ELEC-E8101 Digital and Optimal Control

Exercise 10 - Solutions

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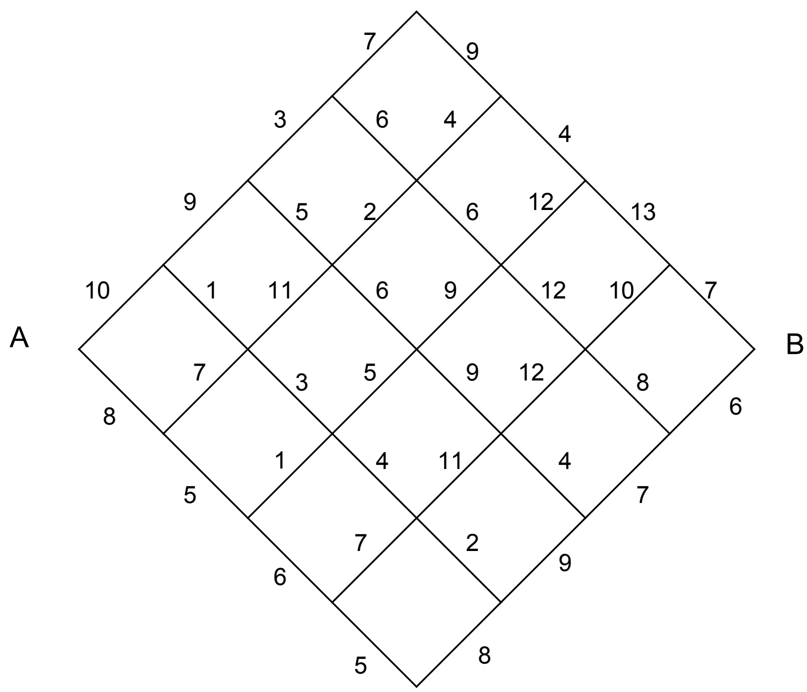
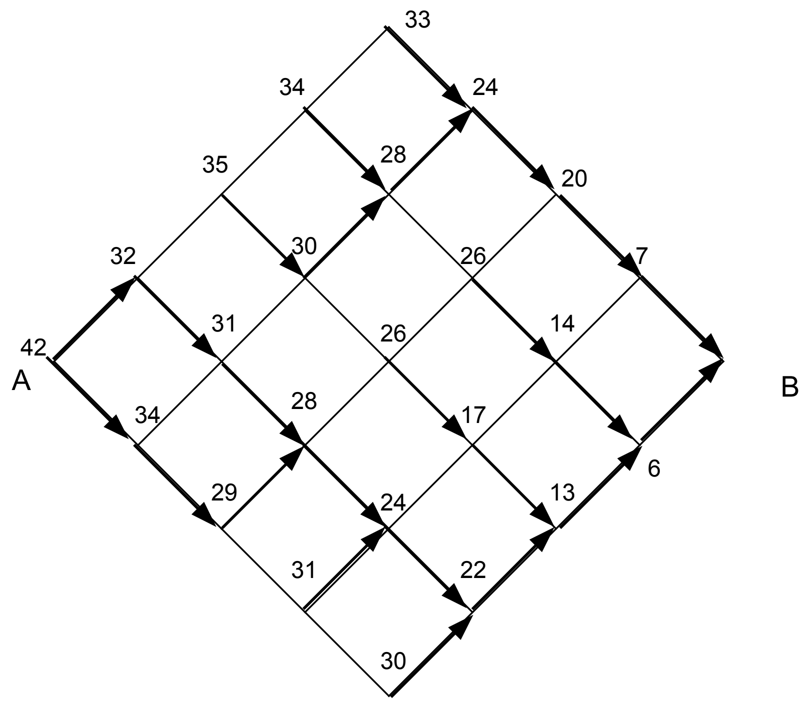
1. In the diagram below, the costs for moving from one state to the other have been marked. Time flows from left to right, and the controls have been restricted to two alternatives: "up right" or "down right". Calculate the optimum cost and path from point A to point B.



Solution. Use dynamic programming and the principle of optimality. Start from the end point B and mark to each state (node) the optimal cost and optimal control.

