Model Solutions 9

1. (a) Car owners care about only their own benefit when choosing a car. Let's first formulate the expected private benefit from choosing a Big car and for choosing a Normal car as a function of the share of big cars (s_B) in the market:

$$\begin{aligned}\text{Big car: } E[V_B] &= P(\text{accident}) \times P(\text{serious}|\text{accident}) \times C_{\text{serious}} - C_B \\ &= 0.1 s_B \times (-1000) - 20 \\ &= -100 s_B - 20\end{aligned}$$

$$\begin{aligned}\text{Normal car: } E[V_N] &= P(\text{accident}) \times P(\text{serious}|\text{accident}) \times C_{\text{serious}} \\ &= 0.1(s_B + 0.5(1 - s_B)) \times (-1000) \\ &= -50 s_B - 50\end{aligned}$$

In equilibrium, the expected benefit of owning a Big car needs to equal that of owning a Normal car ($E[V_B] = E[V_N]$):

$$\begin{aligned}-100 s_B - 20 &= -50 s_B - 50 \\ s_B &= \frac{3}{5}\end{aligned}$$

60% of cars will be Big. Beyond that point, the value of extra safety of the Big car to the owner is lower than the extra cost of buying a Big car.

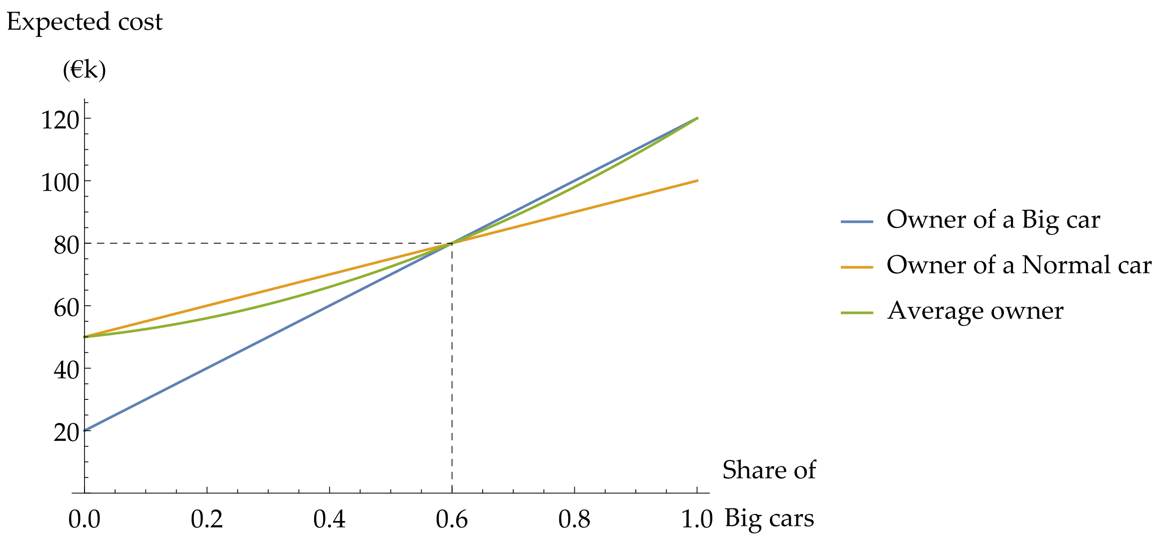

Figure 1: The expected cost of owning a Big vs. a Normal car.

(b) Since Big cars cost more than Normal cars and since all of the extra safety they provide comes at the expense of other road users (a negative externality), it seems evident that the fraction of Big cars should be zero. This can be shown also mathematically:

$$\begin{aligned}\text{Expected Total Surplus: } E[TS] &= s_B(E[V_B]) + (1-s_B)(E[V_N]) \\ &= s_B(-100s_B - 20) + (1-s_B)(-50s_B - 50) \\ &= -50s_B^2 - 20s_B - 50\end{aligned}$$

Differentiate wrt. s_B

$$\frac{\partial E[TS]}{\partial s_B} = -100s_B - 20$$

Since the derivative is always negative for non-negative shares of Big cars, the optimal share is zero.

