Ilmonen/ Shafik/ Pere/ Mellin

6. Theoretical exercises

Demo exercises

6.1 Show that the optimal mean squared error prediction for the stationary and invertible ARIMA(0,1,1) process,

$$Dx_t = \varepsilon_t + \theta_1 \varepsilon_{t-1}, \quad (\varepsilon_t)_{t\in T} \sim \text{iid}\big(0, \sigma^2\big), \tag{1}$$

where ε_s and x_t are independent for $s > t$, satisfies the formula,

$$\hat{x}_{t+1|t} = \alpha x_t + (1-\alpha)\hat{x}_{t|t-1},$$

of exponential smoothing when $|\theta_1| < 1$ and $\alpha = 1 + \theta_1$.

Solution. We consider the optimal mean squared prediction of the ARIMA(0,1,1) process,

$$\hat{x}_{t+1|t} = \mathbb{E}\left[x_{t+1} \mid \varepsilon_t, \varepsilon_{t-1}, ...\right].$$

Since

$$Dx_t = x_t - x_{t-1} = \varepsilon_t + \theta_1 \varepsilon_{t-1}, \quad t \in T,$$

we have that,

$$x_{t+1} = x_t + \varepsilon_{t+1} + \theta_1 \varepsilon_t.$$

The optimal prediction in the sense of the mean squared error is

$$\begin{aligned}\hat{x}_{t+1|t} &= \mathbb{E}\big[x_{t+1} \mid \varepsilon_t, \varepsilon_{t-1}, ...\big] = \mathbb{E}\big[x_t + \varepsilon_{t+1} + \theta_1\varepsilon_t \mid \varepsilon_t, \varepsilon_{t-1}, ...\big] \\ &= x_t + \theta_1 \varepsilon_t.\end{aligned}$$

Then, by combining Equation (1) with $\hat{x}_{t|t-1} = x_{t-1} + \theta_1 \varepsilon_{t-1}$, we get that $\varepsilon_t = x_t - \hat{x}_{t|t-1}$. Thus,

$$\begin{aligned}\hat{x}_{t+1|t} &= x_t + \theta_1 \varepsilon_t = x_t + \theta_1\big(x_t - \hat{x}_{t|t-1}\big) \\ &= (1+\theta_1)x_t - \theta_1 \hat{x}_{t|t-1} = \alpha x_t + (1-\alpha)\hat{x}_{t|t-1},\end{aligned}$$

which concludes the proof.

6.2 Assume that you observe a time series

$$x_0, x_1, x_2, \ldots, x_{7300}, x_{7301}, x_{7302}, x_{7303}, x_{7304}, x_{7305},$$

where

$$\begin{aligned} x_{7300} &= 201.3, \\ x_{7301} &= 219.8, \\ x_{7302} &= 241.4, \\ x_{7303} &= 262.7, \\ x_{7304} &= 281.5 \quad \text{and} \\ x_{7305} &= 300.8. \end{aligned}$$

a) Based on plotting the time series, you observe a linear trend. You manage to stationarize the series by taking a difference. The obtained differenced series is $z_1, z_2, \ldots, z_{7305}$. Give the values of the elements z_{7301}, z_{7302}, z_{7303}, z_{7304} and z_{7305} of the stationarized series.

b) After stationarization, you center the series by subtracting the sample mean $\bar{z} = 20.1$ from the observations $z_1, z_2, \ldots, z_{7305}$. The obtained stationarized and centered time series is $y_1, y_2, \ldots, y_{7305}$. Give the values of the elements y_{7301}, y_{7302}, y_{7303}, y_{7304} and y_{7305}.

c) Based on plotting the stationarized and centered and its estimated autocorrelation and partial autocorrelation -functions, you think that $y_1, y_2, \ldots, y_{7305}$ is an autoregressive process of order 2. You estimate the parameters of the process and the estimated values are $\phi_1 = 0.5$ and $\phi_2 = -0.2$. Give predictions for y_{7306}, y_{7307} and y_{7308}.

d) Using the predictions for y_{7306}, y_{7307} and y_{7308}, give predicted values for x_{7306}, x_{7307} and x_{7308}.

Solution.

a) We have that

$$z_t = Dx_t = x_t - x_{t-1}.$$

Thus

$$\begin{aligned} z_{7301} &= 219.8 - 201.3 = 18.5, \\ z_{7302} &= 241.4 - 219.8 = 21.6, \\ z_{7303} &= 262.7 - 241.4 = 21.3, \\ z_{7304} &= 281.5 - 262.7 = 18.8 \quad \text{and} \\ z_{7305} &= 300.8 - 281.5 = 19.3. \end{aligned}$$

b) We have that

$$y_t = z_t - \bar{z}.$$

Thus

$$\begin{aligned}
y_{7301} &= 18.5 - 20.1 = -1.6,\\
y_{7302} &= 21.6 - 20.1 = 1.5,\\
y_{7303} &= 21.3 - 20.1 = 1.2,\\
y_{7304} &= 18.8 - 20.1 = -1.3 \quad \text{and}\\
y_{7305} &= 19.3 - 20.1 = -0.8.
\end{aligned}$$

c) By Exercise 5.3 we have the following recursive formula for the s-step prediction of stationary AR(2) process,

$$\hat{y}_{t+s|t} = \phi_1 \hat{x}_{t-1+s|t} + \phi_2 \hat{x}_{t-2+s|t}.$$

With above formula we get that

$$\begin{aligned}
\hat{y}_{7306|7305} &= \phi_1 y_{7305} + \phi_2 y_{7304} = 0.5 \cdot (-0.8) - 0.2 \cdot (-1.3) = -0.14,\\
\hat{y}_{7307|7305} &= \phi_1 \hat{y}_{7306|7305} + \phi_2 y_{7305} = 0.5 \cdot (-0.14) - 0.2 \cdot (-0.8) = 0.09, \quad \text{and}\\
\hat{y}_{7308|7305} &= \phi_1 \hat{y}_{7307|7305} + \phi_2 \hat{y}_{7306|7305} = 0.5 \cdot 0.09 - 0.2 \cdot (-0.14) = 0.073.
\end{aligned}$$

d) We have that

$$\hat{z}_{t+s|t} = \hat{y}_{t+s|t} + \bar{z}.$$

Thus

$$\begin{aligned}
\hat{z}_{7306|7305} &= -0.14 + 20.1 = 19.96\\
\hat{z}_{7307|7305} &= 0.09 + 20.1 = 20.19 \quad \text{and}\\
\hat{z}_{7308|7305} &= 0.073 + 20.1 = 20.173
\end{aligned}$$

On the other hand,

$$\hat{x}_{t+s|t} = \hat{z}_{t+s|t} + \hat{x}_{t+s-1|t}.$$

Thus

$$\begin{aligned}
\hat{x}_{7306|7305} &= \hat{z}_{7306|7305} + x_{7305} = 19.96 + 300.8 = 320.76\\
\hat{x}_{7307|7305} &= \hat{z}_{7307|7305} + \hat{x}_{7306|7305} = 20.19 + 320.76 = 340.95 \quad \text{and}\\
\hat{x}_{7308|7305} &= \hat{z}_{7308|7305} + \hat{x}_{7307|7305} = 20.173 + 340.95 = 361.123.
\end{aligned}$$