Example: At time $N-1$ there exist only two possible controls, and the optimal costs in the corresponding nodes are 7 and 6. Then, go to time $N-2$. Now there are three possible states, which are considered separately. The optimal cost from each state is obtained as the sum of the "first" control and cost of the optimal control of the next state (calculated previously). Choose the smallest cost, remember it and mark also the optimal control in that state. For example, in the "state in the middle" there are two possible controls with the costs: 10 and 8. The optimal cost to state B is then the minimum of the sums $(10 + 7)$ and $(8 + 6)$. The optimal control is "down right" and optimal cost is 14. See the solution diagram below. The control has been marked with an arrow and the cost next to the state.

By proceeding in a similar way the starting point in the diagram is eventually reached. Then, the optimal cost is already known, and by following the controls (arrays) the optimal route is found out. By a coincide, two routes can be chosen in the beginning leading to the optimal cost 42.



An alternative solution method would be to calculate all possible routes and then choose the one with the minimum cost. However, this would need a lot of work, and by using dynamic programming the calculation work can be reduced considerably. To demonstrate this point consider a similar square topology but with n "vertices" (in the actual problem there were 4 vertices). By considering all possible routes we need $(2n)!/(n!)^2$ calculations and by using dynamic programming we need $(n+1)^2 - 1$ calculations.

Number of vertices	3	4	5	6
All paths	20	70	252	924
Dynamic programming	15	24	35	48

2. The difference equation describing a system is

$$x_{k+1} = ax_k + bu_k, \quad a, b \text{ constants.}$$

By using dynamic programming calculate the controls $u[k]$, $k = 1, 2, 3$, which minimize the cost

$$J = \sum_{k=1}^{3} \left[x_k^2 + ru_k^2\right], \quad r \text{ constant.}$$

when

a) x_4 is free.

b) $x_4 = 0$.

Solution.

a) We use dynamic programming and hence we start reverse in time.
Step 3: First, we compute the cost-to-go and action at step 3:

$$V_3(x_3) = \min_{u_3} \left\{x_3^2 + ru_3^2\right\}.$$

Since x_4 is free, any action we apply it only adds on the cost. Hence, at this step $u_3 = 0$. Therefore, $V_3(x_3) = x_3^2$.
Step 2: Next, we compute the cost-to-go and action at step 2:

$$\begin{aligned} V_2(x_2) &= \min_{u_2} \left\{x_2^2 + ru_2^2 + V_3(x_3)\right\} \\ &= \min_{u_2} \left\{x_2^2 + ru_2^2 + V_3(ax_2 + bu_2)\right\} \\ &= \min_{u_2} \left\{x_2^2 + ru_2^2 + (ax_2 + bu_2)^2\right\} \\ &= \min_{u_2} \left\{x_2^2 + ru_2^2 + a^2x_2^2 + 2abx_2u_2 + b^2u_2^2\right\} \end{aligned}$$

Take derivative w.r.t. u_2:

$$\left.\begin{aligned} \frac{dV_2(x_2)}{du_2} &= 2ru_2 + 2abx_2 + 2b^2u_2 \\ \text{Minimization: } \frac{dV_2(x_2)}{du_2} &= 0 \end{aligned}\right\} \Rightarrow 2ru_2 + 2abx_2 + 2b^2u_2 = 0 \Rightarrow u_2 = -\frac{ab}{b^2 + r}x_2$$

We substitute u_2 into cost-to-go $V_2(x_2)$:

$$\begin{aligned} V_2(x_2) &= x_2^2 + r\left(\frac{ab}{b^2 + r}\right)^2 x_2^2 + a^2x_2^2 - 2abx_2\frac{ab}{b^2 + r}x_2 + b^2\left(\frac{ab}{b^2 + r}\right)^2 x_2^2 \\ &= \left(1 + \frac{ra^2b^2}{(b^2 + r)^2} + a^2 - \frac{2a^2b^2}{b^2 + r} + \frac{a^2b^4}{(b^2 + r)^2}\right)x_2^2 \\ &= \left(1 + \frac{ra^2b^2 + a^2(b^2 + r)^2 - 2a^2b^2(b^2 + r) + a^2b^4}{(b^2 + r)^2}\right)x_2^2 \\ &= \underbrace{\left(1 + \frac{a^2r}{b^2 + r}\right)}_{\triangleq c} x_2^2 \end{aligned}$$