(c) Now the expected private benefit from owning a Big car is higher, since it provides comfort and not only additional safety. Here it is useful to understand that we can order the consumers based on their valuation for the added comfort provided by a Big car, starting from those with the highest valuations. Then, we can formulate the expected benefit of owning a Big car for the marginal consumer:

$$\begin{aligned}E[V_B] &= -100s_B - 20 + \underbrace{40(1-s_B)}_{\text{the premium on comfort for the marginal consumer}} \\ &= -140s_B + 20\end{aligned}$$

The expected private benefit of owning a Normal car is the same as before. Let's solve for the equilibrium share of Big cars:

$$\begin{aligned}-140s_B + 20 &= -50s_B - 50 \\ s_B &= 7/9 \approx 78\%\end{aligned}$$

2. This is a network externality exercise, where the expected value of the country club for each members increases in the number of members.[1]

(a) Let's first formulate the valuation of a potential member with ith highest valuation:

$$v(i) = 2 - \frac{2i}{1000}$$

With n users the nth highest user valuation is:

$$p^d(n) = nv(n) = 2n - \frac{2n^2}{1000}$$

[1] The exercise is solved for valuations that vary between \$0 and \$2. However, full points are also awarded for a correctly solved exercise with valuations between \$0 and \$1. There you'd need to show that the candidate optimum club size would be 612, but this is not enough for the club to break even, so it closes down.

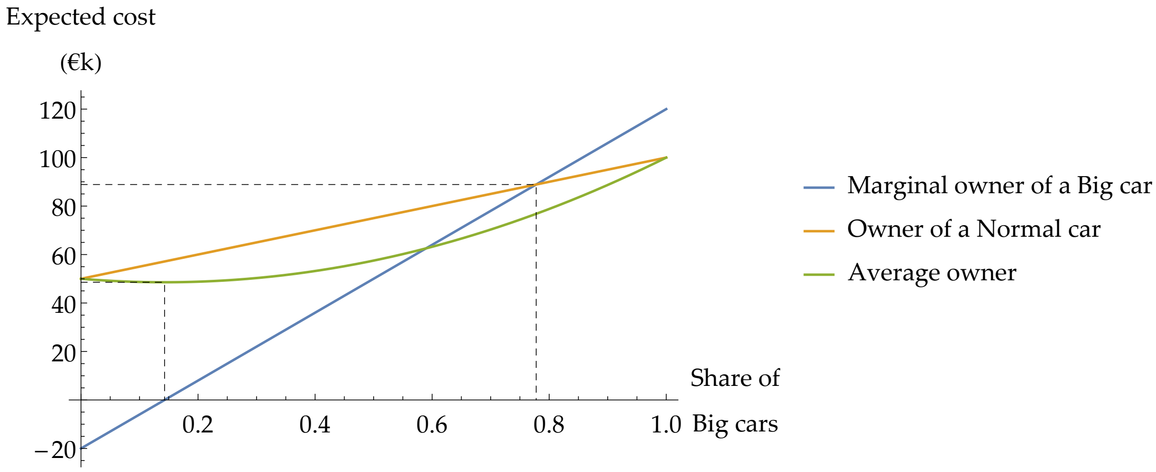

Figure 2: The expected cost of owning a Big vs. a Normal car, when consumers have different valuations for comfort.

This is how much the nth user is willing to pay for club membership, given that there are already n-1 members in the club. Let's then formulate the revenue that the club gets from n users:

$$R(n) = np^d(n) = 2n^2 - \frac{2n^3}{1000}$$

The country club profits are:

$$\Pi(n) = R(n) - \text{VC}(n) - \text{FC} = 2n^2 - \frac{2n^3}{1000} - 100n - 100\,000$$

Let's differentiate wrt. n and solve the resulting quadratic equation to get the profit-maximizing number of members:

$$\frac{\partial \Pi(n)}{\partial n} = 4n - \frac{6n^2}{1000} - 100 = 0$$
$$\implies n^* \approx 641$$

And the profit-maximizing price of club membership:

$$p^* = p^d(641) = 2 \times 641 - \frac{2 \times 641^2}{1000} = \$460.43$$

(b) This is an average cost pricing problem, where it is optimal that the non-profit club prices the membership so that it makes a zero profit. Positive profits would be inefficient, because they would require a higher membership price, which would reduce total surplus because it would be socially optimal for the club to have more members at the margin.

