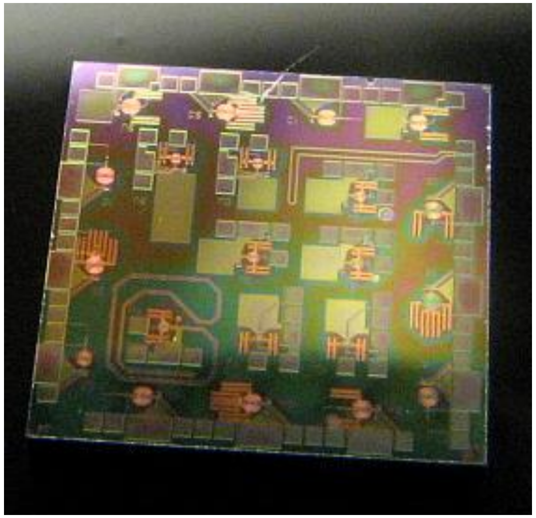
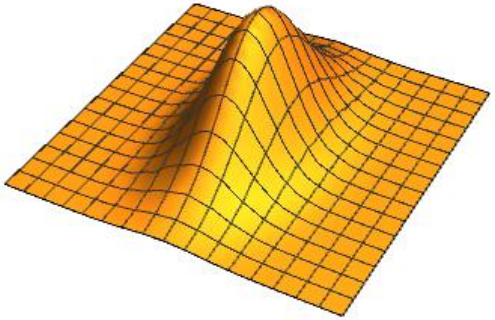
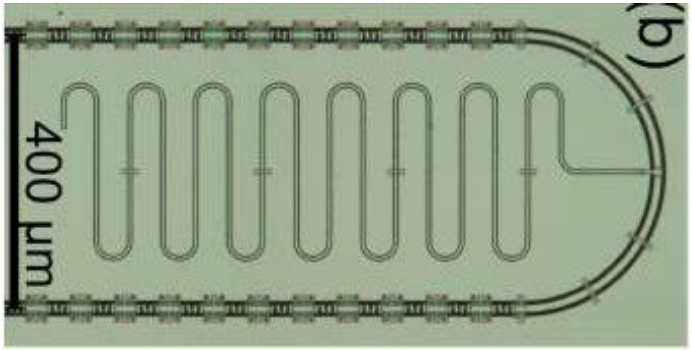
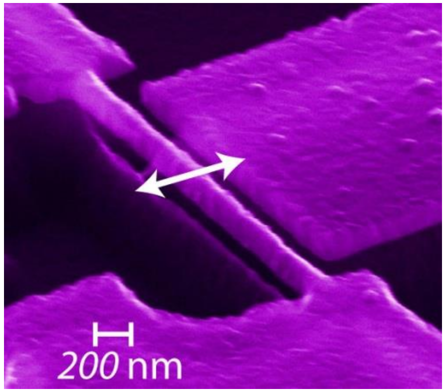
OUTLINE:

- Harmonic oscillator
  - Standard quantum limit
- Squeezing
- Introduction to
  - quantum amplifiers
  - noise temperature
  - parametric amplifiers
- Parametric oscillator as amplifier (pumped SQUIDs)
- Traveling wave parametric amplifier
- Mechanical parametric amplifier



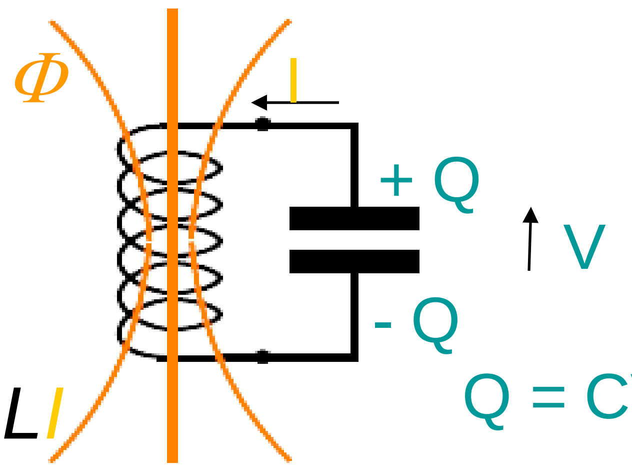






Flux and charge in LC oscillator

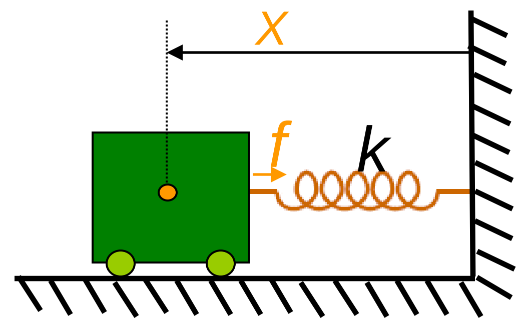
Electrical world



$\Phi = L I$

$Q = CV$

Mechanical world



$$X = \frac{1}{k} f \qquad P = MV$$

$$C \frac{\partial^2 \Phi}{\partial t^2} + \frac{1}{L} \Phi = I$$

$$V = \frac{\partial \Phi}{\partial t}$$

position variable: $\Phi \leftrightarrow X$

momentum variable. $Q \leftrightarrow P$

generalised force: $I \leftrightarrow f$

generalised velocity. $V \leftrightarrow V$

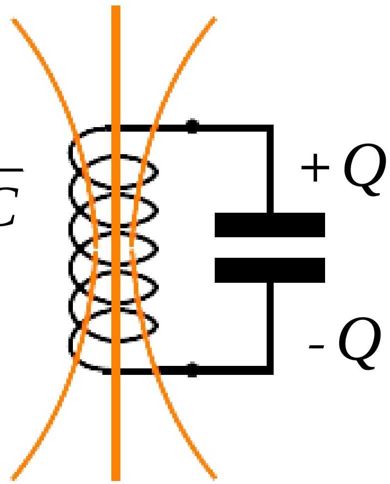
generalised mass: $C \leftrightarrow M$

generalised spring constant: $1/L \leftrightarrow k$

$$M \frac{\partial^2 x}{\partial t^2} + kx = f$$

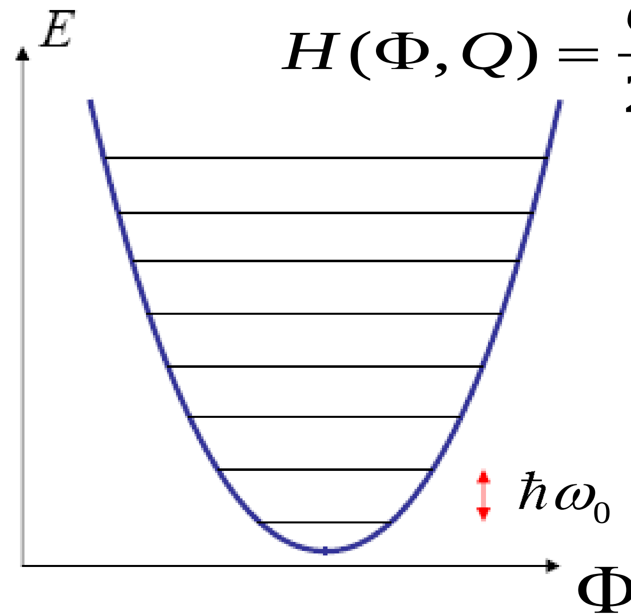
$$\left[X , P \right] = i\hbar$$

LC circuit as quantum harmonic oscillator



$$\omega_0 = 1/\sqrt{LC}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$



$$H(\Phi, Q) = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

annihilation and creation operators

$$H = \hbar\omega_0\left(a^\dagger a + 1/2\right)$$

$$a = \frac{\Phi}{\Phi_r} + i\frac{Q}{Q_r} = \hat{\phi} + i\hat{q}$$

$$a^\dagger = \frac{\Phi}{\Phi_r} - i\frac{Q}{Q_r} = \hat{\phi} - i\hat{q}$$

$$\left[\Phi, Q\right] = i\hbar$$

$$\Phi = \Phi_r \frac{a + a^\dagger}{2} \qquad \Phi_r = \sqrt{2\hbar\omega_0 L}$$

$$Q = Q_r \frac{a - a^\dagger}{2i} \qquad Q_r = \sqrt{2\hbar\omega_0 C}$$

Noise: $\delta\Phi^2 = \left\langle \Phi^\dagger \Phi \right\rangle - \left\langle \Phi^\dagger \right\rangle \left\langle \Phi \right\rangle$

$\left\langle \delta\Phi^2 \right\rangle = \frac{\hbar Z_0}{2}; \ \left\langle \delta Q^2 \right\rangle = \frac{\hbar}{2Z_0}$ *SQL*

Standard quantum limit

$$\left\langle \delta\Phi^2 \right\rangle = \frac{\hbar Z_0}{2}; \quad \left\langle \delta Q^2 \right\rangle = \frac{\hbar}{2Z_0}$$

$$\omega_0 = 1/\sqrt{LC}$$

$$S_\Phi = \frac{\hbar Z_0}{2\omega}; \quad S_Q = \frac{\hbar}{2Z_0\omega}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

$$S_V = \frac{\hbar Z_0}{2}\omega; \quad S_I = \frac{\hbar}{2Z_0}\omega$$

$$S_V S_I = \left(\frac{\hbar\omega}{2}\right)^2$$

$Z_0 = 50\ \Omega$ at 1 GHz

$$\sqrt{\left\langle \delta v^2 \right\rangle} \sim 4\,\mathrm{pV}/\sqrt{\mathrm{Hz}} \qquad \sqrt{\left\langle \delta i^2 \right\rangle} \sim 80\,\mathrm{fA}/\sqrt{\mathrm{Hz}}$$

$Z_0 = 5000\ \Omega$ at 1 GHz

$$\sqrt{\left\langle \delta v^2 \right\rangle} \sim 40\,\mathrm{pV}/\sqrt{\mathrm{Hz}} \qquad \sqrt{\left\langle \delta i^2 \right\rangle} \sim 8\,\mathrm{fA}/\sqrt{\mathrm{Hz}}$$

