Aalto University
School of Electrical Engineering

ELEC-E8101: Digital and Optimal Control

Lecture 11
Introduction to Stochastic Optimal Control

Gökhan Alcan
Office number: 2575, Maarintie 8 Building
Email: gokhan.alcan@aalto.fi

Slides based on ELEC-E8101 material by Themistoklis Charalambous

In the previous lecture…

We:

- **Understood** the principle of optimality
- **Understood** the Dynamic Programming
- **Designed optimal controllers** based on LQ problem formulation

Principle of Optimality

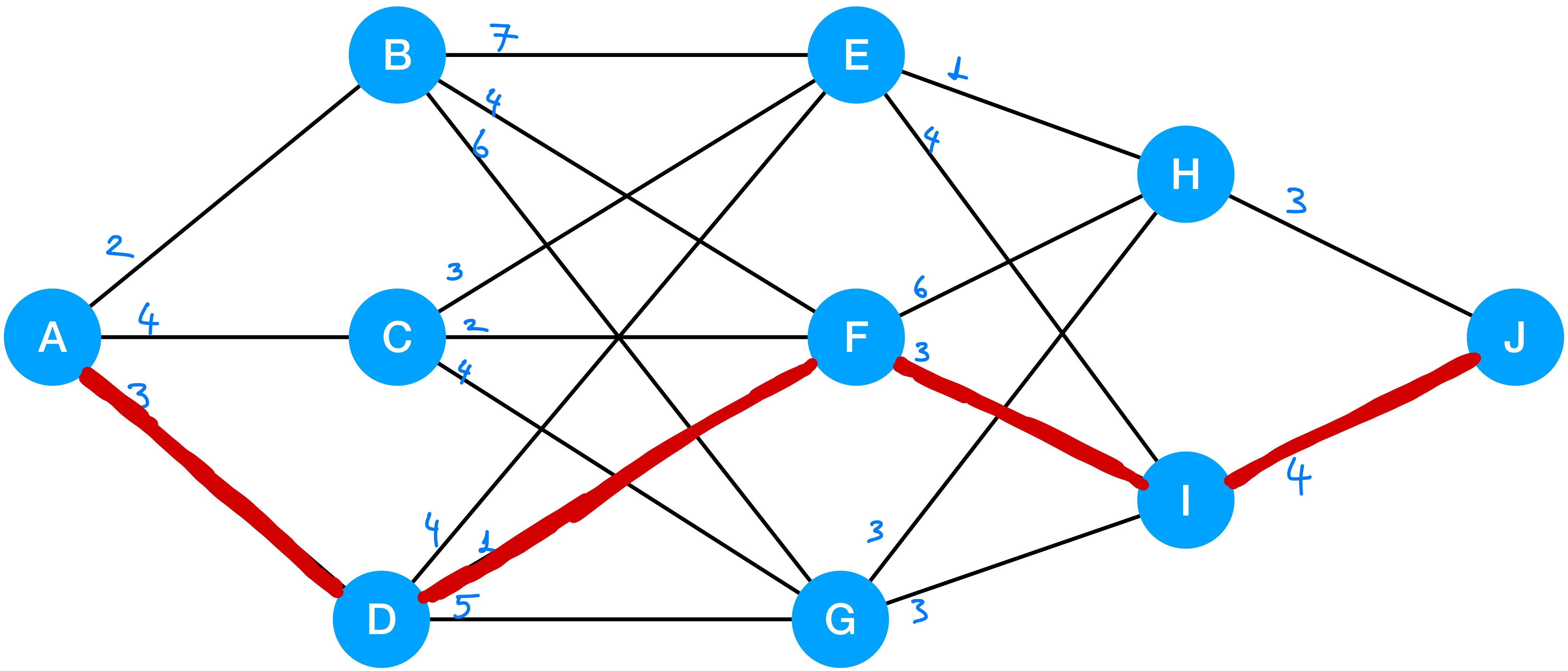
$F(J) = 0$

$F(H) = 3$
$F(I) = 4$

$F(E) = 4$
$F(F) = 7$
$F(G) = 6$

$F(B) = 11$
$F(C) = 7$
$F(D) = 8$

$F(A) = 11$



$F(.)$: cost-to-go

Principle of optimality

- Consider the following process (plant): $x_{k+1} = f_k(x_k, u_k)$
- Criterion/Cost to be minimized: $J_i(x_i) = \min_{u_i,\ldots,u_{N-1}} \left[\sum_{k=i}^{N-1} g_k(x_k, u_k) + g_N(x_N) \right]$
- Note that when the final state is free, there can be an additional cost related to that state.
- Let's use the principle of optimality:

$$J_k(x_k) = \min_{u_k} \left[g_k(x_k, u_k) + J^*_{k+1}(x_{k+1}) \right]$$

We want to find u_k such that the expression is minimized and we get the optimal cost at time k.

Solution of the discrete-time LQ problem
using Dynamic Programming

- We obtain the general solution:

$$L_k = (\Gamma^T S_{k+1} \Gamma + R)^{-1} \Gamma^T S_{k+1} \Phi$$

$$u_k^* = -L_k x_k$$

$$S_k = (\Phi - \Gamma L_k)^T S_{k+1} (\Phi - \Gamma L_k) + Q + L_k^T R L_k \quad \text{(Riccati equation)}$$

$$J_k^* = \frac{1}{2} x_k^T S_k x_k$$

- **Remarks:**
  - The Riccati equation can be written in a way that is independent of L_k

$$S_k = \Phi^T [S_{k+1} - S_{k+1} \Gamma (\Gamma^T S_{k+1} \Gamma + R)^{-1} \Gamma^T S_{k+1}] \Phi + Q$$

  - Note that S_k and L_k are calculated *"backwards in time"*. They can be calculated in advance and saved to be used when control starts at time k_0
  - The procedure matches exactly the principle of optimality!

Introduction

- **Optimal control theory:** a new and direct approach to the synthesis of these complex systems

- **Objective:** determine the control signals that allow a process to satisfy the physical constraints and at the same time optimize some performance criterion

- **In this lecture:**

  - the process is still assumed to be linear, but it may be time-varying

  - the process may have several inputs and outputs

  - *process* and *measurement noise* are introduced in the models

- The problem is formulated to minimize a criterion: a quadratic function of the states and the control signals - Linear Quadratic (LQ) control problem

- If Gaussian stochastic disturbances are allowed in the process models, then the problem is called Linear Quadratic Gaussian (LQG) control problem

Learning outcomes

By the end of *this* lecture, you should be able to:

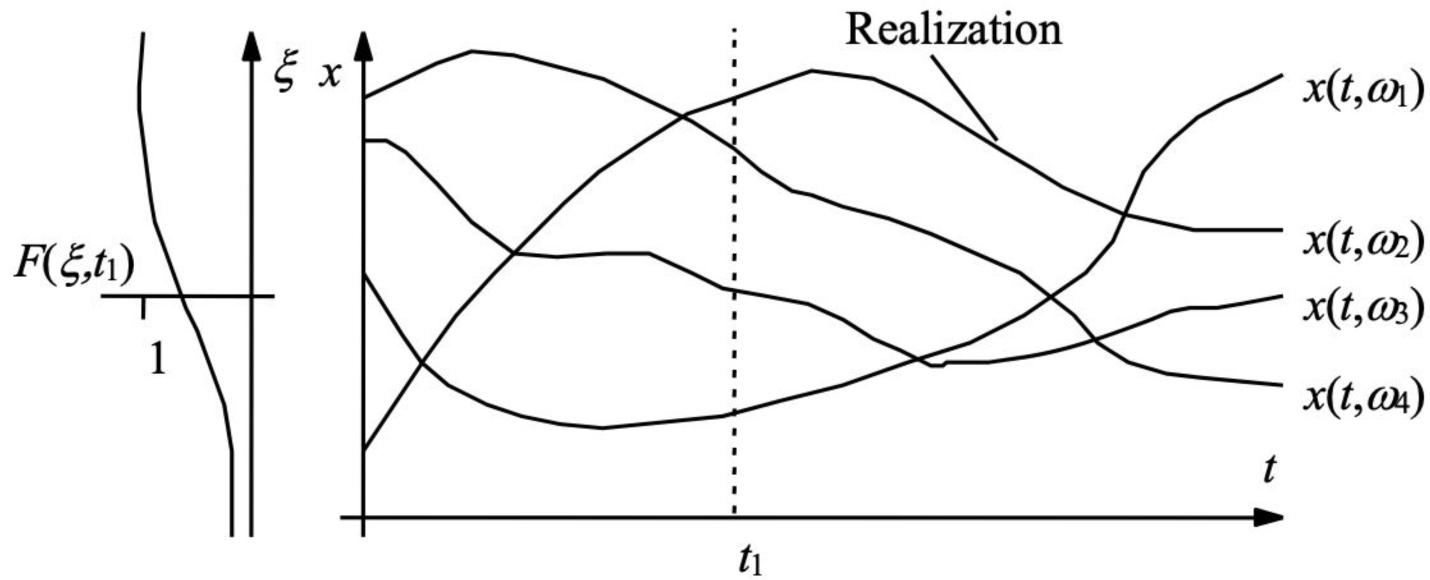
- Compute the mean and covariance matrices of dynamical processes
- Use Kalman filter and optimal control to tackle various estimation and control problems for linear systems
- Understand the Separation Principle

Stochastic models of disturbances (*recap*)

- Natural to use stochastic (random) concepts to describe disturbances
  - possible to describe a wide class of disturbances → permits good formulation of prediction problems

- A stochastic process (random process, random function) can be regarded as a family of stochastic variables $\{x[k], k \in T\}$. In this context, T is the time index

- A stochastic process may be considered as a function of 2 variables,
  - If variable ω is fixed, $x[\bullet,\omega]$ is called realization
  - If variable k is fixed, $x[k,\bullet]$ is a random variable



Concepts of stochastic processes (*recap*)

- For computing the mean and variance, the density function, can be used where

$$\int_{-\infty}^{+\infty} p(x)dx = 1$$

- The expected (or mean) value of a stochastic process x is simplified to

$$m[k] \triangleq E\{x[k]\} = \int_{-\infty}^{+\infty} xp(x)dx$$

- The variance is simplified to

$$\sigma_x^2 \equiv \mathrm{var}\{x\} = E\Big\{\big(x - E\{x\}\big)^2\Big\} = \int_{-\infty}^{+\infty} \big(x - E\{x\}\big)^2 p(x)dx$$