Thus, the optimal number of members is where profits are zero:

$$\Pi(n) = 2n^2 - \frac{2n^3}{1000} - 100n - 100\,000 = 0$$

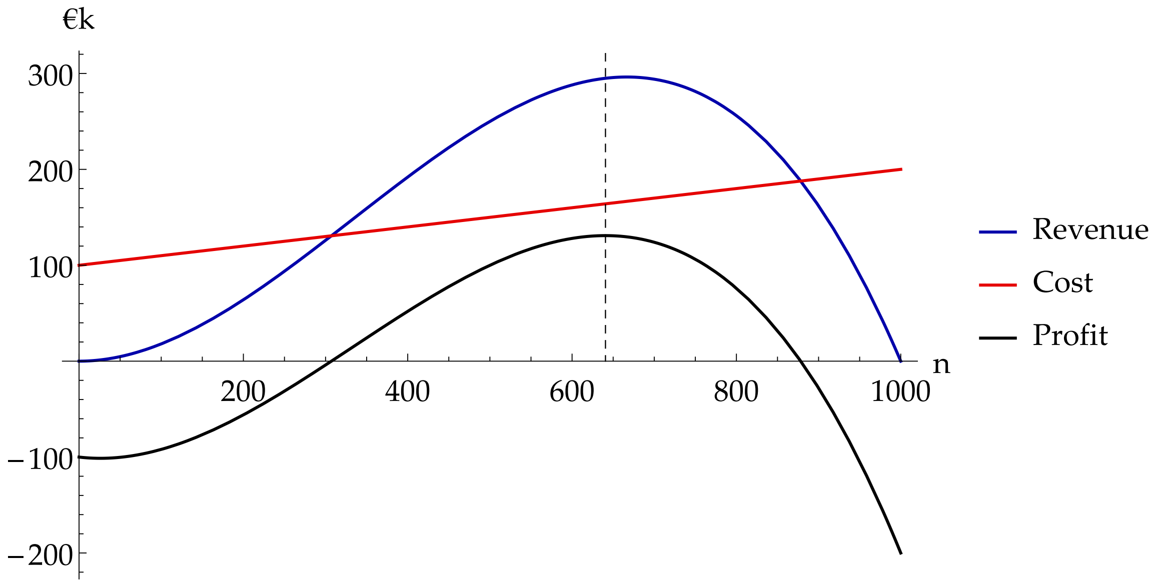

Figure 3: Revenues, costs and profit as a function of the size of membership for a profit-maximizing club.

This is a third-degree polynomial equation that can be solved with an equation solver. Profits are (slightly above) zero, when $n = 878$. The price of the membership is:

$$p^* = p^d(878) = 2 \times 878 - \frac{2 \times 878^2}{1000} = \$214.23$$

Club membership is higher and the price of membership is lower than with a profit-maximizing club.