"Mode" observables: Quadratures

- Quadrature operators (like x and p): $H_{\mathbf{k}} = \hbar\omega_{\mathbf{k}}(a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}} + \frac{1}{2}))$

$$X_1 = \frac{1}{\sqrt{2}}\left(a^{\dagger} + a\right)$$

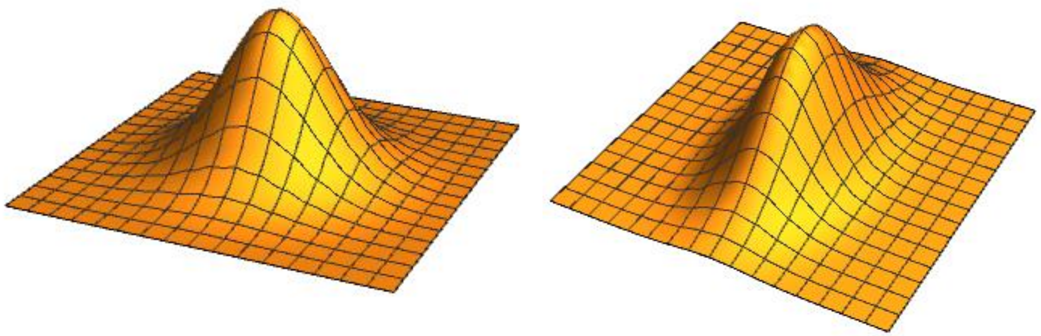
$$X_2 = \frac{i}{\sqrt{2}}\left(a^{\dagger} - a\right)$$

$$X_{\theta} = \frac{1}{\sqrt{2}}\left(ae^{-i\theta} + a^{\dagger}e^{i\theta}\right)$$

- Since $\left[X_1, X_2\right] = i$, there must be an uncertainty relation

$$\Delta X_1 \Delta X_2 \geq \frac{1}{2}$$

- Correlation of quadratures can be manipulated



Single mode squeezing

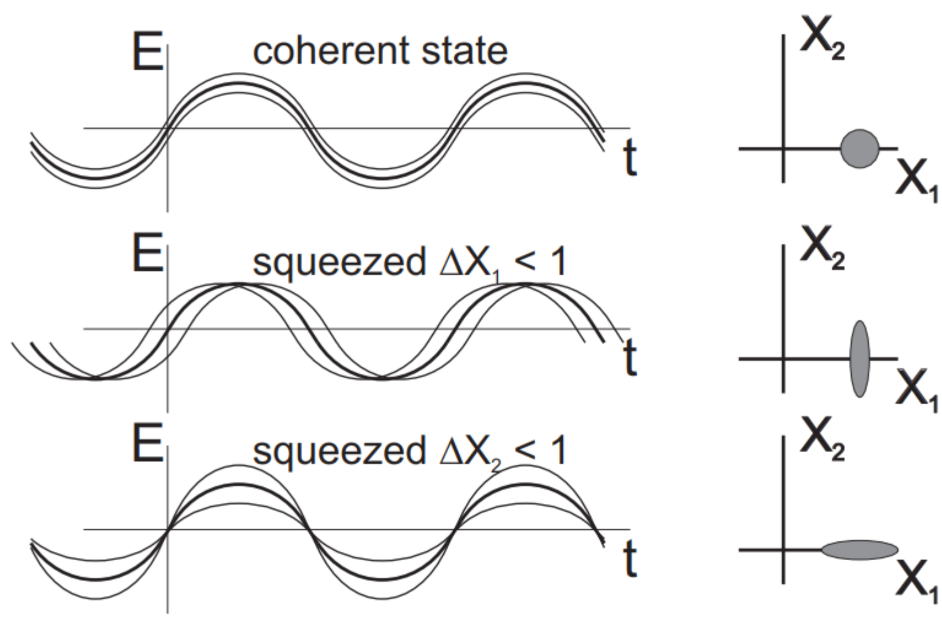
- Squeezing operator

$$S = \exp\left(\frac{1}{2}\xi a^{\dagger 2} - \frac{1}{2}\xi^* a^2\right) \qquad \xi = re^{i\theta} \qquad |\xi\rangle = S|0\rangle$$

$$\langle \Delta X_1{}^2 \rangle = \frac{1}{2}e^{2r}$$

$$\langle \Delta X_2{}^2 \rangle = \frac{1}{2}e^{-2r}$$

$$\Delta X_1 \Delta X_2 = \frac{1}{2}$$



Basic correlator:

$$\langle aa \rangle = \cosh r \sinh r e^{i\theta}$$

Two-mode squeezing

- Two mode squeezing operator

$$S_2 = \exp\left(\xi^* ab - \xi a^\dagger b^\dagger\right) \qquad \xi = re^{i\theta}$$

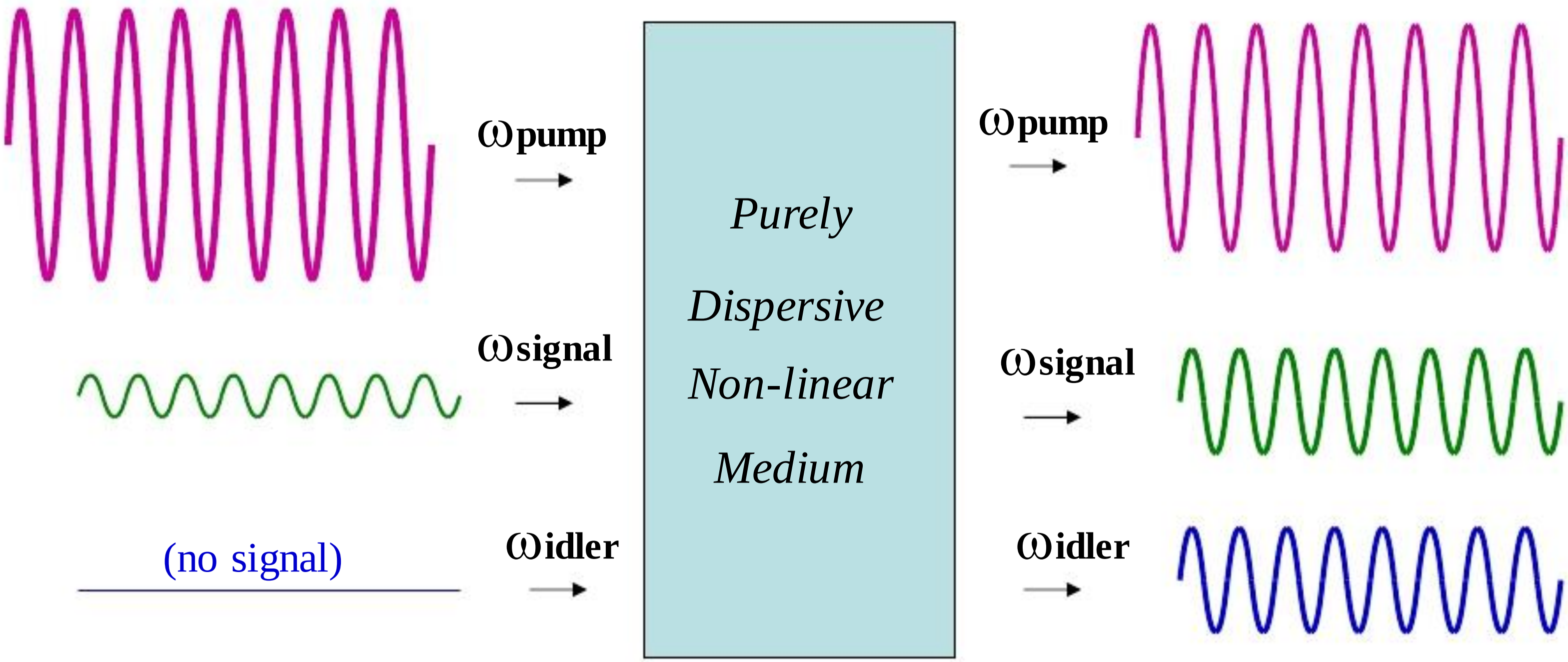
$$\langle ab \rangle = \cosh r \sinh r e^{i\theta} \qquad \langle ab^\dagger \rangle = 0$$

Maps to single mode case by defining operator

$$d = \frac{1}{\sqrt{2}}(a+b) \qquad \left[d, d^\dagger\right] = 1$$

$$X_\theta^d = \frac{1}{\sqrt{2}}\left(de^{-i\theta} + d^\dagger e^{i\theta}\right) \quad \left\langle \Delta X_1^{d\,2} \right\rangle = \frac{1}{2}e^{2r} \quad \left\langle \Delta X_2^{d\,2} \right\rangle = \frac{1}{2}e^{-2r}$$

MIXING IN NONLINEAR MEDIA



$$\omega_{signal} + \omega_{idler} = \omega_{pump}$$

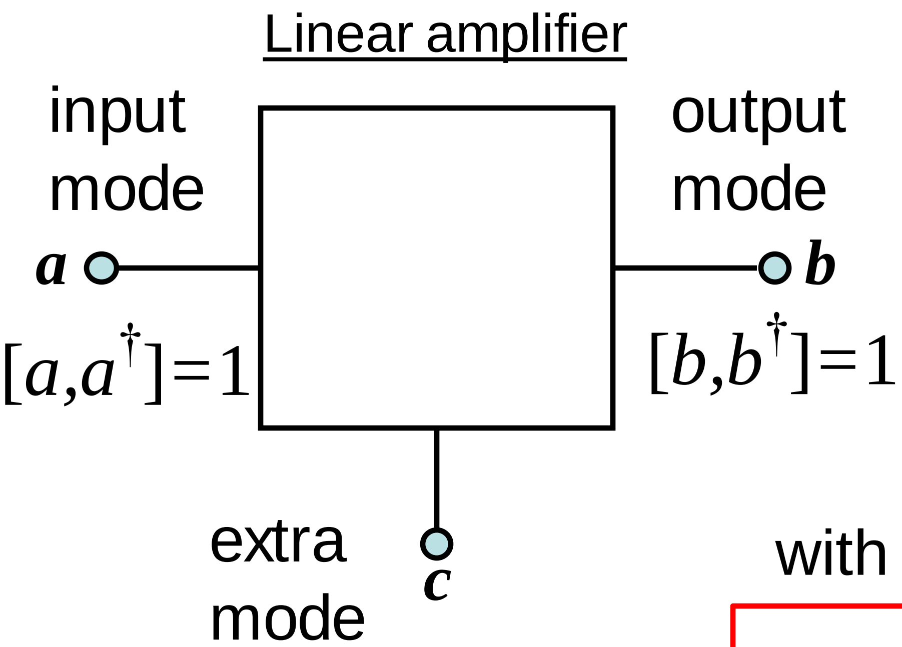
"3-wave process"

$$\omega_{signal} + \omega_{idler} = 2\omega_{pump}$$

"4-wave process"

Phase preserving linear quantum amplifier

- Commutation relations have to be conserved:



if $b=\sqrt{G}a$, $b^\dagger=\sqrt{G}a^\dagger$ photon number gain, G

then $[b,b^\dagger]=G\,[a,a^\dagger]\neq 1$

with extra mode:

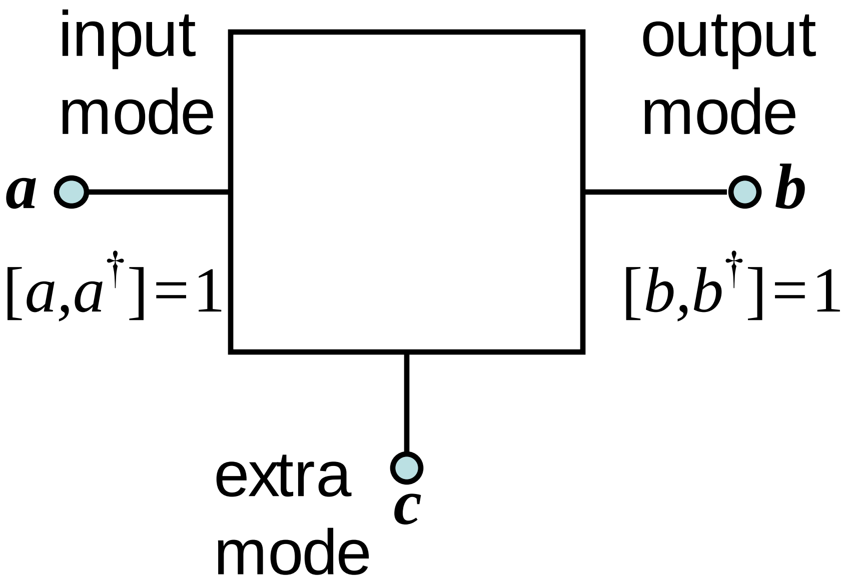
$$b=\sqrt{G}a+\sqrt{G-1}\;c^\dagger$$
$$b^\dagger=\sqrt{G}a^\dagger+\sqrt{G-1}\;c$$

No $a^\dagger$ because of phase preserving condition

$$[b,b^\dagger]=G\,[a,a^\dagger]+(G\text{-}1)\,[c^\dagger,c]=1$$

Caves, 1982
Haus' book

Quantum amplifier: noise (at G »1)



$$b=\sqrt{G}a+\sqrt{G-1}\,c^\dagger$$

$$b^\dagger=\sqrt{G}a^\dagger+\sqrt{G-1}\,c$$

$$(\Delta a)^2\equiv\frac{1}{2}\left\langle\left\{a,a^\dagger\right\}\right\rangle-\left|\langle a\rangle\right|^2$$

$$(\Delta a)^2=\frac{1}{2}\left\langle aa^\dagger+a^\dagger a\right\rangle=n_a+\frac{1}{2}$$

$$(\Delta b)^2=\frac{1}{2}\left\langle bb^\dagger+b^\dagger b\right\rangle$$

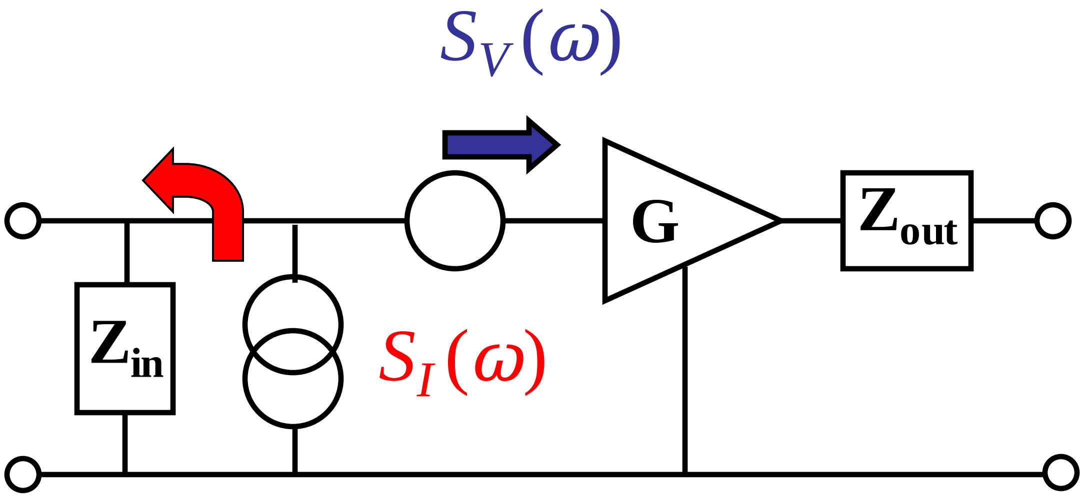
$$=G\left(n_a+\frac{1}{2}+n_c+\frac{1}{2}\right)\geq G\left(n_a+1\right)$$

amplified input vacuum

added noise

Equivalent circuit of an amplifier

H. Rothe and W. Dahlke, Proc. IRE **44**, 811 (1956).



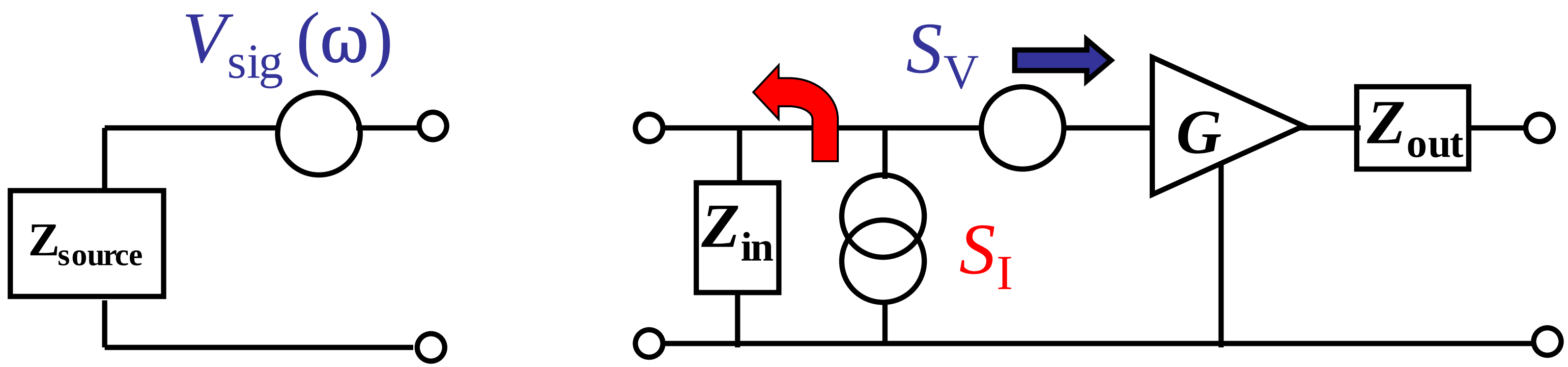
$S_V(\omega)$ = output noise referred to input

$S_I(\omega)$ = a real ”back action” noise (A^2/Hz)

may be strongly correlated with S_V

Noise Temperature of an Amplifier

- Beware: definition varies



Total noise at the input : $S_V^{tot} = S_V + S_I \left| \frac{Z_{in} Z_S}{Z_{in} + Z_S} \right|^2$

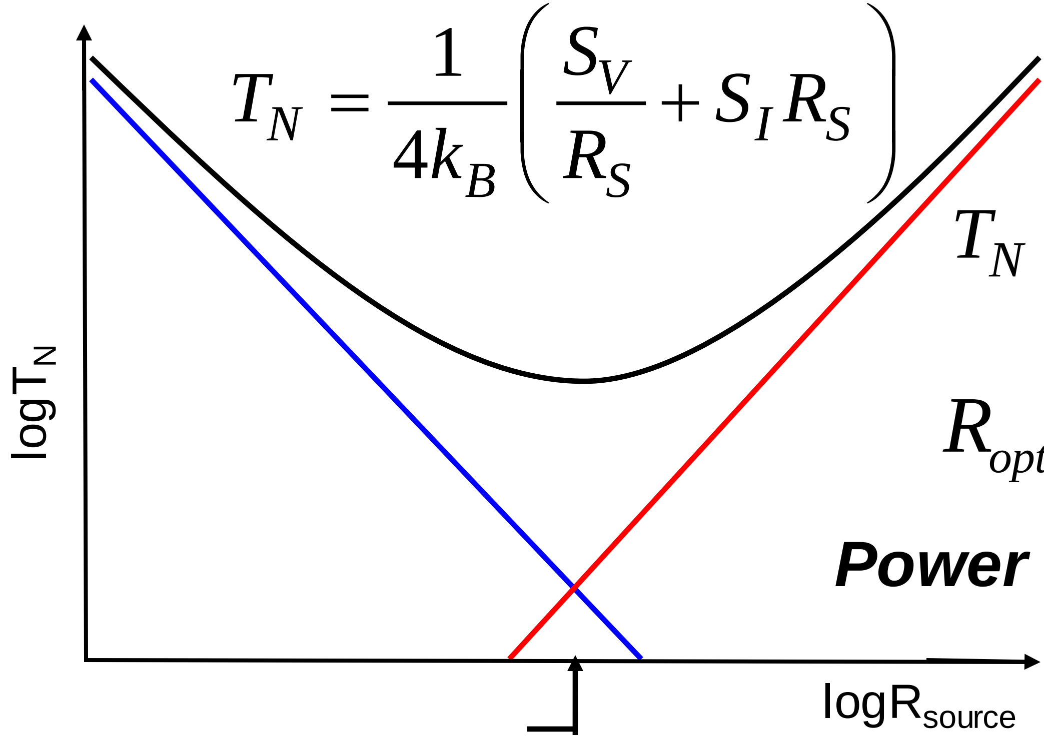
Thermal noise of the source : $S_V^{tot} = 4kT_N \operatorname{Re}\left[Z_S\right]$

Assume: $Z_{in} = R_{in} >> R_S = Z_S$

$$T_N = \frac{1}{4k_B}\left(\frac{S_V}{R_S} + S_I R_S\right)$$

Optimum Noise Temperature

$$T_N = \frac{1}{4k_B}\left(\frac{S_V}{R_S} + S_I R_S\right)$$



$$T_N = \frac{T_N^{opt}}{2}\left(\frac{R_{opt}}{R_S} + \frac{R_S}{R_{opt}}\right)$$

$$R_{opt} = \sqrt{S_V / S_I}$$

Power gain not optimized:

$$R_s = R_{in}$$

$$T_N^{opt} = \sqrt{S_V S_I} \big/ 2k_B$$

$$E_N^{opt} = kT_N^{opt}$$

E_N is the signal energy that can be detected with SNR = 1

Quantum mechanics:

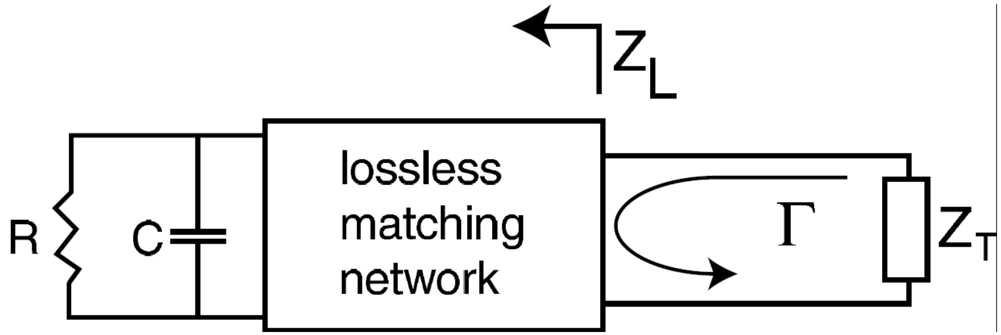
$$E_N \geq \hbar\omega / 2$$

Matching

- How to preserve the band width when you connect your measuring apparatus to your sensor?