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Some useful properties (*recap*)

- Suppose a is constant, x and y are stochastic variables. Then

$$E\{a\} = a$$
$$E\{ax\} = aE\{x\}$$
$$E\{x + y\} = E\{x\} + E\{y\}$$

$$\mathrm{var}\{a\} = 0$$
$$\mathrm{var}\{ax\} = a^2\mathrm{var}\{x\}$$

- If x and y are independent random variables, then

$$E\{xy\} = E\{x\}E\{y\}$$

$$\mathrm{var}\{x + y\} = \mathrm{var}\{x\} + \mathrm{var}\{y\}$$

Concepts of stochastic processes

- The definitions of mean and variance are extended to vector functions; the variance is extended to *covariance*

- The expected (or mean) value of a stochastic process $\mathbf{x}$ is given by

$$\mathbf{m}[k] \triangleq E\{\mathbf{x}[k]\}$$

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$$\begin{aligned}\operatorname{var}\{\mathbf{x}[k]\} &\triangleq E\{(\mathbf{x}[k]-\mathbf{m}[k])(\mathbf{x}[k]-\mathbf{m}[k])^T\} \\ &= E\Big\{\big(\mathbf{x}[k]-E\{\mathbf{x}[k]\}\big)\big(\mathbf{x}[k]-E\{\mathbf{x}[k]\}\big)^T\Big\}\end{aligned}$$

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- The covariance function is given by

$$\begin{aligned}\mathbf{r}_{\mathbf{xx}}(s,k) \equiv \operatorname{cov}\{\mathbf{x}[s],\mathbf{x}[k]\} &\triangleq E\{(\mathbf{x}[s]-\mathbf{m}[s])(\mathbf{x}[k]-\mathbf{m}[k])^T\} \\ &= E\Big\{\big(\mathbf{x}[s]-E\{\mathbf{x}[s]\}\big)\big(\mathbf{x}[k]-E\{\mathbf{x}[k]\}\big)^T\Big\}\end{aligned}$$

Stochastic difference equations

- Consider the representation

$$\mathbf{x}[k+1] = \Phi\mathbf{x}[k] + \mathbf{v}[k]$$

where $\mathbf{v}[\mathbf{k}]$ is an independent zero-mean random variable with covariance $\boldsymbol{R_1}$

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- Suppose that the initial state has the mean $\mathbf{m_0}$ and $\mathbf{R_0}$ covariance .

Consider the behavior of $\mathbf{m}[\mathbf{k}]$ as a function of time: $\mathbf{m}[k] = E\{\mathbf{x}[k]\}$

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- The mean value behaves exactly according to system dynamics!

Stochastic difference equations

- As for the covariance function, use a new variable: $\tilde{\mathbf{x}}[k] = \mathbf{x}[k] - \mathbf{m}[k]$

- For the state covariance :

$$P[k] = \text{cov}\{\mathbf{x}[k], \mathbf{x}[k]\} = E\{\tilde{\mathbf{x}}[k]\tilde{\mathbf{x}}^T[k]\}$$

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- We want to see how the state covariance evolves over time. Towards this end:

$$\tilde{\mathbf{x}}[k+1]\tilde{\mathbf{x}}^T[k+1] = \big(\Phi\tilde{\mathbf{x}}[k] + \mathbf{v}[k]\big)\big(\Phi\tilde{\mathbf{x}}[k] + \mathbf{v}[k]\big)^T$$

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&= \left(\Phi\tilde{\mathbf{x}}[k] + \mathbf{v}[k]\right)\left(\tilde{\mathbf{x}}^T[k]\Phi^T + \mathbf{v}^T[k]\right) \\
&= \Phi\tilde{\mathbf{x}}[k]\tilde{\mathbf{x}}^T[k]\Phi^T + \Phi\tilde{\mathbf{x}}[k]\mathbf{v}^T[k] + \mathbf{v}[k]\tilde{\mathbf{x}}^T[k]\Phi^T + \mathbf{v}[k]\mathbf{v}^T[k]
\end{aligned}$$

- Take expectations in both sides:

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x and y are independent

$$E\{xy\} = E\{x\}E\{y\}$$

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$$+ \underbrace{E\{\mathbf{v}[k]\mathbf{v}^T[k]\}}_{=R_1}$$

Stochastic difference equations

- Therefore, we obtain a dynamic equation for the covariance:

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$$
\Rightarrow P[k+1] = \Phi P[k]\Phi^T + R_1, \quad P[0] = R_0
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- Consider the state auto-covariance for different values of k. For example, if :

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\mathbf{r}_{\mathbf{xx}}(k+1,k) = E\{\tilde{\mathbf{x}}[k+1]\tilde{\mathbf{x}}^T[k]\}
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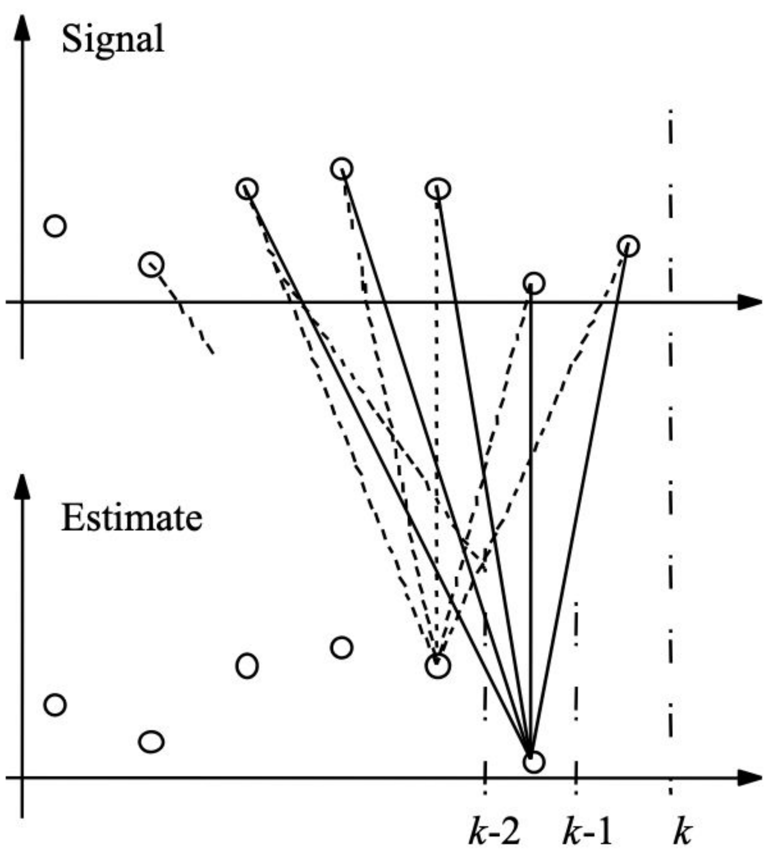
- By repeating for any value of

$$\mathbf{r}_{\mathbf{xx}}(k+\tau,k) = \Phi^\tau P[k], \quad \tau \geq 0$$

Linear Quadratic Gaussian (LQG) Control

- Three different cases can be considered:



$$\hat{x}((k-m)h \mid kh)$$

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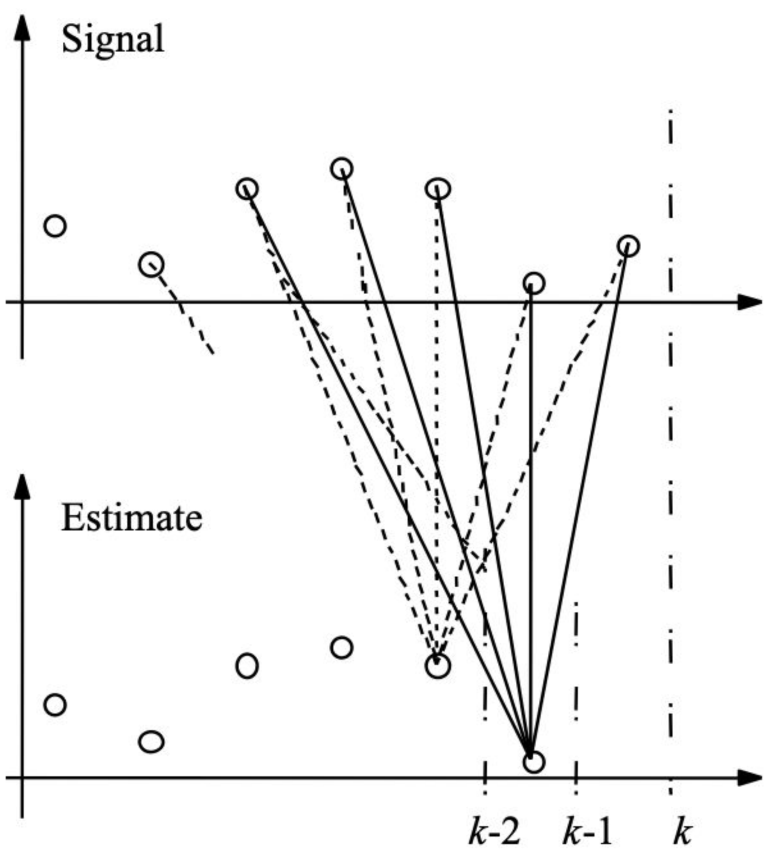