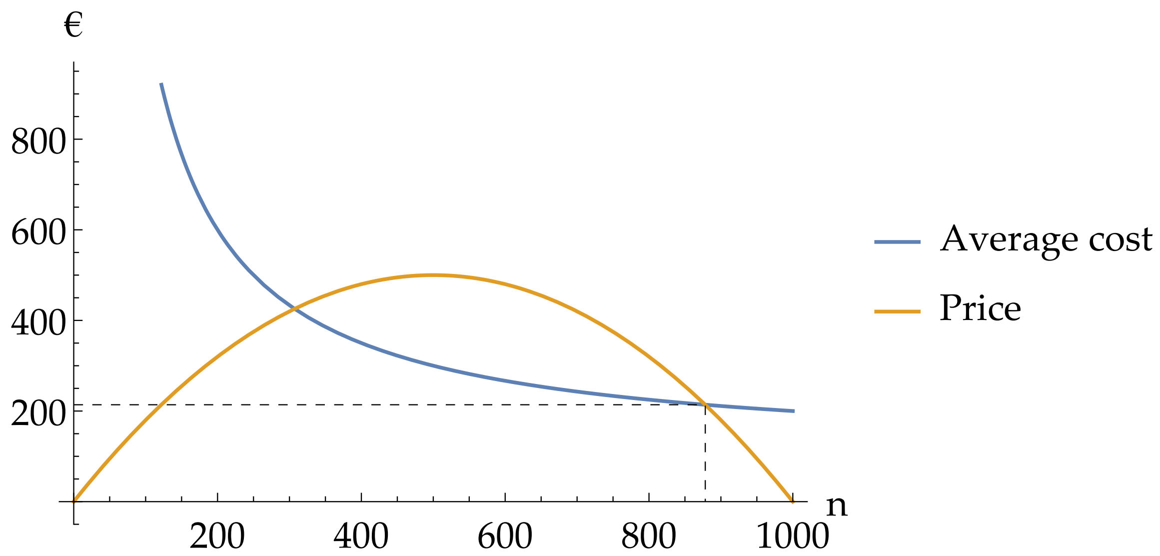

Figure 4: Average cost and price as functions of the size of club membership.

Since profits are zero at the optimum, the net total surplus is equal to the aggregate consumer surplus at the efficient number of club members ($n = 878$). (There is a tiny deviation from zero due to the integer constraint, but that is not economically interesting). Let's formulate the consumer surplus function, when the club has n members:

$$\text{CS}(n) = \underbrace{n}_{\text{number of members}} \times \underbrace{n}_{\text{potential meetings per member}} \times \underbrace{\frac{2 + (2 - \frac{2n}{1000})}{2}}_{\text{average valuation per member}}$$

$$= n^2(2 - \frac{n}{1000}) = 2n^2 - \frac{n^3}{1000}$$

At $n = 878$, consumer surplus is $CS(878) \approx \$865\,000$

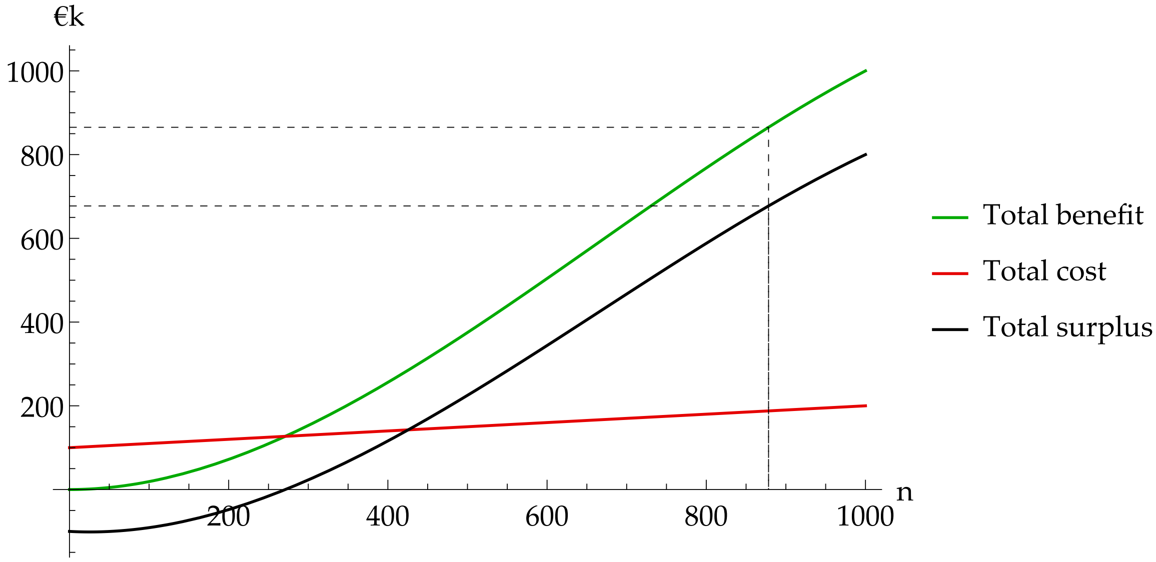

Figure 5: Consumer surplus, total cost and total surplus as a function of membership size.