The theoretical maximum bandwidth:

$$\int_0^\infty \ln \frac{1}{|\Gamma(\omega)|} d\omega \leq \frac{\pi}{RC},$$



Bode--Fano criterion

$$\Gamma(\omega) = \frac{Z_L(\omega) - Z_T}{Z_L(\omega) + Z_T}$$

”You cannot exceed inverse of RC time constant”

T_n of cascaded amplifiers

C. D. Motchenbacher and J. A. Connelly, *Low noise eletronic system design*

$$T_N = T_{N_1} + T_{N_2} / G_1 + T_{N_3} / G_1 G_2 + \ldots$$

- $\boldsymbol{T_N}$ *of the first amplifier dominates if it has sufficient gain*

$T_{N_1} = 100\ \mathrm{mK}$ SQUID amplifier

$T_{N_2} = 10\ \mathrm{K}$ HEMT amplifier

Desirable to have the gain of the SQUID amplifier ~30 dB

Current State of the Art LNAs

Band	Substrate	Technology	Freq (GHz)	Noise Temp(K)	Year	LeadAuthor/ Organisation
S	InP	MMIC	2	1.2(0.09)	2017	LNF
C	InP	MMIC	7	3.0(0.34)	2012	Schleeh
X	InP	MIC	8.4	4.0(0.40)	2001	Bautista
Ka	InP	MIC	30	5.0(1.44)	2009	PLANCK
W	InP	MMIC	85	22(4.08)	2009	Bryerton

Quantum Noise Limit

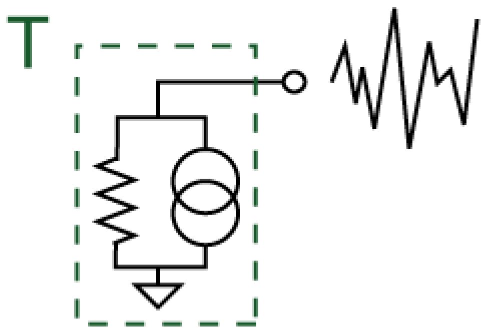
MMIC - Monolithic microwave integrated circuit
MIC - Microwave integrated circuits

M. McCulloch, Manchester

Measurement time speed-up

- Dicke Radiometer Formula:

$$\delta T = \frac{T + T_{noise}}{\sqrt{B\tau}}$$



- Thus

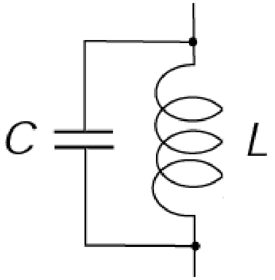
$$\tau = \frac{(T + T_{noise})^2}{(\delta T)^2 B}$$

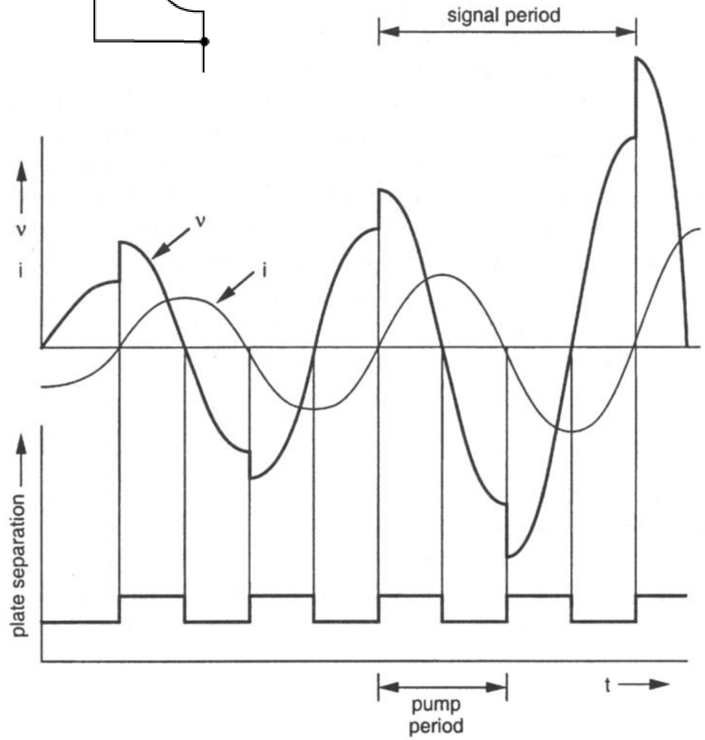
- 40x lower T_N gives 1600x speedup in measurement times!

Comes from Poisson statistics!

Parametric amplifiers

The **Botafumeiro** is a famous thurible found in the **Santiago de Compostela Cathedral**. Incense is burned in this swinging metal container, or "incensory". The name "Botafumeiro" means "smoke expeller" in Galician.





L. Blackwell and K. Kotzebue, Semiconductor-Diode Parametric Amplifiers (1961)

Small signal model of parametric circuits

$$v_1 = V_1 \cos \omega t \qquad v_p = V_p \cos \omega_p t \qquad v_p \gg v_1$$

$$q(v) = q(v_p) + \frac{dq}{dv}(v_p) v_1$$

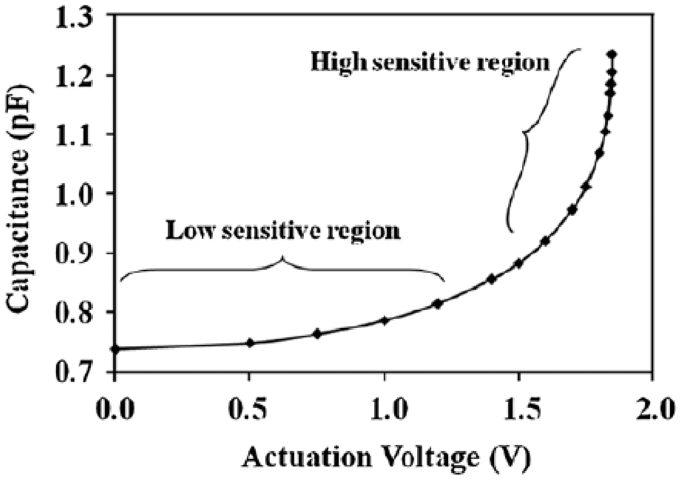
$$C(v_p) = \frac{dq}{dv}(v_p)$$ ***Pumped capacitance***

$$i = \frac{dq}{dt} = \frac{d}{dt} q(v_p) + \frac{d}{dt}\left[C(v_p) V_1 \cos \omega_1 t \right]$$

$$C(v_p) = \sum_{n=0}^{\infty} C_n \cos n\omega_p t$$ ***Time dependent linear capacitance***

$$\frac{d}{dt}\left[C(v_p) V_1 \cos \omega_1 t \right] \Rightarrow \frac{d}{dt}\left[C(t) v_1(t) \right]$$

- Circuit simulators in time domain applicable

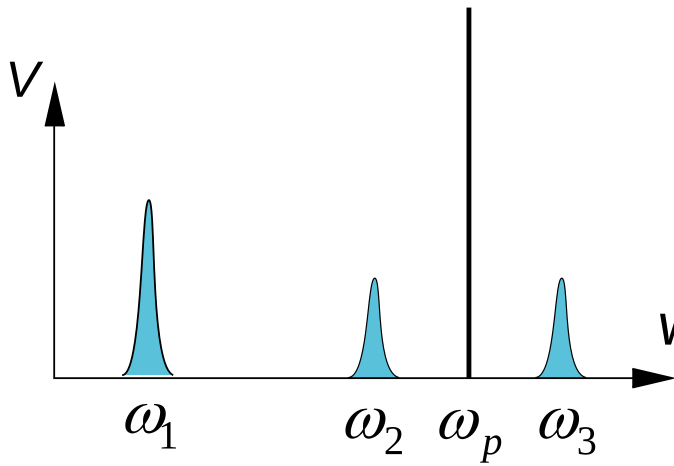


MEMS parallel-plate tunable capacitor with structural nonlinearity in the supporting beams, J. Micromech. Microeng. 22 (2012) 025022

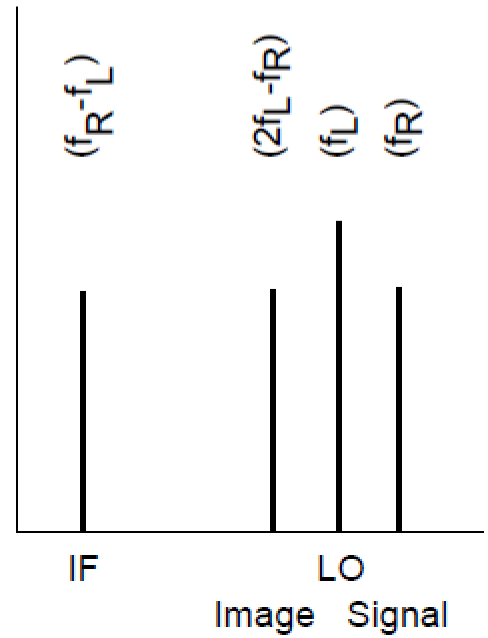
Conversion matrix for parametric circuits

$$i = \frac{d}{dt}\left[C(t)v(t)\right]$$

$$v = V_1 \exp(-j\omega_1 t) + V_2 \exp(-j\omega_2 t) + V_3 \exp(-j\omega_3 t)$$



Mixer:



$$\begin{pmatrix} I_2^* \\ I_1 \\ I_3 \end{pmatrix} = \begin{pmatrix} -j\omega_2 C_0 & -j\omega_2 C_0 M & 0 \\ j\omega_1 C_0 M & j\omega_1 C_0 & j\omega_1 C_0 M \\ 0 & j\omega_2 C_0 M & j\omega_3 C_0 \end{pmatrix} \begin{pmatrix} V_2^* \\ V_1 \\ V_3 \end{pmatrix}$$

$$C(t) = C_0(1 + M\cos\omega_p t)$$

Manley Rowe relations

After some algebra:

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \frac{nP_{nm}}{n\omega_1 + m\omega_2} = 0$$

$$\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{mP_{nm}}{n\omega_1 + m\omega_2} = 0$$