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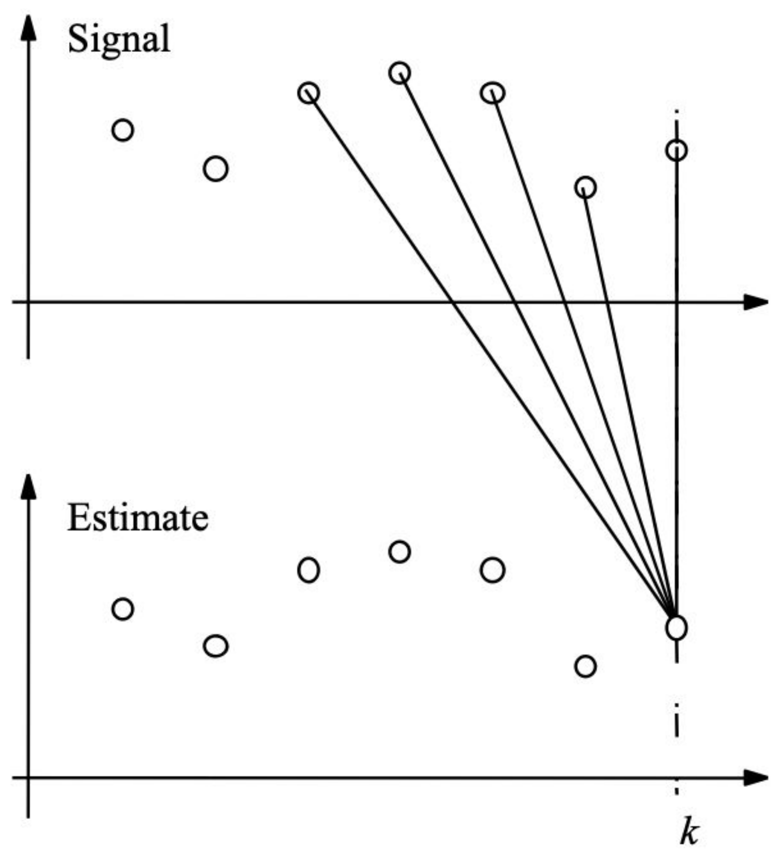
Smoothing



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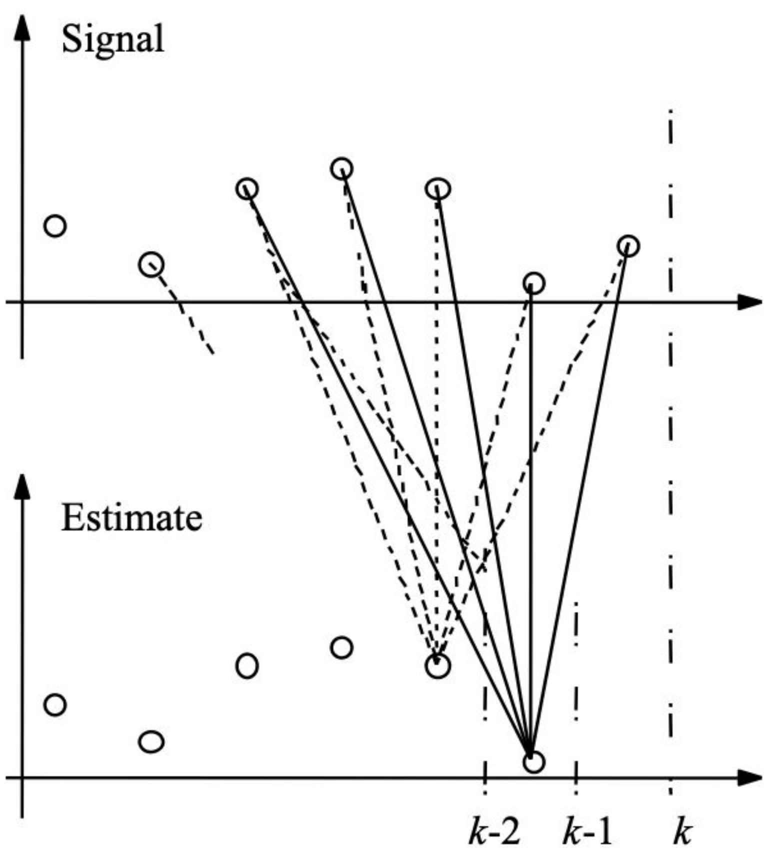

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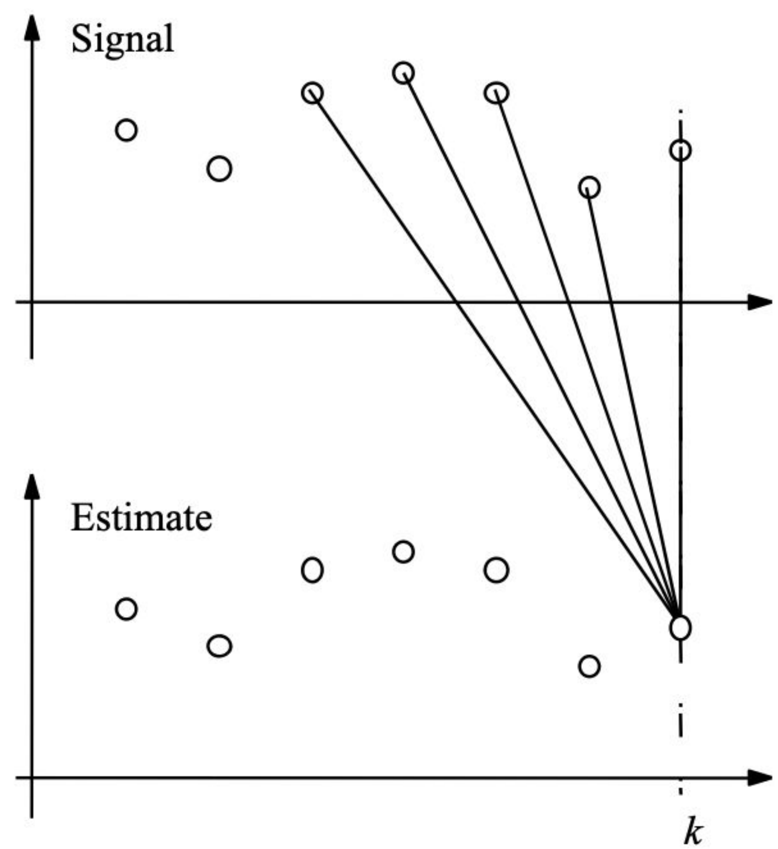

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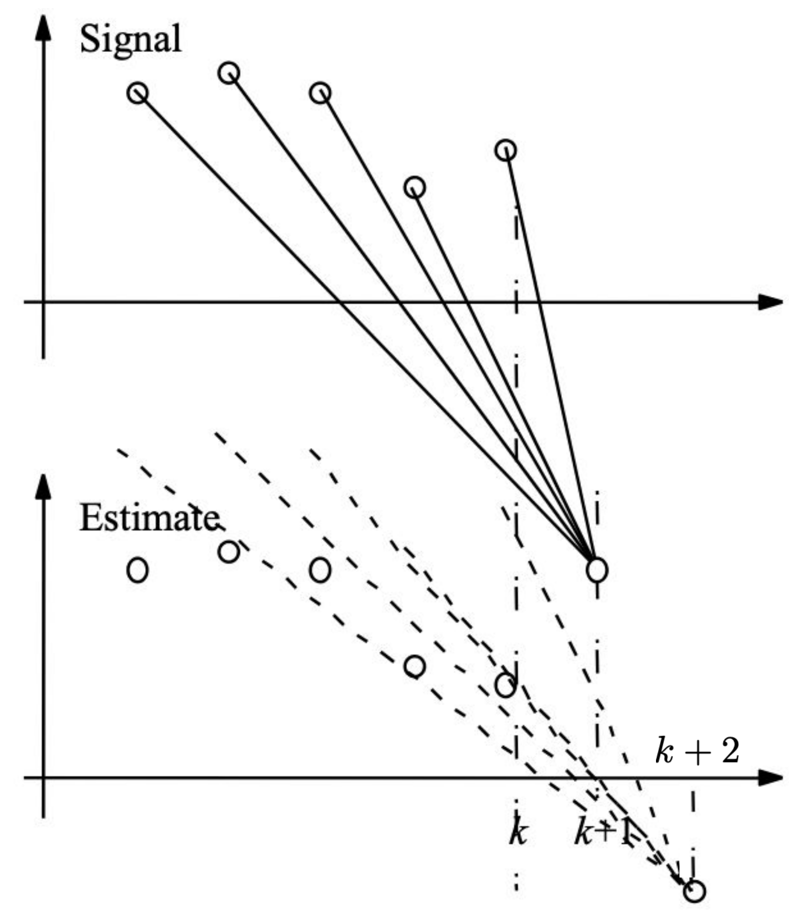
Filtering



$$\hat{x}((k+m)h \mid kh)$$

Linear Quadratic Gaussian (LQG) Control

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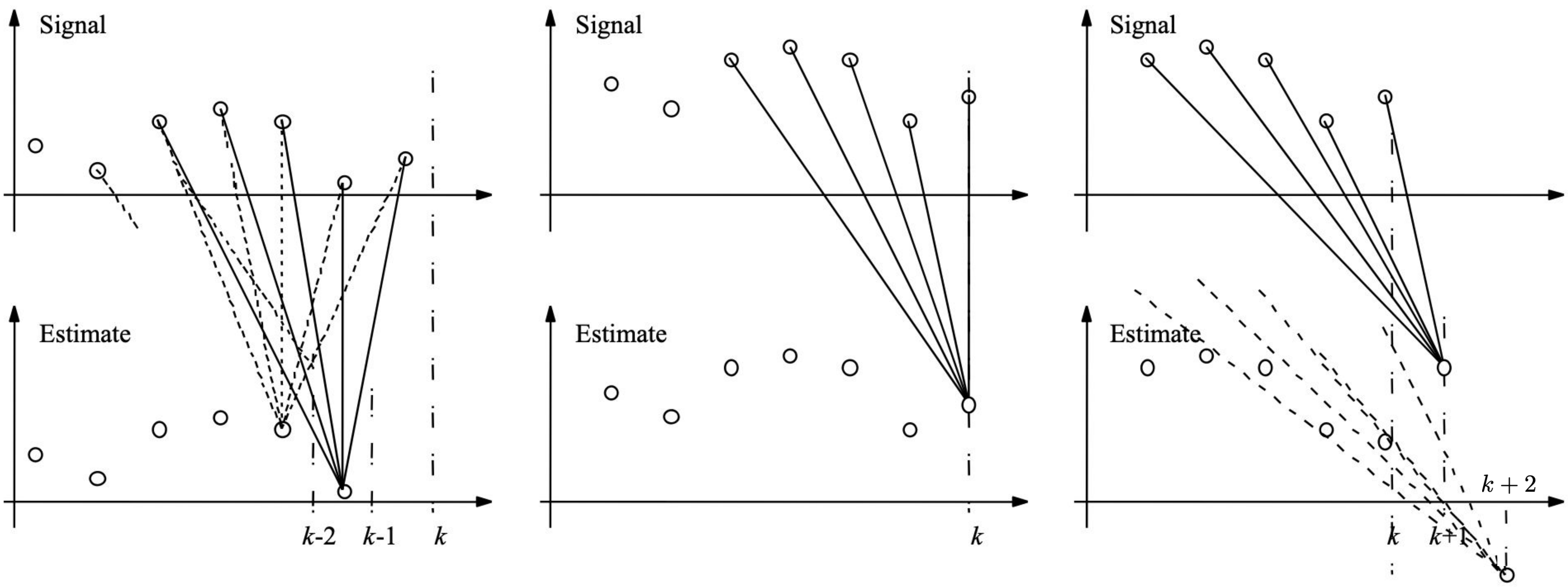

$$\hat{x}((k-m)h \mid kh) \qquad \hat{x}(kh \mid kh) \qquad \hat{x}((k+m)h \mid kh)$$