(c) The question is what happens to the price of membership as population declines when the objective is to maximize total surplus subject to a balanced budget. In 2b, we saw that a non-profit club uses average cost pricing. Notice that the distribution of preferences is not affected by population size, there are just fewer potential members. Thus a larger number of members just allows the fixed cost to be divided between more people, lowering the average cost and hence the price. When the population declines then the size of membership will also decline and thus the average cost and price of membership will go up.

Additional comment. A decline in the town population would cause the profit-maximizing price to go down. The relation of prices and population is illustrated in Figure 6. Regardless of what the club's objective is, the population has to satisfy a particular minimum scale required for the club to be able to cover its fixed costs anymore. This minimum scale is the same for a profit-maximizing club and a non-profit club because, at the minimum scale, both are making zero profits. If the population is below the minimum scale then the club cannot be sustained.

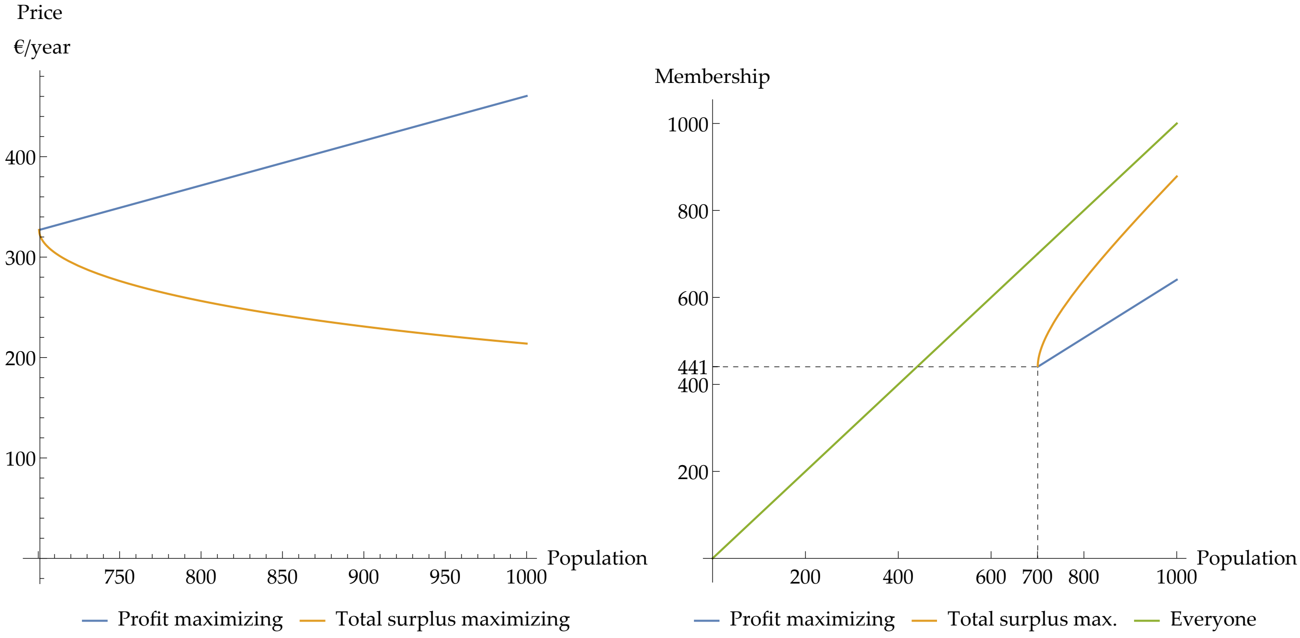

Figure 6: The optimal price of membership for a profit-maximizing and a non-profit networking club as a function of town population.

