ELEC E-5440 Statistical Signal Processing.
Homework set #2
due December 30, 2022

I. MAP Estimator

Suppose that Θ is a random parameter and that, given $\Theta = \theta$, the real random variable X is distributed according to the Lévi density:

$$f(x|\theta) = \sqrt{\frac{\theta}{2\pi}}\, x^{-\frac{3}{2}} \exp\Big(-\frac{\theta}{2x}\Big), \quad x > 0. \tag{1}$$

Suppose further that Θ has the prior density:

$$p(\theta) = \frac{1}{\theta}, \quad 1 < \theta < e \tag{2}$$

Given n statistically independent measurements $x_1, \dots x_n$, $(n > 2)$, find the maximum a posteriori estimate of θ.

II. MS and MAP Estimators

Consider a communication scenario where a random PAM (Pulse Amplitude Modulation) signal is transmitted over an additive white Gaussian channel. The PAM signal takes values $\{-1, 0, 1\}$ with probabilities $\{(1-\alpha)/2, \alpha, (1-\alpha)/2\}$, respectively and $0 < \alpha < 1$. The random noise N is additive white Gaussian noise, with zero mean and variance σ^2, independent from the random signal X.

a) Derive the minimum mean-square estimator of X based on the observation Y.

b) Derive the maximum a posteriori estimator of X based on the observation Y.

c) Questions for extra-points: Explain the result obtained for the MAP estimator in the following scenarios:

   c1) The parameter $\alpha = 1/3$. How does the noise power influence the MAP estimator in this case?

   c2) The parameter $\alpha > 1/3$. The noise variance tends to zero, and to infinity, respectively.

c3) The parameter α tends to one.

Hints:

- the additive model is: $Y = X + N$
- the pdf of the data is given by $f_X(x) = \frac{1-\alpha}{2}\delta(x+1) + \alpha\delta(x) + \frac{1-\alpha}{2}\delta(x-1)$, where $\delta(\cdot)$ is the unit impulse function.
- the pdf of the noise is given by $f_N(n) = \frac{1}{\sigma\sqrt{2\pi}}e^{-n^2/2\sigma^2}$

III. MS Estimator

Consider the array signal model

$$\mathbf{x} = \mathbf{A}(\boldsymbol{\theta})\mathbf{s} + \mathbf{v}, \tag{3}$$

where:
$\mathbf{x}$ is the $M \times 1$ received signal vector,
$\mathbf{A}(\boldsymbol{\theta})$ is an $M \times K$ array steering matrix $K < M$,
$\mathbf{s}$ is a $K \times 1$ transmitted signal vector and
$\boldsymbol{\theta} = [\theta_1, \ldots, \theta_K]^T$ is a known vector of DOAs that determines the matrix $\mathbf{A}(\boldsymbol{\theta})$.
Let $\mathbf{v}$ be complex Gaussian noise with mean ($E[\mathbf{v}] = \mathbf{0}$) and covariance matrix $E[\mathbf{v}\mathbf{v}^H] = \sigma^2\mathbf{I}_M$. The noise is uncorrelated to the signal. Assign a prior density $\mathbf{s} \sim \mathcal{CN}(0, \mathbf{P})$ for the transmitted signal (complex zero mean and covariance $E[\mathbf{s}\mathbf{s}^H] = \mathbf{P}$). Show that the mean square estimate of $\mathbf{s}$ based on $\mathbf{x}$ is:

$$\hat{\mathbf{s}}_{MS} = \mathbf{P}\mathbf{A}^H(\mathbf{A}\mathbf{P}\mathbf{A}^H + \sigma^2\mathbf{I}_M)^{-1}\mathbf{x}, \tag{4}$$

Hint: In your derivation you may use the following result:

Result

Assuming that

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \sim \mathcal{CN}\begin{bmatrix} \boldsymbol{\mu}_\mathbf{x} \\ \boldsymbol{\mu}_\mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathbf{C}_\mathbf{xx} & \mathbf{C}_\mathbf{xy} \\ \mathbf{C}_\mathbf{yx} & \mathbf{C}_\mathbf{yy} \end{bmatrix} \tag{5}$$

where $\boldsymbol{\mu}_\mathbf{x} = E[\mathbf{x}]$, $\boldsymbol{\mu}_\mathbf{y} = E[\mathbf{y}]$,
$\mathbf{C}_\mathbf{xx} = E[(\mathbf{x} - \boldsymbol{\mu}_\mathbf{x})(\mathbf{x} - \boldsymbol{\mu}_\mathbf{x})^H]$, $\mathbf{C}_\mathbf{yy} = E[(\mathbf{y} - \boldsymbol{\mu}_\mathbf{y})(\mathbf{y} - \boldsymbol{\mu}_\mathbf{y})^H]$, $\mathbf{C}_\mathbf{yx}E[(\mathbf{y} - \boldsymbol{\mu}_\mathbf{y})(\mathbf{x} - \boldsymbol{\mu}_\mathbf{x})^H]$, then the posterior density $p(\mathbf{y}|\mathbf{x})$ is also Gaussian with mean:

$$E[\mathbf{y}|\mathbf{x}] = \boldsymbol{\mu}_\mathbf{y} + \mathbf{C}_\mathbf{yx}\mathbf{C}_\mathbf{xx}^{-1}(\mathbf{x} - \boldsymbol{\mu}_\mathbf{x}) \tag{6}$$

IV. Experiment on the Kalman filter and state variable model

You are tracking a target moving along a straight line. The state vector consists of *the position* and *the velocity* of the target. The sampling interval is $T = 1$. The system evolves with constant velocity and the state equation is:

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} T^2/2 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{w}(k)$$

and the measurement equation is

$$y(k) = [1 \quad 0]\mathbf{x}(k) + v(k)$$

Do the tracking over 200 time steps using the following assumptions:

- initial "true" state $\mathbf{x}(0) = [0.5 \quad 10.0]^T$
- initial state estimate $\hat{\mathbf{x}}(0|0) = [0.5 \quad 11.0]^T$
- variance of the state noise $\sigma_w^2 = 9.0$
- variance of the measurement noise $\sigma_v^2 = 1.0$
- all noises are zero mean, white, Gaussian

The initial state covariance is

$$\mathbf{P}(0|0) = \begin{bmatrix} \sigma_v^2 & \sigma_v^2/T \\ \sigma_v^2/T & 2\sigma_v^2/T^2 \end{bmatrix}$$

where $T = 1$. In the simulation each new "true state" is generated by

$$\mathbf{x}(k) = \mathbf{F}\mathbf{x}(k-1) + \mathbf{G}\mathbf{w}(k)$$

and the new measurement

$$y(k) = \mathbf{H}\mathbf{x}(k) + v(k).$$

As an example, for the first step they are generated by $\mathbf{x}(1) = \mathbf{F}\mathbf{x}(0) + \mathbf{G}\mathbf{w}(1)$ and $y(1) = \mathbf{H}\mathbf{x}(1) + v(1)$.

Requirements:

Plot the tracking results as follows:

a) the position and the velocity of the target and the corresponding tracking result, in the same plot (position on the horizontal axis, velocity on the vertical axis).

b) the predicted and the estimated velocity error variances as a function of time, on the same plot

c) the predicted and the estimated position error variances as a function of time, on the same plot

d) the Kalman gains as a function of time

e) in your answer, include also the Matlab codes.

V. DoA estimation experiment

Write a Matlab program implementing the following DoA (Direction of Arrival) estimation methods: the classical beamformer, the MVDR (Capon) beamformer and the MUSIC algorithm. Test your function by using a 6-element uniform linear antenna array with $\lambda/2$ inter-element spacing. Consider two mutually uncorrelated QPSK signal sources in additive white Gaussian noise. The signal-to-noise ratio is assumed to be 20 dB. The signals arrive from directions:
(a) 38 and 93 degrees
(b) 88 and 93 degrees.
A total of 512 snapshots are collected. Estimate the DoA's by using the three methods mentioned above. Compare the results from 25 independent experiments, by plotting the spatial spectrum estimates in dB. How do they compare in terms of angular resolution? In your simulation assume the number of signals to be known. Repeat the same simulations at SNR 2 dB. How did SNR impact the results?

In your answer include the Matlab codes as well.

Your questions related to the homework exercises will be answered during the tutoring sessions dedicated to this matter (Wednesdays 14-16). It is highly recommended to attend these sessions, since not too much help will be provided by e-mail (just in special cases). No late homeworks will be accepted. Please return your answers before the deadline.