Smoothing | **Filtering** | **Prediction**

Linear Quadratic Gaussian (LQG) Control

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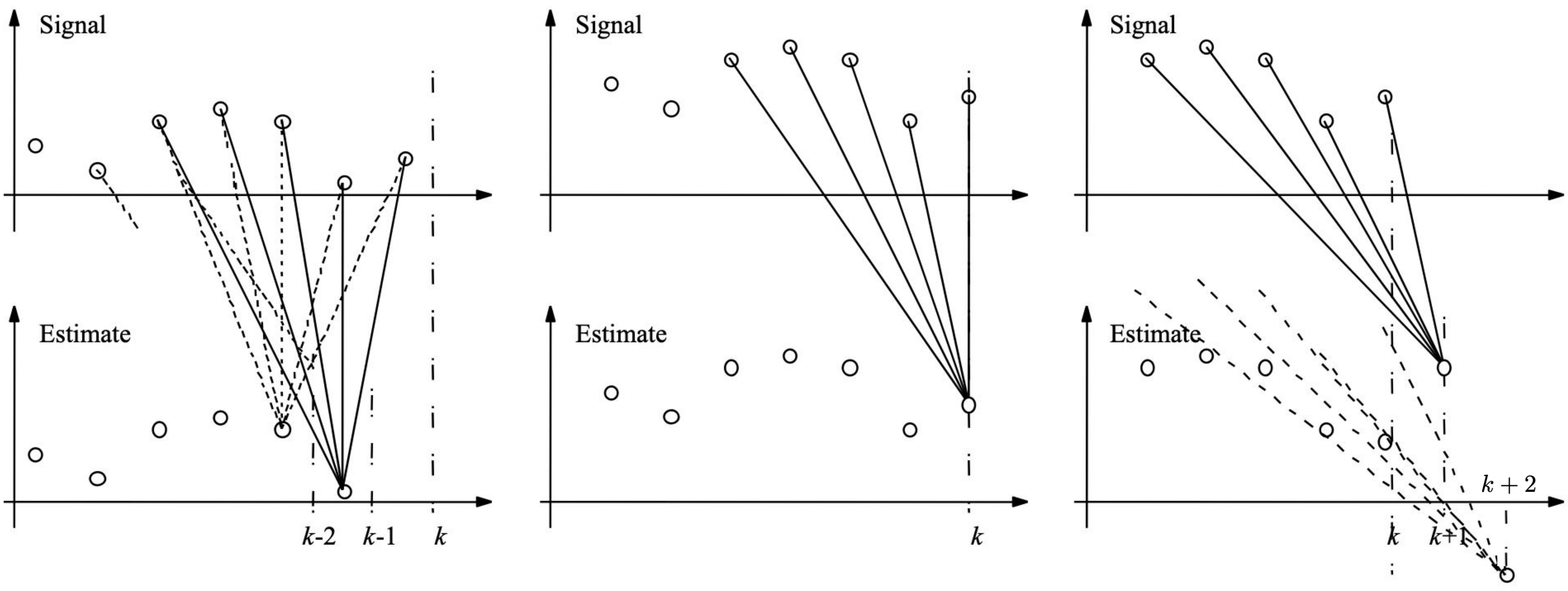


$$\hat{x}((k-m)h \mid kh)$$

Smoothing

$$\hat{x}(kh \mid kh)$$

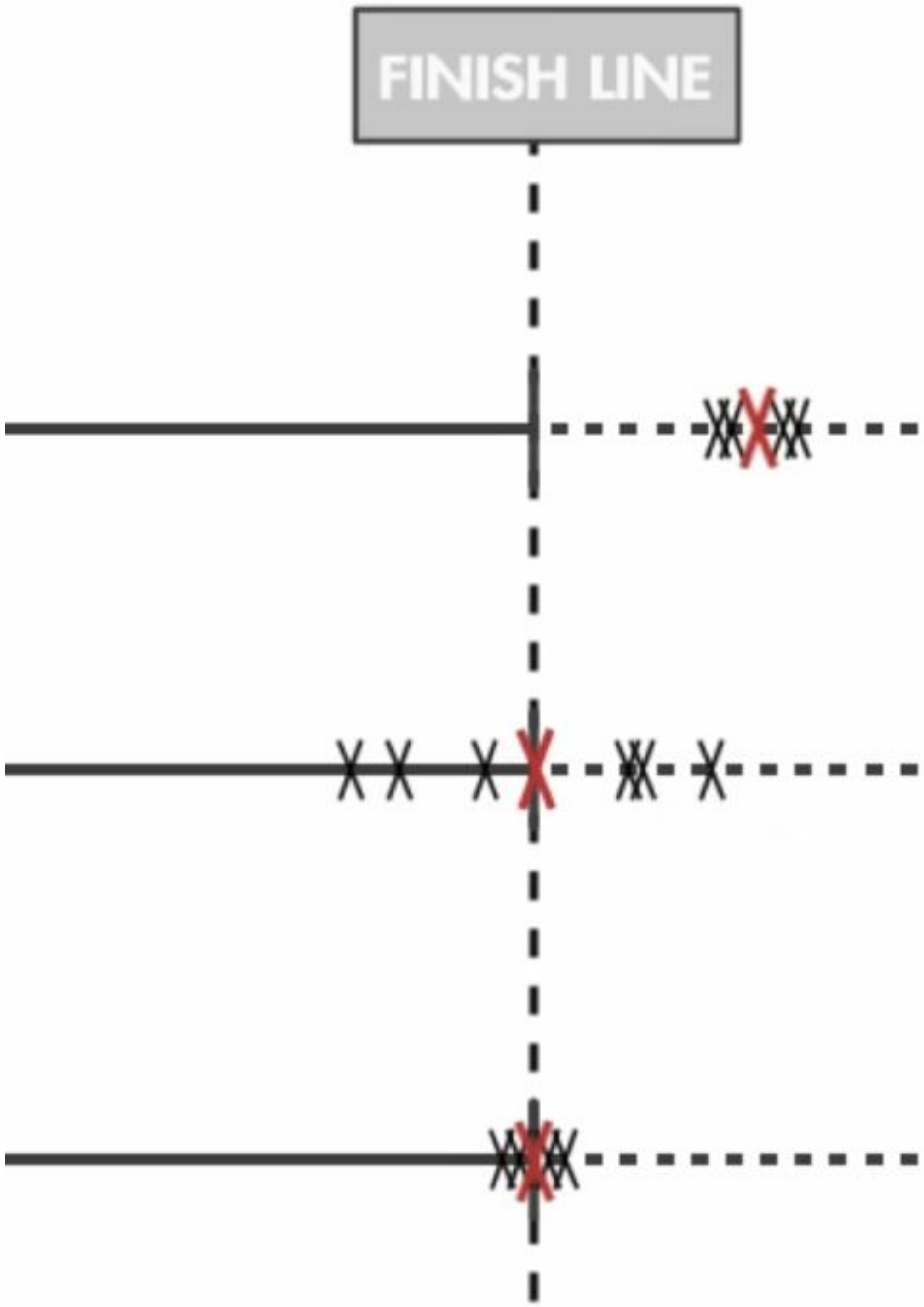
Filtering

$$\hat{x}((k+m)h \mid kh)$$

Prediction

- In LQG control, the Kalman filter is used as a predictor.

Which predictor would you choose?



