - P(S,t) + d(t,T)K = S$$

- Thus, with $d(t, T) = e^{-r(T-t)}$ we have:

$$\begin{aligned} & P(S,t) = C(S,t) - S + d(t,T)K \\ \Leftrightarrow\ & P(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) - S + Ke^{-r(T-t)} \\ \Leftrightarrow\ & P(S,t) = S(N(d_1) - 1) - Ke^{-r(T-t)}(N(d_2) - 1) \end{aligned}$$

Delta Δ

- **Delta** Δ measures how sensitive the value of the derivative (e.g., an option) is with respect to changes in the price of the underlying asset

$$\Delta = \frac{\partial f(S,t)}{\partial S} \approx \frac{\Delta f(S,t)}{\Delta S}$$

- The Black-Scholes call option formula implies that the delta of European call on a non-dividend-paying stock is

$$\Delta = N(d_1)$$
$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

- Note that delta depends on S and t

Delta hedging

- You can delta-hedge any derivative by
  1. Buying (or, selling) the derivative
  2. Selling (or, buying) Δ shares of the underlying asset
- The combined portfolio of the derivative and the shares is **delta-neutral**, i.e., immune to **very small** changes in the value of the underlying asset
- For example, for a long position in a derivative:

$$\frac{\partial}{\partial S}\left(f(S,t) - \Delta S\right) = \Delta - \Delta = 0$$

Delta hedging

- Recall that delta depends on S and t
  - $\Rightarrow$ In theory, after every very small change in S and t, the portfolio must be rebalanced
  - $\Rightarrow$ *Continuous* hedging / rebalancing
- Impossible in practice, but maybe daily rebalancing could be possible
- In reality, the hedging requirements of the Black-Scholes setting can only be roughly approximated

Gamma Γ and theta Θ

- **Gamma** is the second derivative of the value of the derivative with respect to the price of the underlying:

$$\Gamma = \frac{\partial^2 f(S,t)}{\partial S^2}$$

- **Theta** Θ is the change in the value of a derivative with respect to time

$$\Theta = \frac{\partial f(S,t)}{\partial t}$$

  - Over time, the value of the option approaches the value that is has on the expiry date
  - Time value diminishes $\Rightarrow$ Θ is negative for options

Approximation using the greeks

- The approximation for the change in the value of a derivative is

$$\delta f \approx \Delta \cdot \delta S + \frac{1}{2}\Gamma \cdot (\delta S)^2 + \Theta \cdot \delta t$$

- Note that this parallels Itô's lemma:

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S}\sigma S dz$$

Approximation using the greeks

- Let $S = 43$ €, volatility $\sigma = 0.20$, and risk-free rate $r = 0.10$
- Consider a European call which expires in $T - t = 6$ months with strike price $K = 40$ €
- The option price at present using the Black-Scholes call option formula is

$$\begin{aligned} d_1 &= 0.936, \quad d_2 = 0.794 \\ \Rightarrow C &= 5.56 \end{aligned}$$

- The option price using the call option formula in two weeks' time after the stock has moved up 1 euro to $S = 44$ is

$$\begin{aligned} d_1 &= 1.109, \quad d_2 = 0.973 \\ \Rightarrow C &= 6.232 \end{aligned}$$

Approximation using the greeks

- Delta, gamma and theta are

$$\Delta = N(d_1) = 0.825, \quad \Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T-t}} = 0.143$$

$$\Theta = -\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-rT}N(d_2) = -6.127$$

- As the price of stock rises by one euro in two weeks, the value of the option becomes

$$C' \approx C + \delta C = 5.56 + \Delta \cdot 1 + \frac{1}{2}\Gamma \cdot 1^2 + \Theta \cdot \frac{2}{52} = 6.22$$

- Not exactly accurate result (6.232), but very close

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Synthetic options

- A return identical to an option (or any other derivative) can be obtained from a (replicating) portfolio of the underlying asset and the risk-free asset
  - This replicating portfolio value tracks the value of the derivative, but the portfolio must be continuously rebalanced
  - Sometimes called a **replicating trading strategy** for the derivative
- A **synthetic option** (= continuously rebalanced replicating portfolio) can be constructed as follows:
  1. Define the value C of an option (e.g., using binomial lattice or Black-Scholes)
  2. Invest ΔS in the underlying asset and the rest $C - \Delta S$ at the risk-free rate
  3. Rebalance the portfolio frequently so that the portfolio has the required Δ

Synthetic option example

Weeks remaining	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
XON price	35.50	34.63	33.75	34.75	33.75	33.00	33.88	34.50	33.75	34.75	34.38	35.13	36.00	37.00	36.88	38.75	37.88	38.00	38.63	38.50	37.50
Call price	2.62	1.96	1.40	1.89	1.25	0.85	1.17	1.42	0.96	1.40	1.10	1.44	1.94	2.65	2.44	4.10	3.17	3.21	3.76	3.57	2.50
Delta	0.701	0.615	0.515	0.618	0.498	0.397	0.494	0.565	0.456	0.583	0.522	0.624	0.743	0.860	0.858	0.979	0.961	0.980	0.998	1.000	
Portfolio value	2.62	1.96	1.39	1.87	1.22	0.81	1.14	1.41	0.96	1.38	1.13	1.49	2.00	2.69	2.53	4.08	3.16	3.22	3.76	3.57	2.50
Stock portfolio	24.89	21.28	17.37	21.47	16.79	13.09	16.74	19.48	15.39	20.27	17.94	21.92	26.74	31.80	31.65	37.92	36.39	37.25	38.56	38.50	
Bond portfolio	-22.27	-19.32	-15.98	-19.59	-15.58	-12.28	-15.60	-18.07	-14.43	-18.89	-16.81	-20.43	-24.75	-29.11	-29.12	-33.84	-33.23	-34.03	-34.79	-34.93	

- A synthetic call option on Exxon stock with a strike price of 35 € and with 20 weeks to expiration is constructed by buying the stock and selling the risk-free asset at 10%
- The portfolio is adjusted each week based on the value of delta Δ at that time

Exotic options

- Some options are more complicated than the American and European options we have treated
  1. **Bermudan option**: Early exercise possible on specific dates before expiry
  2. **Compound option**: An option on another option
  3. **Chooser option**: The holder specifies after a given time whether the option is a call or a put
  4. **CAP**: Automatically exercised if the price of underlying asset exceeds the specified given limit
     - E.g., if a 20 € CAP-call option has strike 60 €, it will be automatically exercised when the stock price exceeds 80 €
  5. **Knockout option**: Expires if the price of underlying asset reaches the specified level
     - Call expires if price of underlying asset below knockout level ("down and out")
     - Put expires if price of underlying asset above knockout level ("up and out")

Exotic options

6. **Discontinuous option**: Profit depends discontinuously on the price of the underlying asset
   - E.g., return 100 € if the price of the underlying asset is above the strike price at expiry; otherwise 0
7. **Digital option**: Has a payoff 1 € if the corresponding European option is in the money and 0 € otherwise
8. **Lookback option**: Exercise price is determined by the minimum and maximum values obtained by the underlying asset during the period of the option
   - Put option exercise price = highest value of the underlying asset during the period
   - Call exercise price = lowest value of the underlying asset during the option period

   $\Rightarrow$ Lookback options have always a positive value at expiry

   $\Rightarrow$ They are expensive
9. **Asian option**: Profit depends on the average underlying asset price S_{avg} during the period of the option

Overview

Stochastic processes

Black-Scholes equation

Application of Black-Scholes equation

Synthetic and exotic options