Manley and Rowe 1956

$$\frac{P_{10}}{\omega_1} + \frac{P_{11}}{\omega_1 + \omega_2} = 0$$

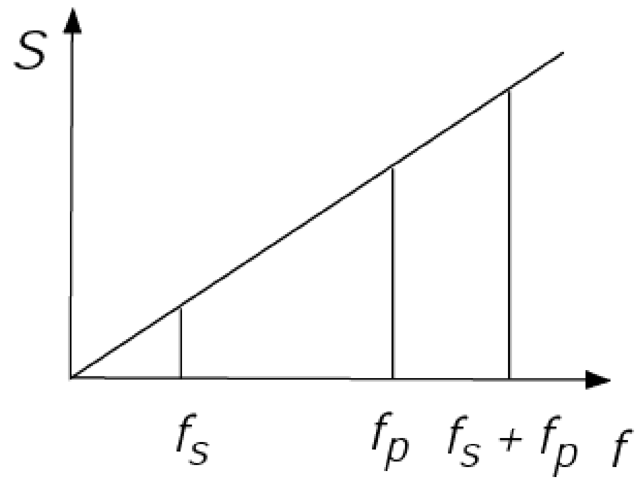
$$\frac{P_{01}}{\omega_2} + \frac{P_{11}}{\omega_1 + \omega_2} = 0$$

Take only three frequencies:

f_1: nm=10, f_2: nm=01, $f_1 + f_2$: nm=11,

$$-\frac{P_{11}}{P_{10}} = \frac{\omega_1 + \omega_2}{\omega_1} = \frac{\omega_3}{\omega_1} = 1 + \frac{\omega_2}{\omega_1}$$

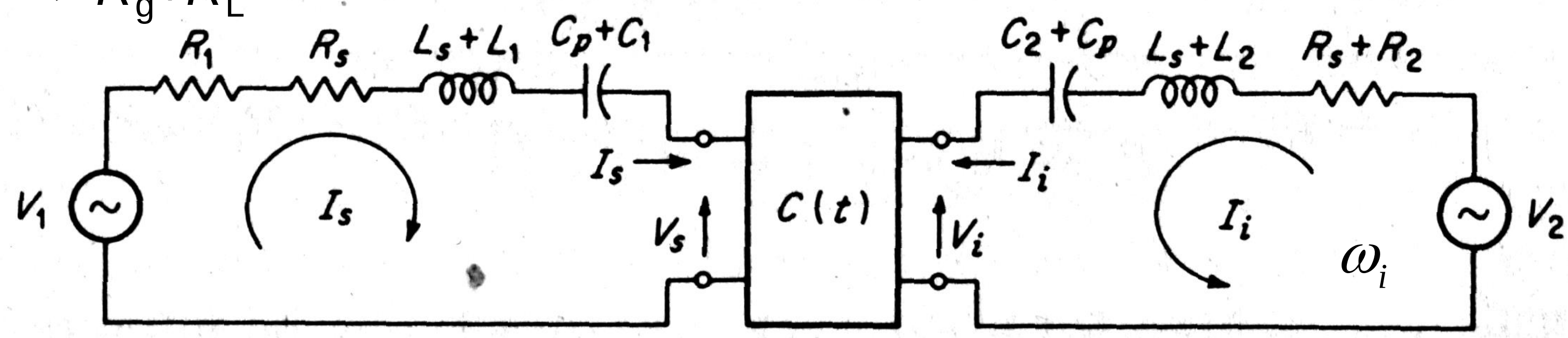
Maximum gain for a parametric up-converter



Negative resistance amplifier

MEASURE at ω_s

=> R_g+R_L



$C(t)=C_0(1+2M\cos\omega_p t)$

Idler circuit

$$\begin{pmatrix} I_s \\ I_i^* \end{pmatrix} = \begin{pmatrix} j\omega_s C_0 & -j\omega_s C_0 M \\ -j\omega_i C_0 M & j\omega_i C_0 \end{pmatrix} \begin{pmatrix} V_s \\ V_i^* \end{pmatrix}$$

Conversion matrix

$$G_0 = \frac{4R_g R_L}{(R_g + R_L + R_s - R)^2}$$

$$R = \frac{M^2}{(R_2 + R_s)\omega_i\omega_s(1 - M^2)^2 C_0^2}$$

If idler frequency blocked, th
no negative resistance

Mixing back to signal frequency from the idler frequency

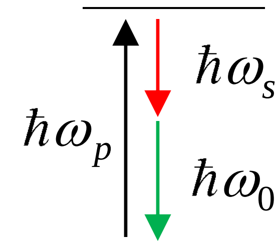
$R \to 0 \; if \;\; R_2 \to \infty$

- Use a circulator to improve stability

- Splits R_g *and* R_L *physically (4 times more gain)*

Parametric amplifier: ”quantum theory”

$$H = \hbar\omega_s a_s^\dagger a_s + \hbar\omega_0 a_0^\dagger a_0 - \hbar\lambda M \left(a_0^\dagger a_s^\dagger + a_s a_0 \right)$$



- Equations of motion yield exponential dependence on time *T* for the operators:

$$\begin{pmatrix} a_s(T) \\ a_0^\dagger(T) \end{pmatrix} = \begin{pmatrix} \cosh(T) & -\sinh(T) \\ -\sinh(T) & \cosh(T) \end{pmatrix} \begin{pmatrix} a_s(0) \\ a_0^\dagger(0) \end{pmatrix}$$

$$G = \cosh^2(\delta T)$$

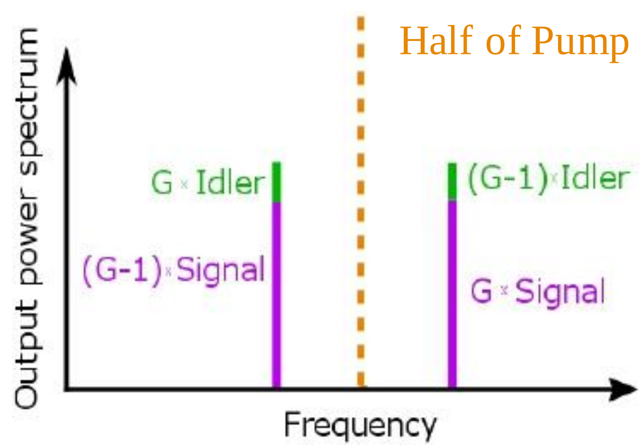
$$G - 1 = \sinh^2(\delta T)$$

$$b \triangleq a_s(T) \quad a \triangleq a_s(0) \quad c \triangleq a_0(0)$$

$$\delta T \propto \xi \qquad S_2 = \exp\left(\xi^* ac - \xi a^\dagger c^\dagger \right)$$

$$b = \sqrt{G}a + \sqrt{G-1}\, c^\dagger$$

$$b^\dagger = \sqrt{G}a^\dagger + \sqrt{G-1}\, c$$

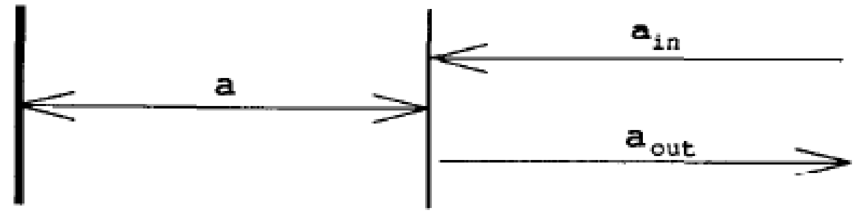


phase-preserving amplifier

Dissipation into account using **input/output formalism** of quantum optics

Input/Output theory

$$\tilde{H}_{\mathrm{RWA}} = \hbar\Delta a^\dagger a - \frac{\hbar}{2}(\alpha^* a^2 + \alpha a^{\dagger 2})$$



$$\dot{\tilde{a}} = -i\Delta\tilde{a} + i\alpha\tilde{a}^\dagger - \frac{\kappa}{2}\tilde{a} - \sqrt{\kappa}\tilde{a}_{\mathrm{in}}$$

$$\tilde{a}^\dagger(\nu) = \int_{-\infty}^{\infty} dt \exp(i\nu t)\tilde{a}^\dagger(t) = [\tilde{a}(-\nu)]^\dagger$$

$$a(t) = \tilde{a}(t)\exp[-i\omega_d t/2]$$

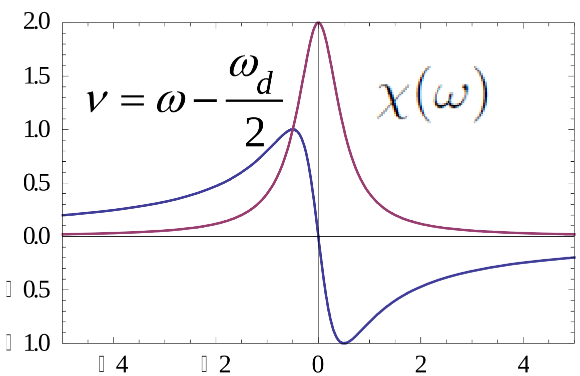
$$\Delta = \omega_{\mathrm{res}} - \omega_d/2$$

$$\tilde{a}_{\mathrm{out}}(\nu) = \left[1 - \frac{\kappa\chi\left(\frac{\omega_d}{2}+\nu\right)}{\mathcal{N}(\nu)}\right]\tilde{a}_{\mathrm{in}}(\nu) - \frac{i\alpha\kappa}{\mathcal{N}(\nu)}\chi\left(\frac{\omega_d}{2}+\nu\right)\chi\left(\frac{\omega_d}{2}-\nu\right)^* \tilde{a}^\dagger_{\mathrm{in}}(\nu)$$

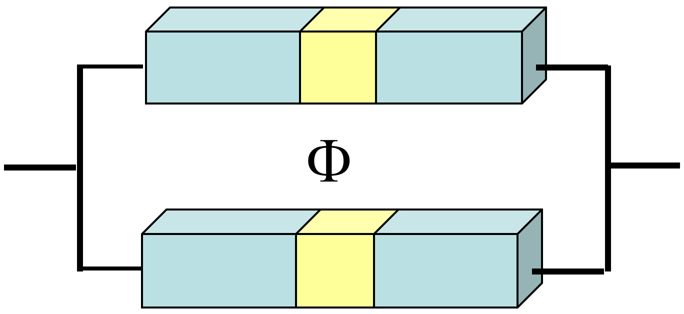
$$\mathcal{N}(\nu) = 1 - |\alpha|^2\chi\left(\frac{\omega_d}{2}+\nu\right)\chi\left(\frac{\omega_d}{2}-\nu\right)^*$$

$$\tilde{a}_{out} = \cosh\lambda\,\tilde{a}_{in} - \sinh\lambda\,\tilde{a}^\dagger_{in} \quad (\alpha \propto \tanh\lambda/2)$$