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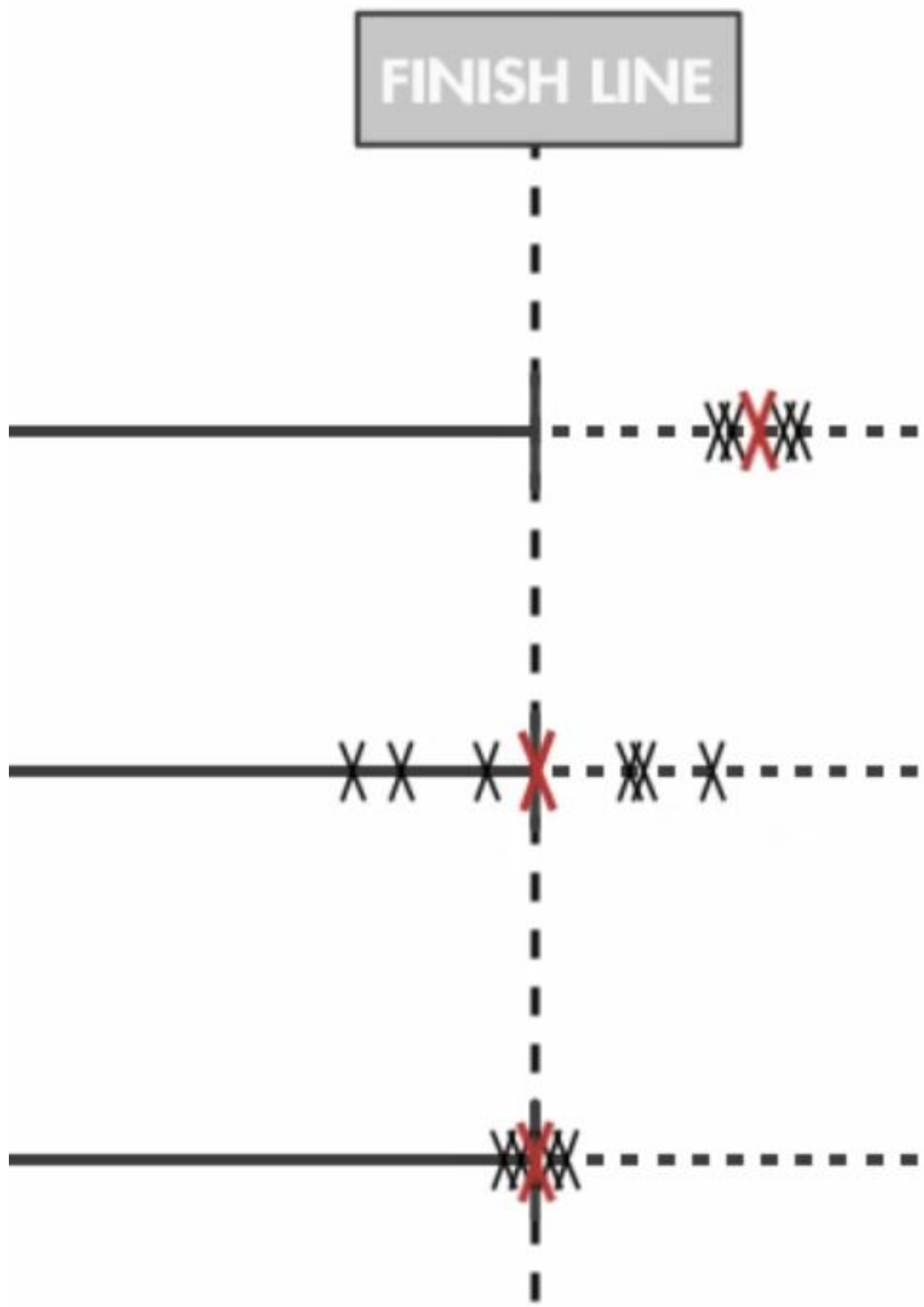
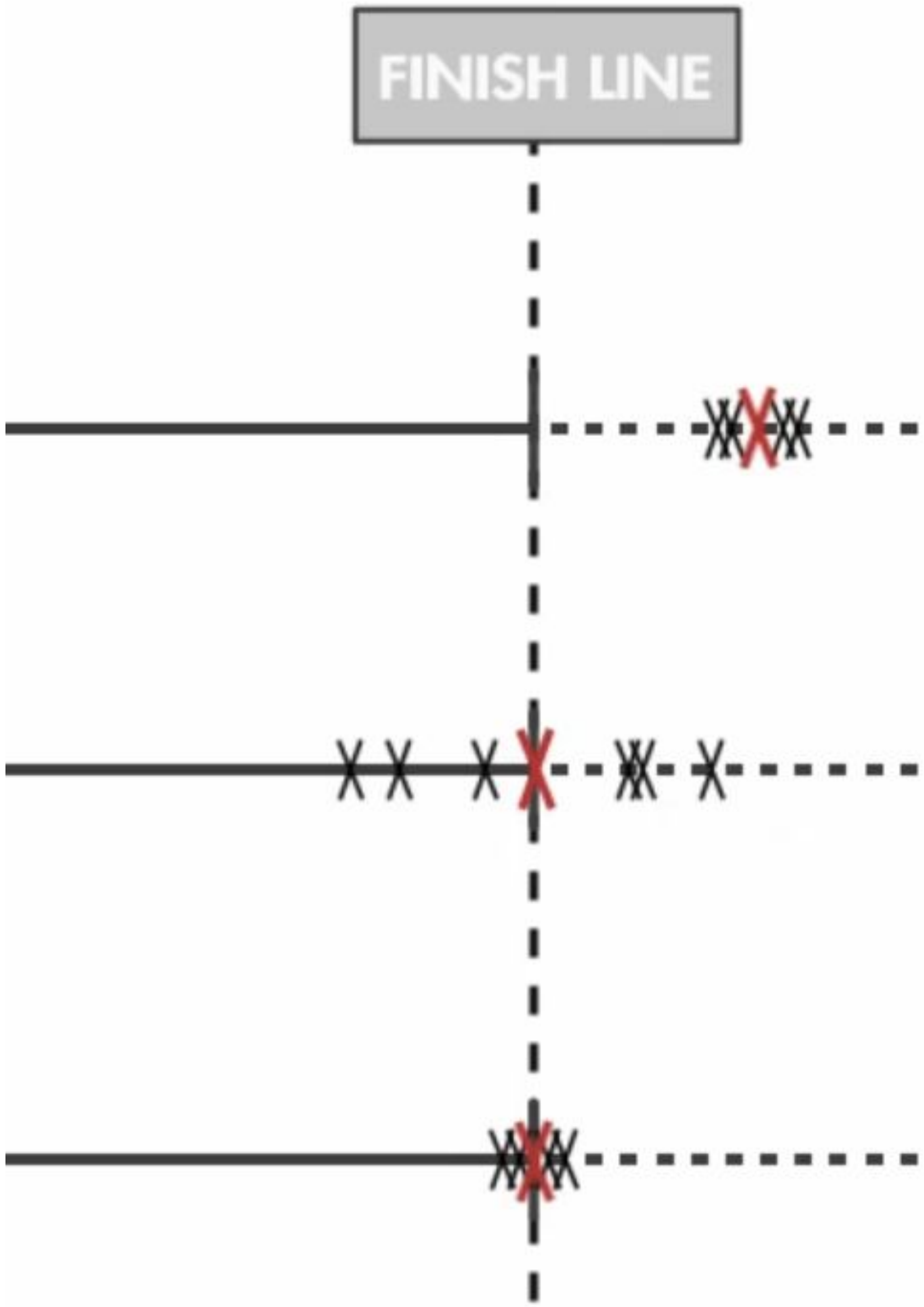


Biased, Low Variance

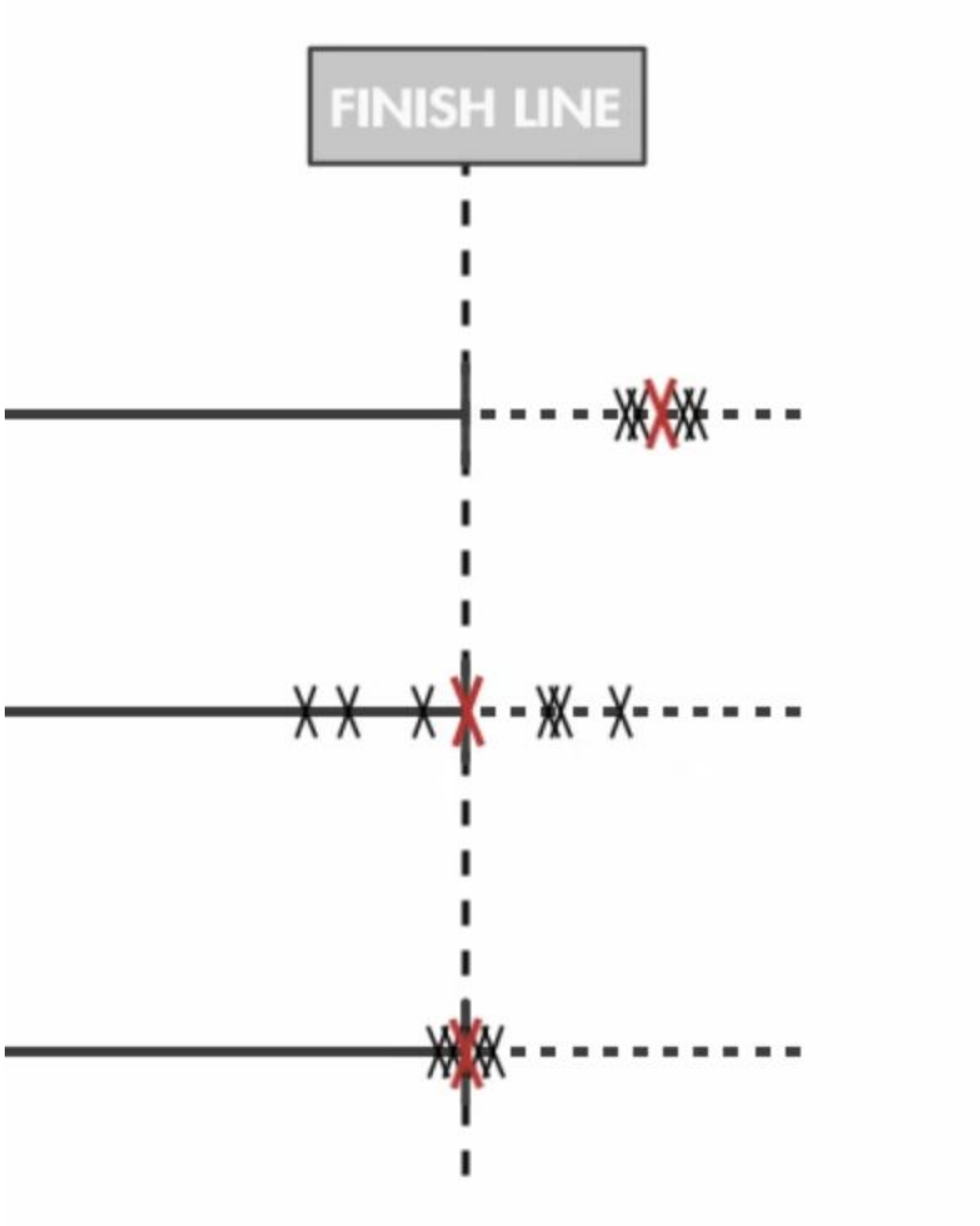
Which predictor would you choose?



Biased, Low Variance

No Bias, High Variance

Which predictor would you choose?