Step 1: Next, we compute the cost-to-go and action at step 1:

$$\begin{aligned} V_1(x_1) &= \min_{u_1} \left\{ x_1^2 + r u_1^2 + V_2(x_2) \right\} \\ &= \min_{u_1} \left\{ x_1^2 + r u_1^2 + V_2(a x_1 + b u_1) \right\} \\ &= \min_{u_1} \left\{ x_1^2 + r u_1^2 + c(a x_1 + b u_1)^2 \right\} \end{aligned}$$

Take derivative w.r.t. u_1:

$$\left.\begin{aligned} \frac{dV_1(x_1)}{du_1} &= 2 r u_1 + 2bc(a x_1 + b u_1) \\ \text{Minimization: } \frac{dV_2(x_1)}{du_1} &= 0 \end{aligned}\right\} \Rightarrow 2 r u_1 + 2bc(a x_1 + b u_1) = 0 \Rightarrow u_1 = -\frac{abc}{b^2 c + r} x_1$$

We can substitute c in u_1:

$$u_1 = -\frac{ab(b^2 + r + a^2 r)}{r(b^2 + r) + b^2(b^2 + r + a^2 r)} x_1$$

Hence the optimal controls are:

$$\begin{aligned} u_1 &= -\frac{ab(b^2 + r + a^2 r)}{r(b^2 + r) + b^2(b^2 + r + a^2 r)} x_1 \\ u_2 &= -\frac{ab}{b^2 + r} x_2 \\ u_3 &= 0 \end{aligned}$$

b) Now the final state is fixed to $x_4 = 0$, and the direct solution formulas cannot be used, dynamic programming can still be used.
Step 3: From the end condition:

$$x_4 = a x_3 + b u_3 = 0 \Rightarrow u_3 = -\frac{a}{b} x_3.$$

Therefore,

$$V_3(x_3) = \min_{u_3} \left\{ x_3^2 + r u_3^2 \right\} = x_3^2 + r\left(-\frac{a}{b} x_3\right)^2 = \underbrace{\left(1 + \frac{r a^2}{b^2}\right)}_{\triangleq B} x_3^2$$

Step 2: Next, we compute the cost-to-go and action at step 2:

$$\begin{aligned} V_2(x_2) &= \min_{u_2} \left\{ x_2^2 + r u_2^2 + V_3(x_3) \right\} \\ &= \min_{u_2} \left\{ x_2^2 + r u_2^2 + V_3(a x_2 + b u_2) \right\} \\ &= \min_{u_2} \left\{ x_2^2 + r u_2^2 + B(a x_2 + b u_2)^2 \right\} \end{aligned}$$

Take derivative w.r.t. u_2:

$$\left.\begin{aligned} \frac{dV_2(x_2)}{du_2} &= 2ru_2 + 2bB(ax_2 + bu_2) \\ \text{Minimization: } \frac{dV_2(x_2)}{du_2} &= 0 \end{aligned}\right\} \Rightarrow 2ru_2 + 2bB(ax_2 + bu_2) = 0 \Rightarrow u_2 = -\underbrace{\frac{abB}{b^2B + r}}_{\triangleq C} x_2$$

We substitute u_2 into cost-to-go $V_2(x_2)$:

$$V_2(x_2) = \ldots = Dx_2^2,$$

where D is a constant.

Step 1: Next, we compute the cost-to-go and action at step 1:

$$\begin{aligned} V_1(x_1) &= \min_{u_1} \left\{x_1^2 + ru_1^2 + V_2(x_2)\right\} \\ &= \min_{u_1} \left\{x_1^2 + ru_1^2 + V_2(ax_1 + bu_1)\right\} \\ &= \min_{u_1} \left\{x_1^2 + ru_1^2 + D(ax_1 + bu_1)^2\right\} \end{aligned}$$

Take derivative w.r.t. u_1:

$$\left.\begin{aligned} \frac{dV_1(x_1)}{du_1} &= 2ru_1 + 2bD(ax_1 + bu_1) \\ \text{Minimization: } \frac{dV_2(x_1)}{du_1} &= 0 \end{aligned}\right\} \Rightarrow 2ru_1 + 2bD(ax_1 + bu_1) = 0 \Rightarrow u_1 = -\frac{abD}{b^2D + r} x_1$$

Hence the optimal controls are:

$$\begin{aligned} u_1 &= -\frac{abD}{b^2D + r} x_1 \\ u_2 &= -\frac{abB}{b^2B + r} x_2 \\ u_3 &= -\frac{a}{b} x_3. \end{aligned}$$