- Bogolyubov transformation



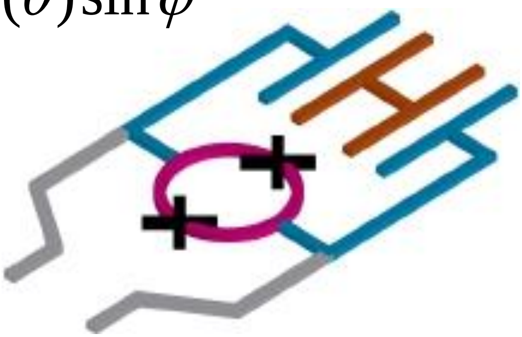
SQUID: A NONLINEAR L

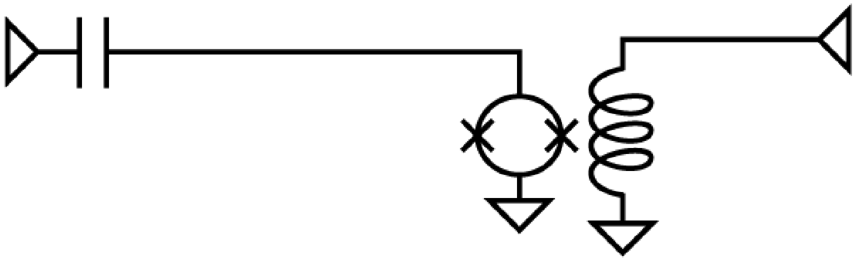


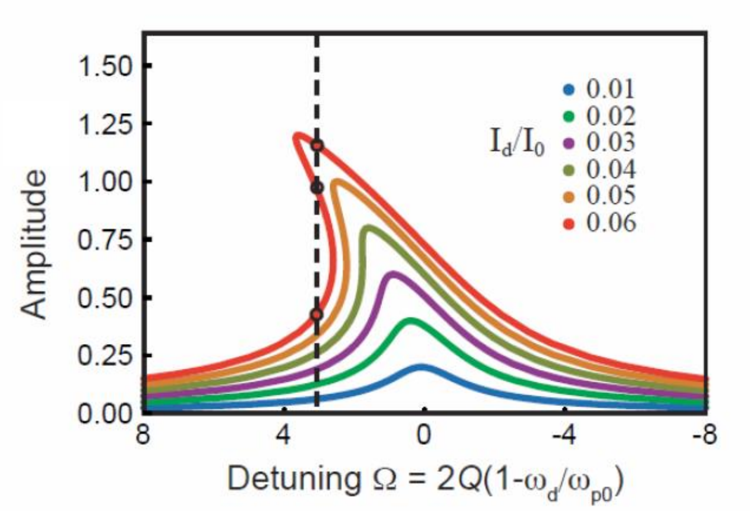
SQUID loop with

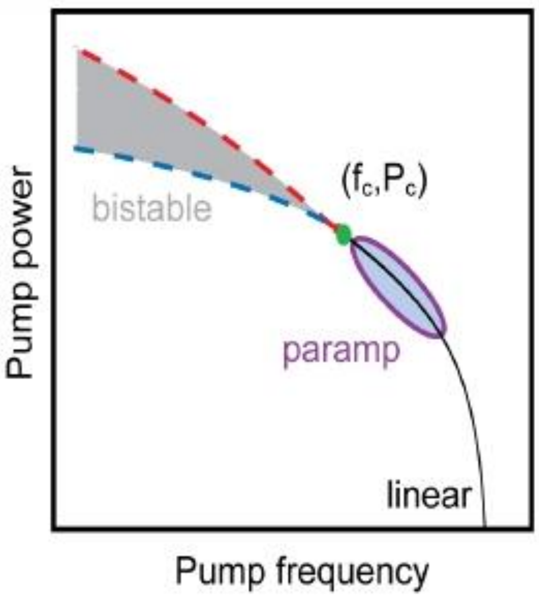
$$\theta = \pi \frac{\Phi}{\Phi_0}$$

$$I = I_c(\theta)\sin\varphi$$



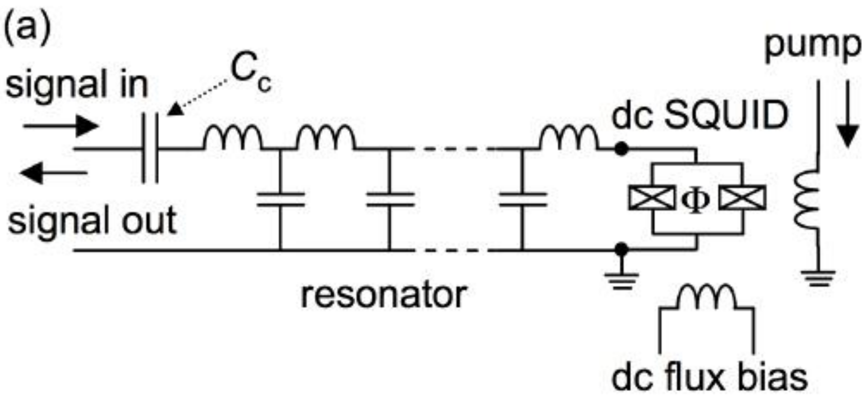






Josephson Parametric Amplifiers

Flux-driven JPA
[Yamamoto et al, 2008]

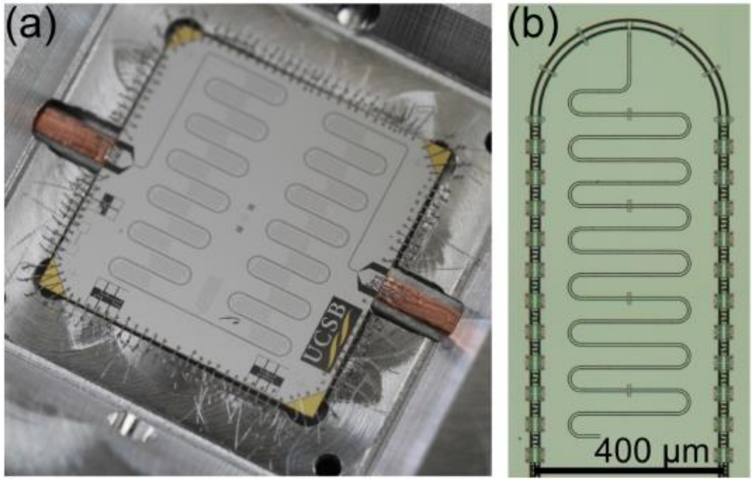


Band: 20 MHz

Insufficient characteristics

Easy in fabrication

Travelling wave parametric amplifier
[Macklin et al, 2015]



Band: 3 GHz

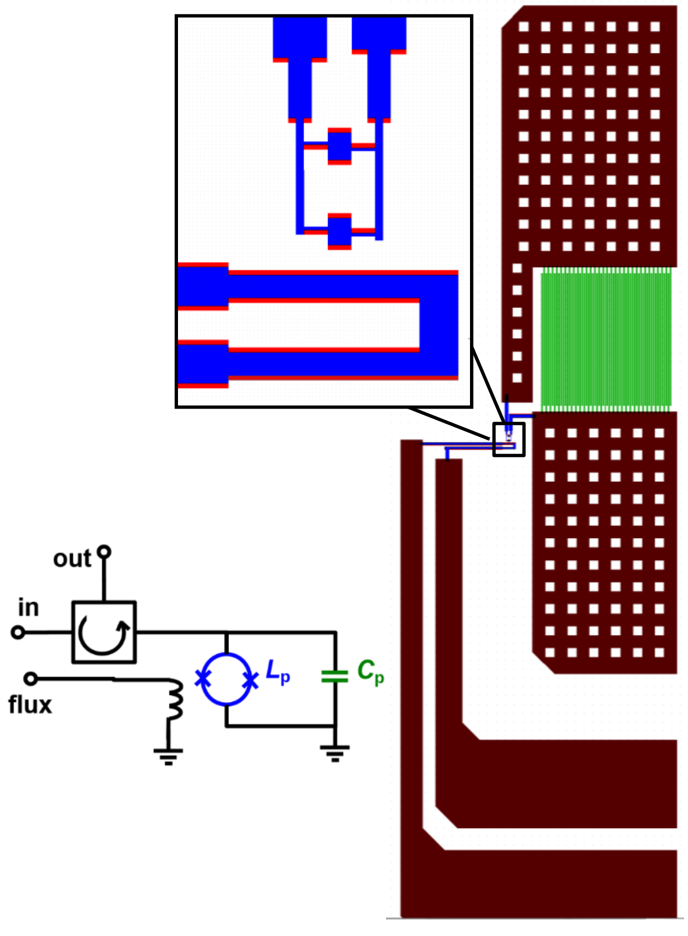
Impressive characteristics

High complexity of manufacturing

How to combine the best qualities

JPA Design

Lumped element design



Design Parameters:

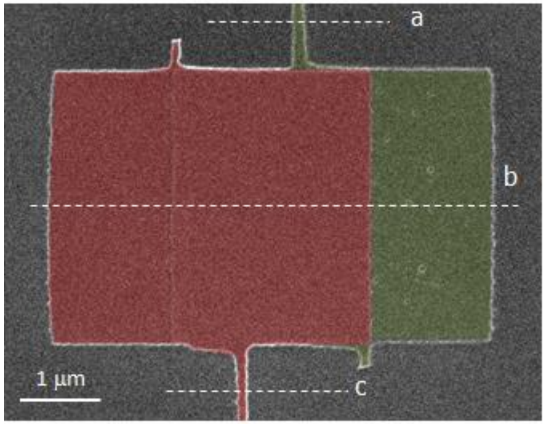
- Critical current of SQUID: 1.2 μA
- Josephson inductance (Φ = 0): 275 pH
- Interdigital capacitor: 1.2 pF
- Resonance frequency: 7.5 GHz
- Fluxline for DC and RF
- Pump at double the signal frequency