Biased, Low Variance

No Bias, High Variance

No Bias, Low Variance

System corrupted with process and measurement noise

- The full LQG model assumes **linear dynamics**, **quadratic costs** and **Gaussian noise**. Imperfect observation is the most important point. The model is:

$$\begin{aligned}\mathbf{x}[k+1] &= \Phi\mathbf{x}[k] + \Gamma\mathbf{u}[k] + \mathbf{v}[k],\\ \mathbf{y}[k] &= C\mathbf{x}[k] + \mathbf{e}[k]\end{aligned}$$

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where $\mathbf{v}$ and $\mathbf{e}$ are discrete-time Gaussian white noise processes with zero-mean value and

$$\left.\begin{aligned}E\{\mathbf{v}\mathbf{v}^T\} &= R_1\\ E\{\mathbf{v}\mathbf{e}^T\} &= R_{12}\\ E\{\mathbf{e}\mathbf{e}^T\} &= R_2\end{aligned}\right\} \Rightarrow \operatorname{cov}\left\{\begin{bmatrix}\mathbf{v}\\ \mathbf{e}\end{bmatrix}\right\} = E\left\{\begin{bmatrix}\mathbf{v}\\ \mathbf{e}\end{bmatrix}\begin{bmatrix}\mathbf{v}\\ \mathbf{e}\end{bmatrix}^T\right\} = \begin{bmatrix}R_1 & R_{12}\\ R_{12}^T & R_2\end{bmatrix}$$

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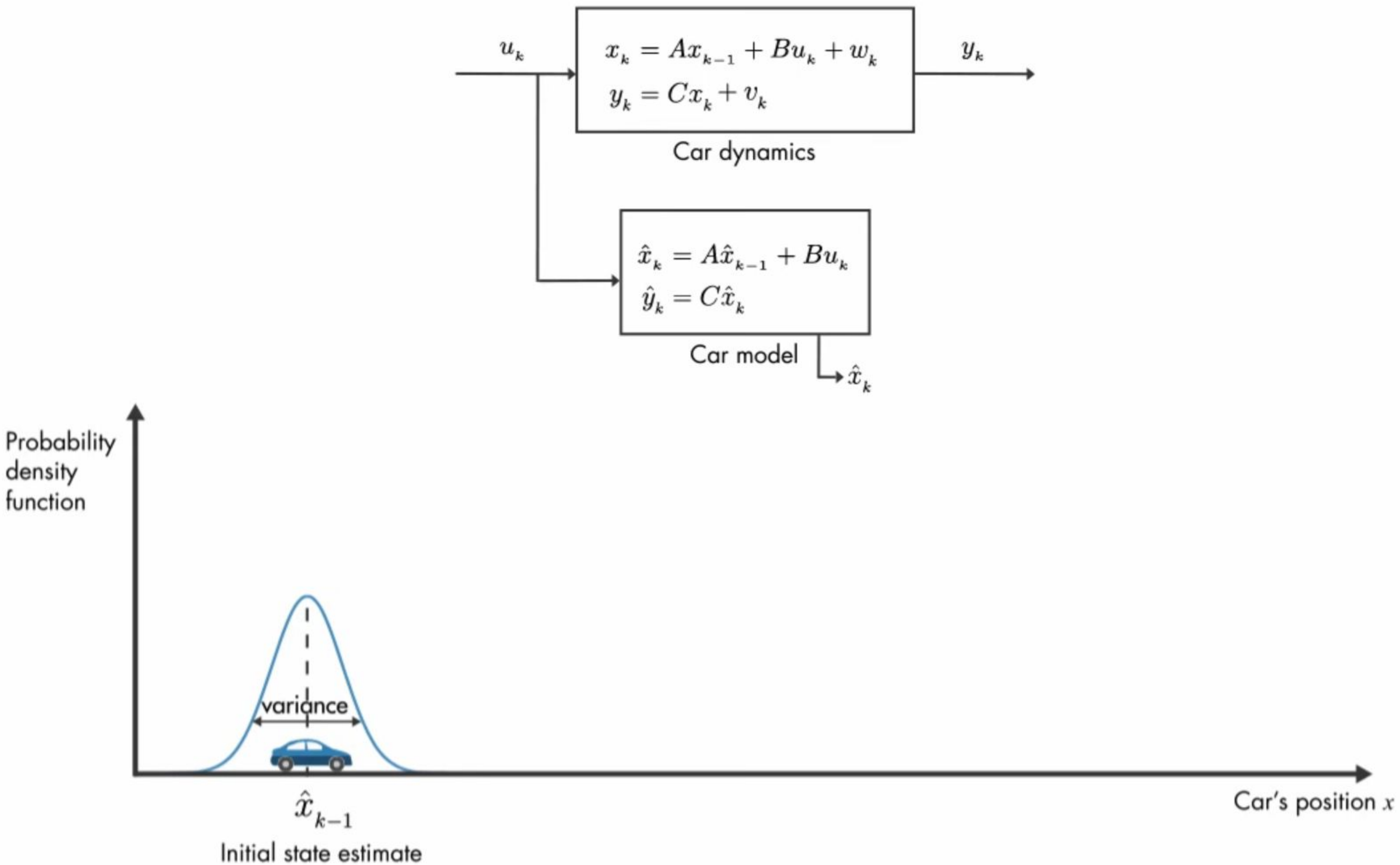
- The initial state $\mathbf{x}[0]$ is assumed to be Gaussian distributed with

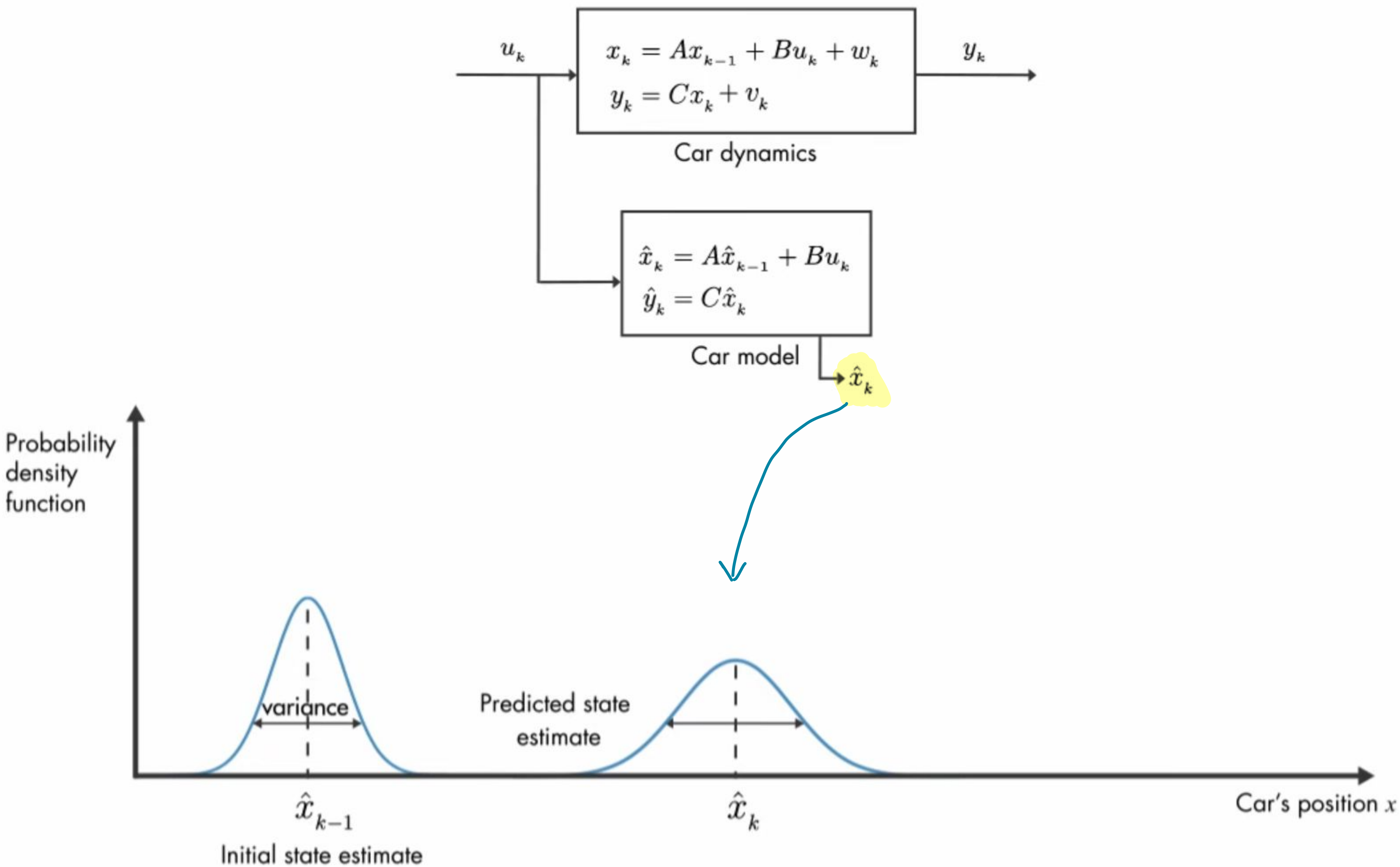
$$E\{\mathbf{x}[0]\} = m_0 \quad \operatorname{cov}\{\mathbf{x}[0]\} = R_0$$

Using standard notation from the literature, we can write $\mathbf{x}[0] \sim \mathcal{N}(m_0, R_0)$