3. First, let's tabulate the productivity and outside options for each type:

\$	Productivity	Outside option 60%	Outside option 70%
High	1500	900	1050
Median	600	360	420
Low	300	180	210

(a) Firms don't know the diligence of applicants beforehand. Since the labor market is competitive, firms' expected profits are zero and thus wage equals expected productivity. First consider the case where everyone applies. The expected value for the firm is:

$$\text{EV}_{\text{all}} = \frac{1}{3}(1500 + 600 + 300) = \$800$$

Compare this to the 60% outside option of high diligence applicants: $\text{EV}_{\text{all}} = \$800 < \$900$. Since the wage offer would be smaller than the outside option, the high diligence technicians will be self-employed. If only median and low diligence technicians apply, $\text{EV}_{\text{median\&low}} = \frac{1}{2}(600 + 300) = \$450$. With this wage offer, both types are hired by the firms. Average earnings are $\frac{1}{3}900 + \frac{2}{3}450 = \$600$.

Adverse selection causes a partial unraveling, because high type technicians opt out of the more productive employment due to asymmetric information. However, for the median and lower types, the outside option is sufficiently low and their productivity sufficiently close to each other, so that even though firms undervalue the median types' productivity they can still offer a wage above what the median types would earn in self-employment.

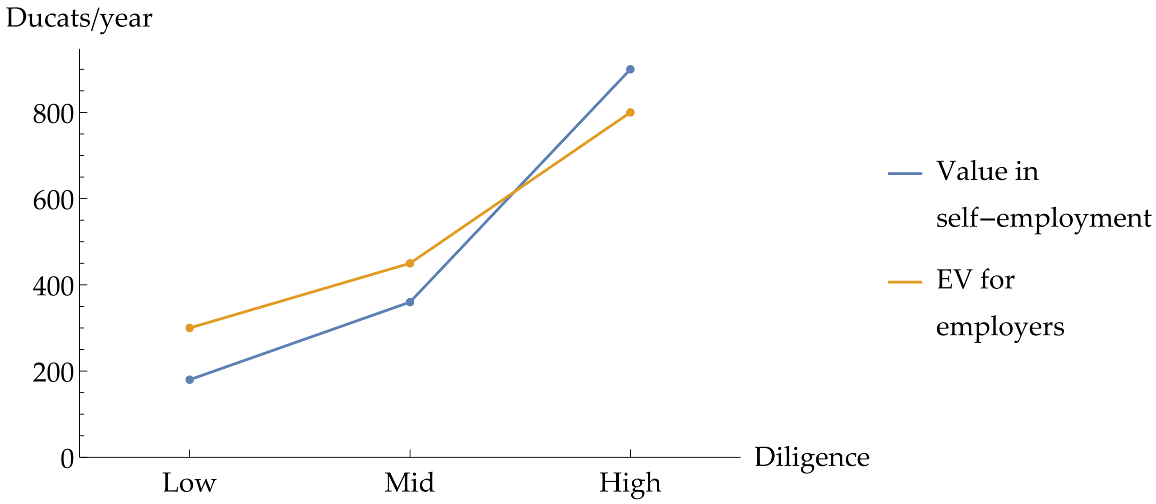

Figure 7: Adverse selection in problem 3a. "EV for employers" shows the expected worker productivity (and wage) if the worker type on horizontal axes and all lower types were pooled together. If a worker type can get a higher pay by "going it alone" in self-employment then this worker type will not in equilibrium accept employment in a firm. Here high diligence types will be self-employed in equilibrium.

(b) The situation is otherwise identical, but the value of the outside option has increased to 70% of the productivity at the firm. High-type employees will still choose the outside option, since $EV_{\text{all}} = \$800 < \$1050$. Median and low type employees prefer the offer of the firm, which is the same as before at \$450, while the outside option is worth only \$420 for the median types. Average earnings are higher: $\frac{1}{3}1050 + \frac{2}{3}450 = \$650$. Again, adverse selection causes a partial unraveling of the labor market.