Large Josephson junctions : 9 μm^2

Large Critical current

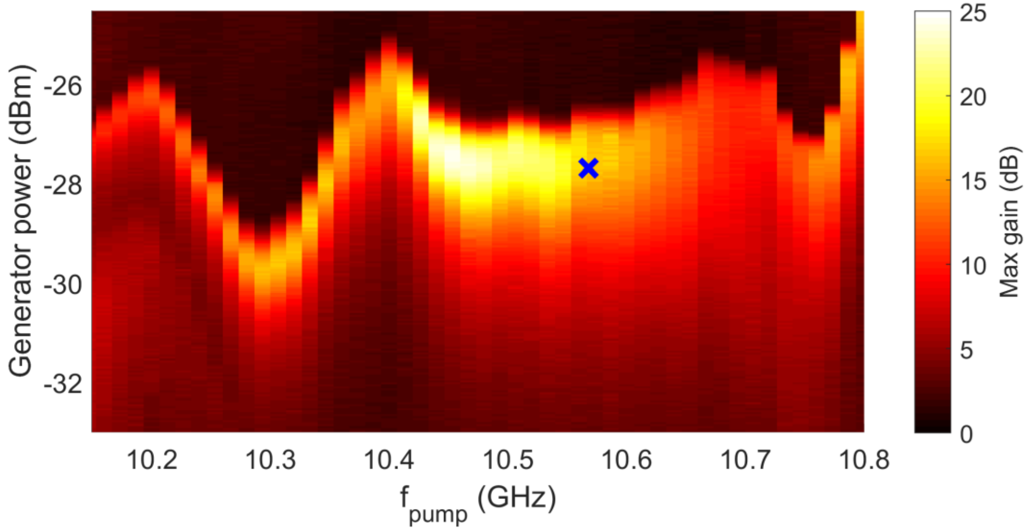
Low impedance of the resonator

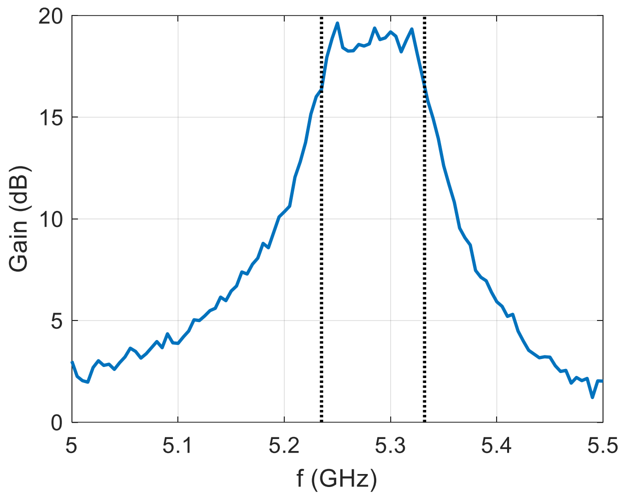
Junctions utilizing AL shadow evaporation **without** a suspended bridge



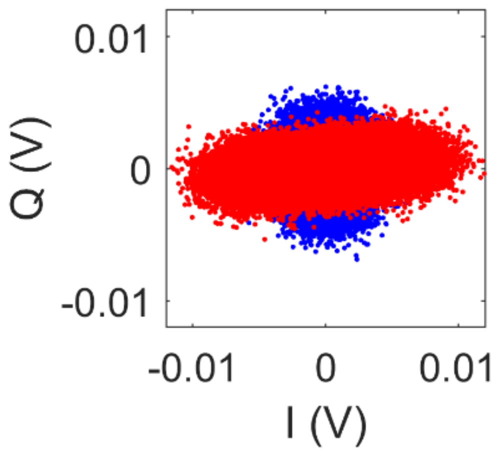
[F. Lecocq *et al.* Nanotechnology, 22, 315302 (2011).]

Results – JPA Performance





- ❑ Good tunability
- ❑ Center frequency 5 – 5.5 GHz
- ❑ Additional tunabilty from DC flux



- ❑ Operating point example:
  20 dB gain
  100 MHz bandwidth
  1 dB compression at -125 dBm

T. Elo et al, APL 2019

Traveling wave parametric amplifier

$$L(x) = L_0\left[1 + \eta \sin(2(\omega t - \beta x))\right]$$

$$C(x) = C_0$$

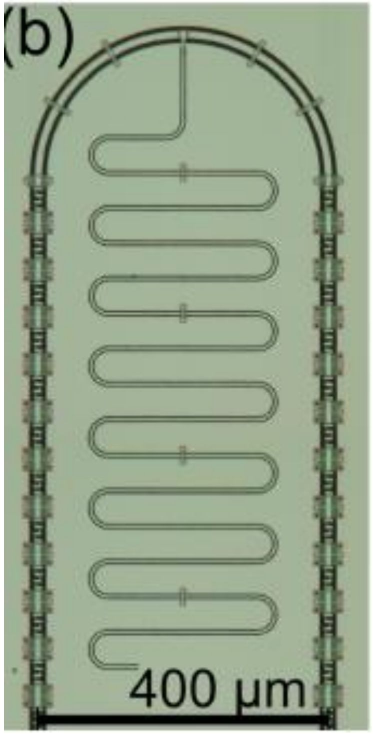
$$L(I) = L_0\left[1 + \frac{1}{2}\frac{I^2}{I_c^2}\right], \; L_0 = \frac{\Phi_0}{2\pi I_c}$$

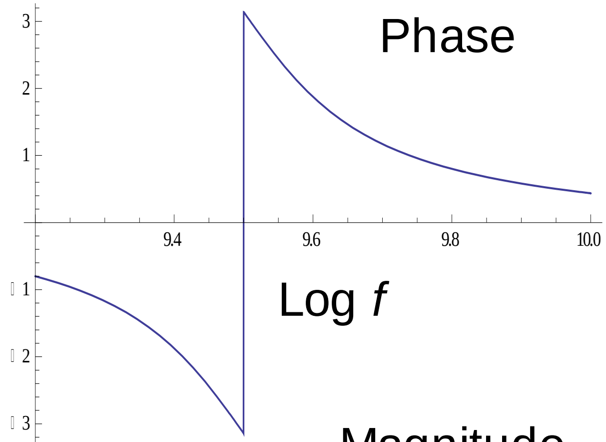
$$i(x) = i_0 \exp(\alpha x) \sin(\omega t - \beta x + \varphi)$$

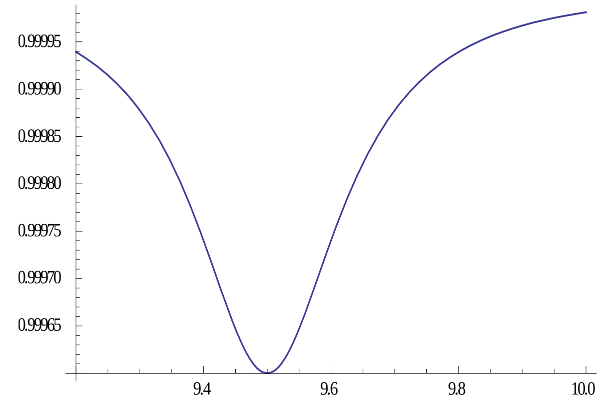
$$\alpha = \omega\sqrt{L_0 C_0}\,\frac{\eta}{4}\cos(2\varphi)$$