Working principles of the Kalman filter

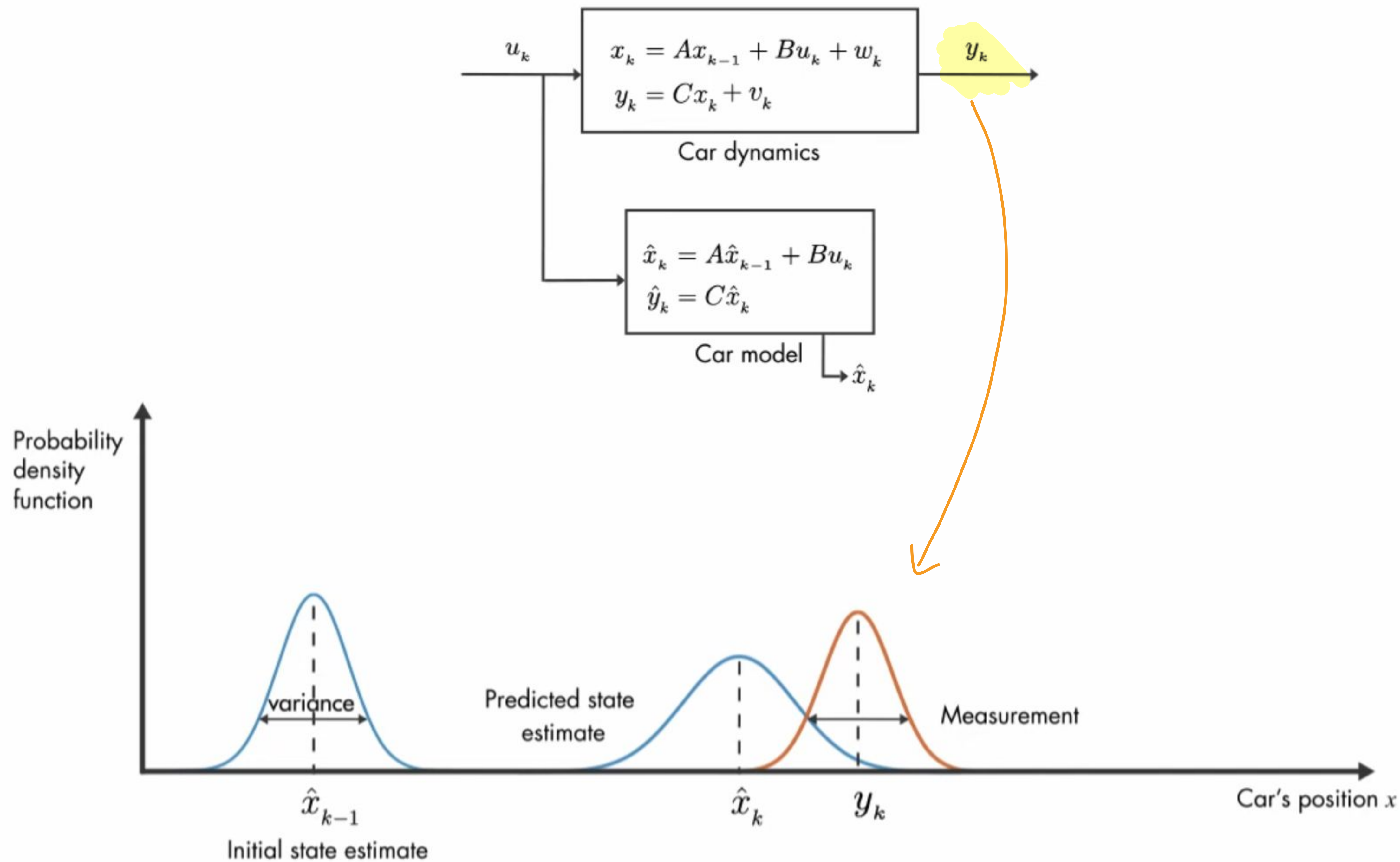


Working principles of the Kalman filter



https://www.youtube.com/watch?v=ul3u2yLPwU0

Working principles of the Kalman filter



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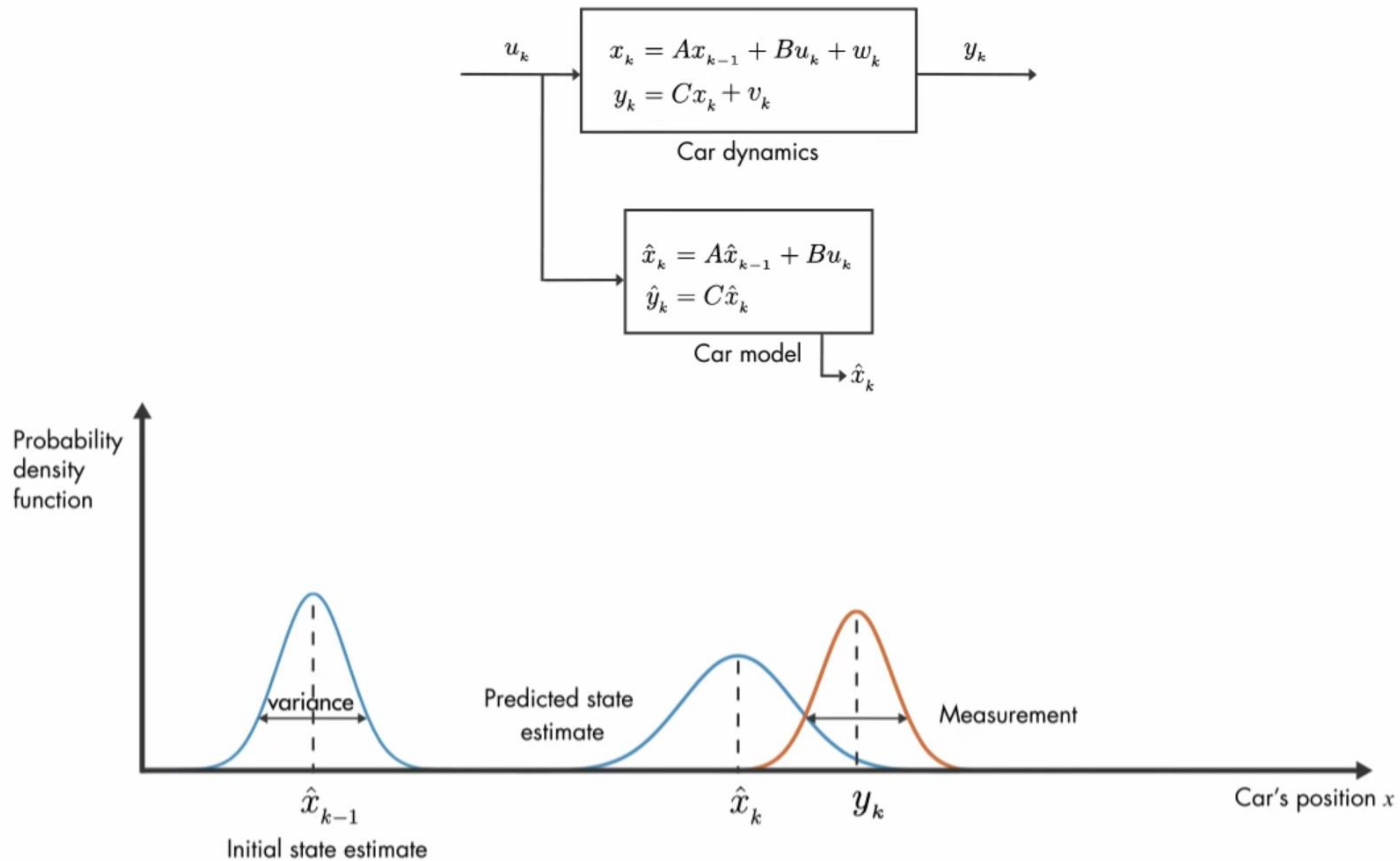

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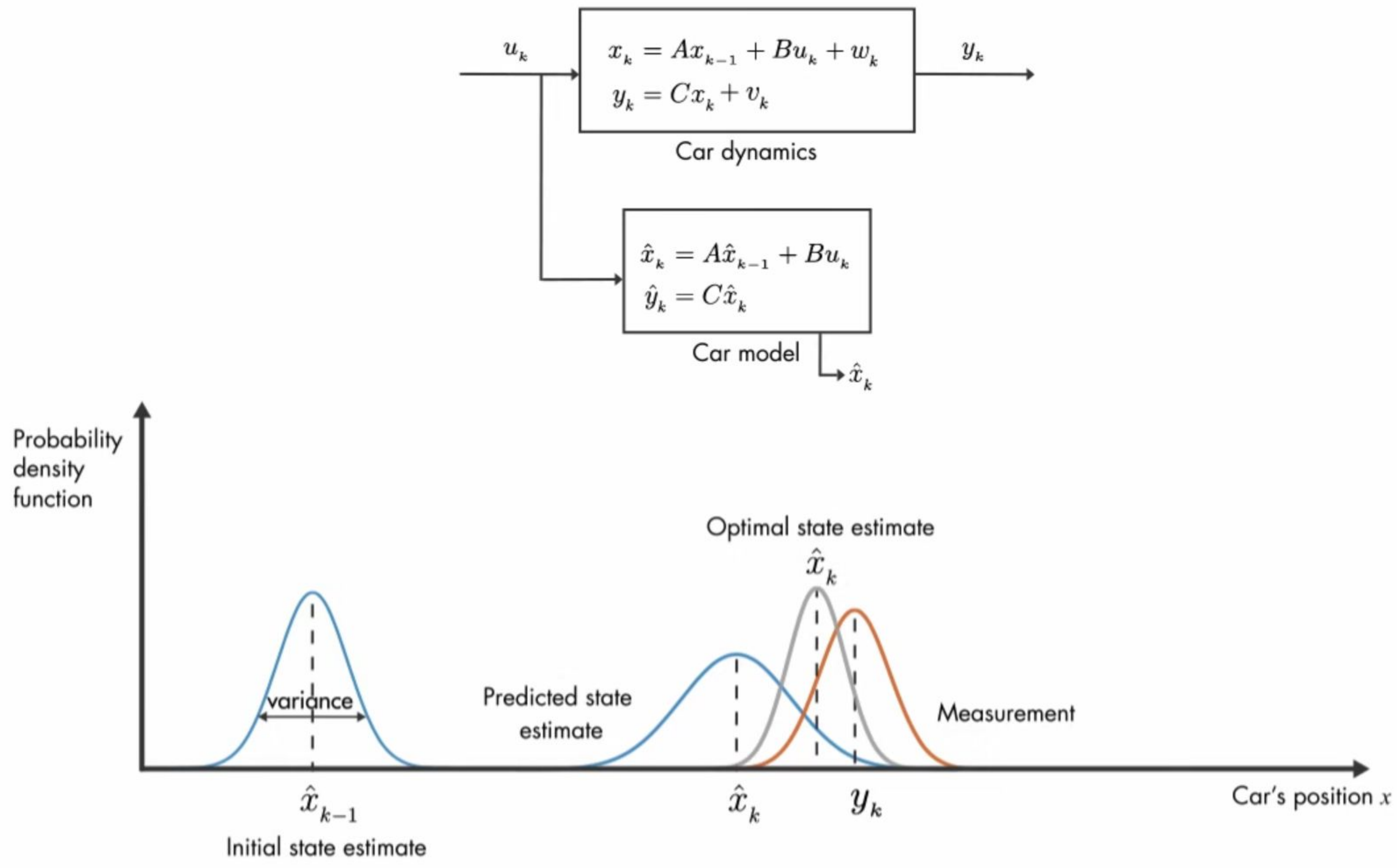

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Recall: state observers

$$\mathbf{x}[k+1] = \Phi\mathbf{x}[k] + \Gamma\mathbf{u}[k]$$
$$\mathbf{y}[k] = C\mathbf{x}[k]$$

- Recall the approach using the observer/estimator

$$\hat{\mathbf{x}}[k+1] = \Phi\hat{\mathbf{x}}[k] + \Gamma u[k] + K\tilde{y}[k]$$

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&= \Phi\hat{\mathbf{x}}[k] + \Gamma u[k] + K(y[k] - C\hat{\mathbf{x}}[k]) \\
&= (\Phi - KC)\underbrace{\hat{\mathbf{x}}[k]}_{\text{Estimation}} + \Gamma \underbrace{u[k]}_{\text{input}} + K\underbrace{y[k]}_{\text{Measurement}}
\end{aligned}$$

K: observer gain

Recall: state observers

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and the performance was studied by comparing the estimate with the real state:

$$\tilde{\mathbf{x}}[k+1] = \mathbf{x}[k+1] - \hat{\mathbf{x}}[k+1]$$

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$$\begin{aligned}
\tilde{\mathbf{x}}[k+1] &= \mathbf{x}[k+1] - \hat{\mathbf{x}}[k+1] \\
&= \overbrace{\left(\Phi\mathbf{x}[k] + \Gamma u[k]\right)} - \overbrace{\left((\Phi - KC)\hat{\mathbf{x}}[k] + \Gamma u[k] + KC\mathbf{x}[k]\right)}
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Input disappeared!