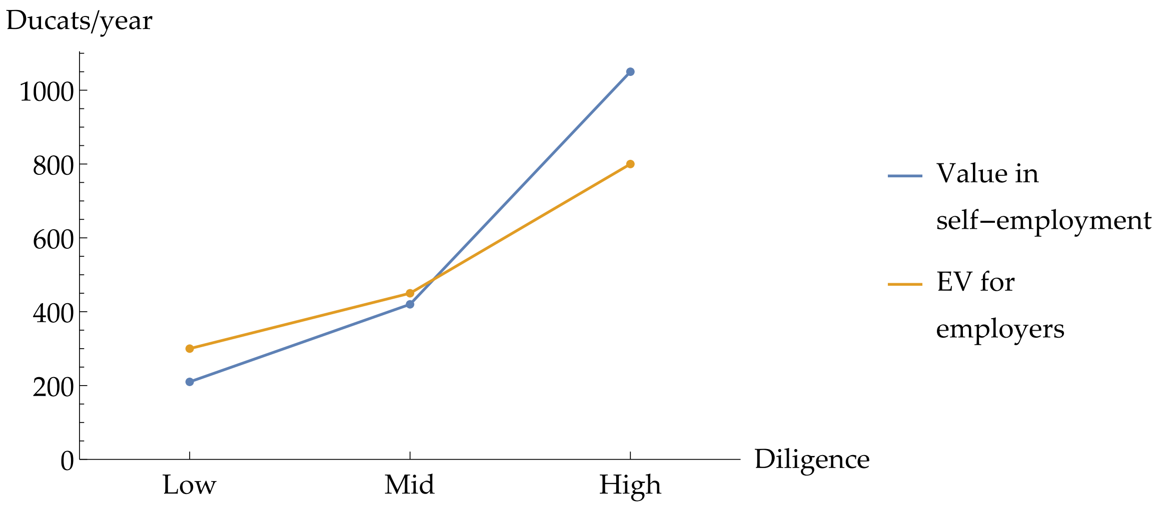

Figure 8: Adverse selection in problem 3b.

4. (a) This is a signaling problem, in which the highly skilled chefs want to signal to potential customers that the cakes they make are delicious. Since decorating a cake is less costly to them than the ordinary chefs, let's show that in equilibrium, the skilled chefs decorate their cakes and the ordinary ones don't.

To show this, let's first verify that ordinary chefs are better off by not decorating their cakes than by trying to imitate the highly-skilled chefs by decorating:

$$\begin{aligned} V_{\text{plain}} - \text{MC} &\geq V_{\text{delicious}} - \text{MC} - C^{\text{O}}_{\text{decoration}} \\ 10 - 5 &\geq 36 - 5 - 30 \\ 5 &\geq 1 \end{aligned}$$

Indeed, ordinary chefs get more profit by selling plain-looking, plain tasting cakes. This means that highly skilled chefs can signal their ability by decorating their cakes. Let's still verify that they earn more by decorating their cakes than by not decorating them:

$$\begin{aligned} V_{\text{delicious}} - \text{MC} - C^{\text{H}}_{\text{decoration}} &\geq E[V_{\text{cake}}] - \text{MC} \\ 36 - 5 - 10 &\geq \frac{36 + 10}{2} - 5 \\ 21 &\geq 18 \end{aligned}$$

Since ordinary chefs are better off by not decorating their cakes and highly skilled are better off by decorating their cakes, there is indeed an equilibrium in which cakes with complex decorations are delicious and plain-looking cakes taste plain.

(b) The equilibrium breaks apart when ordinary chefs have an incentive to also start decorating their cakes. This happens, when:

$$\begin{aligned} V_{\text{plain}} - \text{MC} &< V_{\text{delicious}} - \text{MC} - C^{\text{O}}_{\text{decoration}} \\ 10 - 5 &< V - 5 - 30 \\ V &> 40 \end{aligned}$$

Since ordinary chefs have an incentive to start making complex decorations when buyer valuation for delicious cakes exceeds 40, highly skilled chefs cannot signal the superior taste of their products anymore. Decorating cakes becomes useless so nobody makes complex decorations anymore, buyers cannot distinguish between delicious and plain-tasting cakes and are willing to pay the expected value $\frac{V+10}{2}$ for any cake.