$$\beta = \omega\sqrt{L_0 C_0}\left[1 - \frac{\eta}{4}\sin(2\varphi)\right]$$

$$\eta = 0.1 \rightarrow 1.36\, dB/\lambda$$

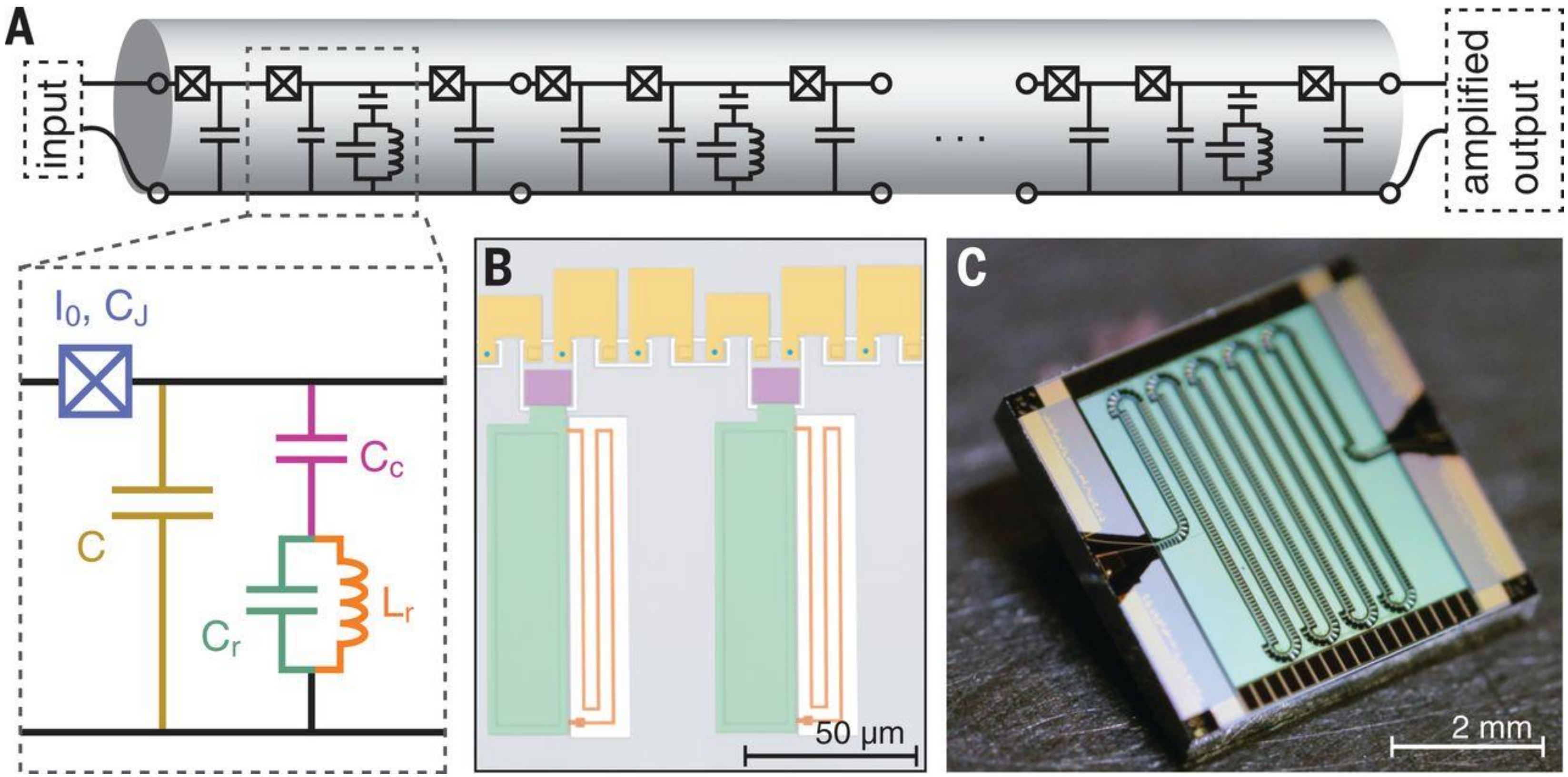






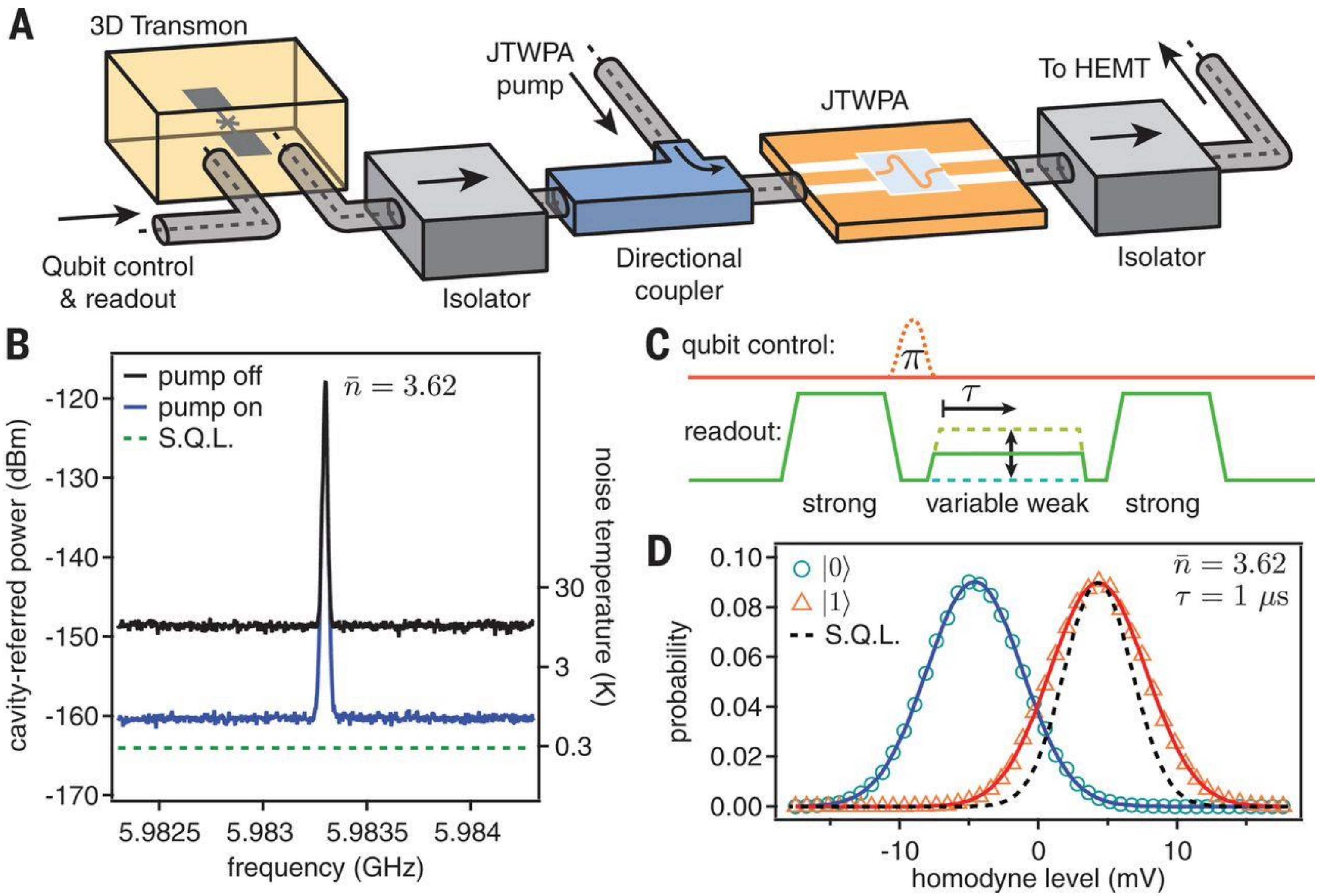
M. Cullen, Nature **181**, 332 (1958)

Josephson traveling-wave parametric amplifier



C. Macklin et al. Science 350, 307 (2015)
White et al. Appl. Phys. Lett. **106**, 242601 (2015)

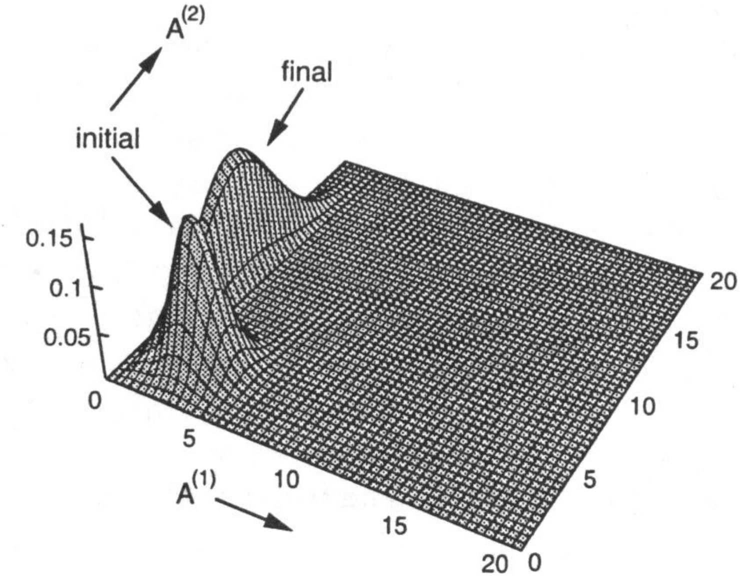
Noise performance of JTWPA

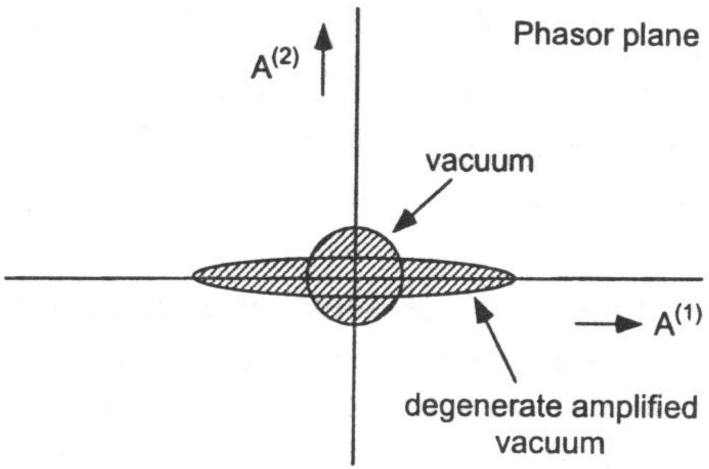


C. Macklin et al. Science 2015;350:307-310

Applications of phase coherent DPAs

- Production of squeezed states
- QND measurements





Quantum treatment:

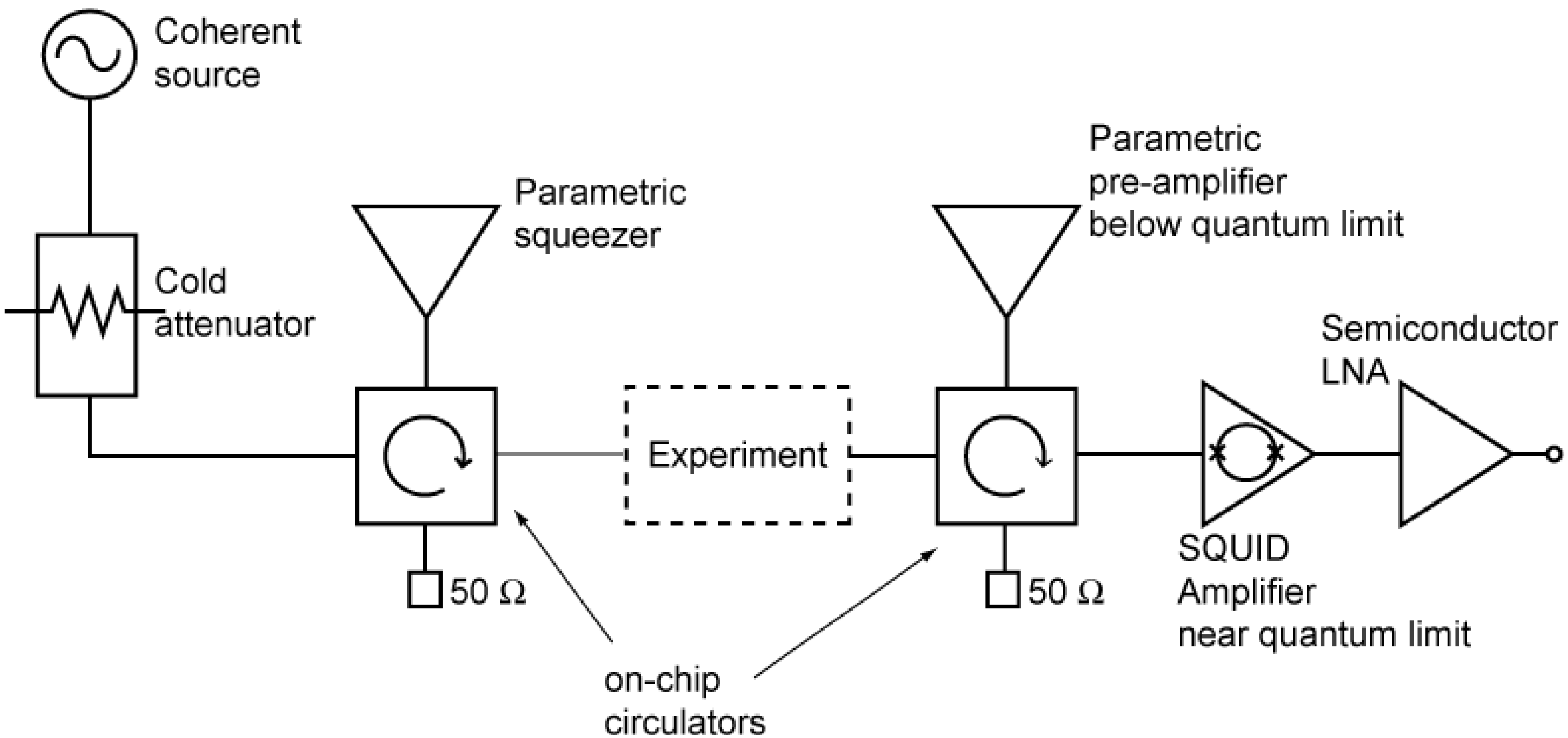
$$G(\varphi) = 2G - 1 + 2G^{1/2}(G-1)^{1/2}\cos 2\varphi$$

$$G_{\max} = 2G - 1 + 2G^{1/2}(G-1)^{1/2}$$

$$G_{\min} = 2G - 1 - 2G^{1/2}(G-1)^{1/2}$$

$$G_{\min}G_{\max} = 1$$

Amplification: Ultimate scheme



B. Yurke in late 80'ies

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H. Haus, *Electromagnetic Noise and Quantum Optical Measurements*

G.S. Agarwal, *Quantum Optics*

L. Blackwell, K. Kotzebue, *Semiconductor-Diode Parametric Amplifiers*