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&= (\Phi - KC)(\mathbf{x}[k] - \hat{\mathbf{x}}[k]) \\
&= \underbrace{(\Phi - KC)}_{\Phi_o}\tilde{\mathbf{x}}[k] = \Phi_o\tilde{\mathbf{x}}[k]
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- Matrix K was chosen such that the eigenvalues of Φ_{o} are at desired places in the complex plane.

The Kalman Filter

- Now, we have the freedom to choose **K[k]** (not just setting the estimation error poles at desired values). Hence, the estimation error dynamics are:

$$\tilde{\mathbf{x}}[k+1] \triangleq \tilde{\mathbf{x}}[k+1|k] = \mathbf{x}[k+1] - \hat{\mathbf{x}}[k+1|k]$$

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&= (\Phi - K[k]C)\underbrace{(\mathbf{x}[k] - \hat{\mathbf{x}}[k|k-1])}_{\tilde{\mathbf{x}}[k]} + \mathbf{v}[k] - K[k]\mathbf{e}[k] \\
&= \begin{bmatrix} I & -K[k] \end{bmatrix} \left(\begin{bmatrix} \Phi \\ C \end{bmatrix} \tilde{\mathbf{x}}[k] + \begin{bmatrix} \mathbf{v}[k] \\ \mathbf{e}[k] \end{bmatrix} \right)
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- **Observation:** the estimation error distribution is independent of the input

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\end{aligned}$$

- **Observation:** the estimation error distribution is independent of the input

- We set the criterion of minimizing the variance of the estimation error:

$$P[k] = E\left\{\left(\tilde{\mathbf{x}}[k] - E\{\tilde{\mathbf{x}}[k]\}\right)\left(\tilde{\mathbf{x}}[k] - E\{\tilde{\mathbf{x}}[k]\}\right)^T\right\}$$

The Kalman Filter

- The mean value of $\tilde{\mathbf{x}}$ is

$$E\{\tilde{\mathbf{x}}[k+1]\} = E\left\{(\Phi - K[k]C)\tilde{\mathbf{x}}[k]\right\} = (\Phi - K[k]C)E\left\{\tilde{\mathbf{x}}[k]\right\}$$

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- Since $E\{\mathbf{x}[0]\} = \mathbf{0}$, if $\hat{\mathbf{x}}[0] = \mathbf{0}$, the mean value of the reconstruction error is zero for all $\mathbf{k}$ independent of $\mathbf{K[k]}$.

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$\Longrightarrow$ No Bias!

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- Since $\tilde{\mathbf{x}}[k]$ is independent of $\mathbf{v}[\mathrm{k}]$ and $\mathbf{e}[\mathrm{k}]$, we obtain the following:

$$P[k+1] = E\left\{\tilde{\mathbf{x}}[k+1]\tilde{\mathbf{x}}^T[k+1]\right\}$$

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$$= E\left\{\begin{bmatrix} I & -K[k] \end{bmatrix} \left(\begin{bmatrix} \Phi \\ C \end{bmatrix} \tilde{\mathbf{x}}[k] + \begin{bmatrix} \mathbf{v}[k] \\ \mathbf{e}[k] \end{bmatrix} \right) \left(\begin{bmatrix} \Phi \\ C \end{bmatrix} \tilde{\mathbf{x}}[k] + \begin{bmatrix} \mathbf{v}[k] \\ \mathbf{e}[k] \end{bmatrix} \right)^T \begin{bmatrix} I \\ -K^T[k] \end{bmatrix} \right\}$$

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$$= \begin{bmatrix} I & -K[k] \end{bmatrix}\left(\begin{bmatrix} \Phi \\ C \end{bmatrix} P[k] + \begin{bmatrix} R_1 & R_{12} \\ R_{12} & R_2 \end{bmatrix}\right)\begin{bmatrix} I \\ -K^T[k] \end{bmatrix}$$

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- Since $\tilde{\mathbf{x}}[k]$ is independent of $\mathbf{v}[\mathrm{k}]$ and $\mathbf{e}[\mathrm{k}]$, we obtain the following:

$$P[k+1] = E\left\{\tilde{\mathbf{x}}[k+1]\tilde{\mathbf{x}}^T[k+1]\right\}$$

$$= E\left\{\begin{bmatrix} I & -K[k]\end{bmatrix}\left(\begin{bmatrix}\Phi \\ C\end{bmatrix}\tilde{\mathbf{x}}[k] + \begin{bmatrix}\mathbf{v}[k] \\ \mathbf{e}[k]\end{bmatrix}\right)\left(\begin{bmatrix}\Phi \\ C\end{bmatrix}\tilde{\mathbf{x}}[k] + \begin{bmatrix}\mathbf{v}[k] \\ \mathbf{e}[k]\end{bmatrix}\right)^T \begin{bmatrix} I \\ -K^T[k]\end{bmatrix}\right\}$$

$$= \begin{bmatrix} I & -K[k]\end{bmatrix}\left(\begin{bmatrix}\Phi \\ C\end{bmatrix} P[k] + \begin{bmatrix} R_1 & R_{12} \\ R_{12} & R_2\end{bmatrix}\right)\begin{bmatrix} I \\ -K^T[k]\end{bmatrix}$$

$$= \begin{bmatrix} I & -K[k]\end{bmatrix}\begin{bmatrix}\Phi P[k]\Phi^T + R_1 & \Phi P[k]C^T + R_{12} \\ CP[k]\Phi^T + R_{12} & CP[k]C^T + R_2\end{bmatrix}\begin{bmatrix} I \\ -K^T[k]\end{bmatrix}, \quad P[0] = R_0$$

The Kalman Filter

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&= \begin{bmatrix} I & -K[k]\end{bmatrix}\left(\begin{bmatrix}\Phi \\ C\end{bmatrix}P[k] + \begin{bmatrix}R_1 & R_{12} \\ R_{12} & R_2\end{bmatrix}\right)\begin{bmatrix} I \\ -K^T[k]\end{bmatrix} \\
&= \begin{bmatrix} I & -K[k]\end{bmatrix}\begin{bmatrix}\Phi P[k]\Phi^T + R_1 & \Phi P[k]C^T + R_{12} \\ CP[k]\Phi^T + R_{12} & CP[k]C^T + R_2\end{bmatrix}\begin{bmatrix} I \\ -K^T[k]\end{bmatrix}, \quad P[0] = R_0
\end{aligned}$$

What about "Low Variance"?

A "Lemma"

- Consider the static quadratic cost function

$$J(\mathbf{x},\mathbf{u}) = \begin{bmatrix}\mathbf{x}^T & \mathbf{u}^T\end{bmatrix}\begin{bmatrix}Q_{\mathbf{x}} & Q_{\mathbf{xu}} \\ Q_{\mathbf{xu}} & Q_{\mathbf{u}}\end{bmatrix}\begin{bmatrix}\mathbf{x} \\ \mathbf{u}\end{bmatrix}$$

where the weight functions $Q_{\mathbf{x}}$ and $Q_{\mathbf{xu}}$ are symmetric and positive semidefinite, $Q_{\mathbf{u}}$ is positive definite.

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and the minimum is

$$J^* = \mathbf{x}^T \left(Q_{\mathbf{x}} - Q_{\mathbf{xu}} Q_{\mathbf{u}}^{-1} Q_{\mathbf{xu}} \right) \mathbf{x}$$

The Kalman Filter

- Now we consider minimizing $\mathbf{a}^T P[k+1]\mathbf{a}$ for any value of $\mathbf{a}$.

$$P[k+1] = \begin{bmatrix} I & -K[k] \end{bmatrix} \begin{bmatrix} \Phi P[k]\Phi^T + R_1 & \Phi P[k]C^T + R_{12} \\ CP[k]\Phi^T + R_{12} & CP[k]C^T + R_2 \end{bmatrix} \begin{bmatrix} I \\ -K^T[k] \end{bmatrix}$$

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- Now we consider minimizing $\mathbf{a}^T P[k+1]\mathbf{a}$ for any value of $\mathbf{a}$.
- By using the "Lemma" we get that the minimizing vector , called the **Kalman gain**, is

$$K[k] = \left(\Phi P[k] C^T + R_{12}\right)\left(C P[k] C^T + R_2\right)^{-1}$$

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Riccati for estimator design!

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which together with

$$\hat{\mathbf{x}}[k+1|k] = \Phi\hat{\mathbf{x}}[k|k-1] + \Gamma u[k] + K[k](y[k] - C\hat{\mathbf{x}}[k|k-1])$$

$$K[k] = \left(\Phi P[k] C^T + R_{12}\right)\left(C P[k] C^T + R_2\right)^{-1}$$

is the celebrated **Kalman filter**.

Example

- Consider the scalar system:

$$
\begin{aligned}
x[k+1] &= x[k] \\
y[k] &= x[k] + e[k]
\end{aligned}
$$

where the measurement is corrupted by noise (zero mean white noise with standard deviation σ; $x[0]$ is assumed to have variance 0.5 (i.e., $P[0]$=0.5). Compute the Kalman filter.

Solution:

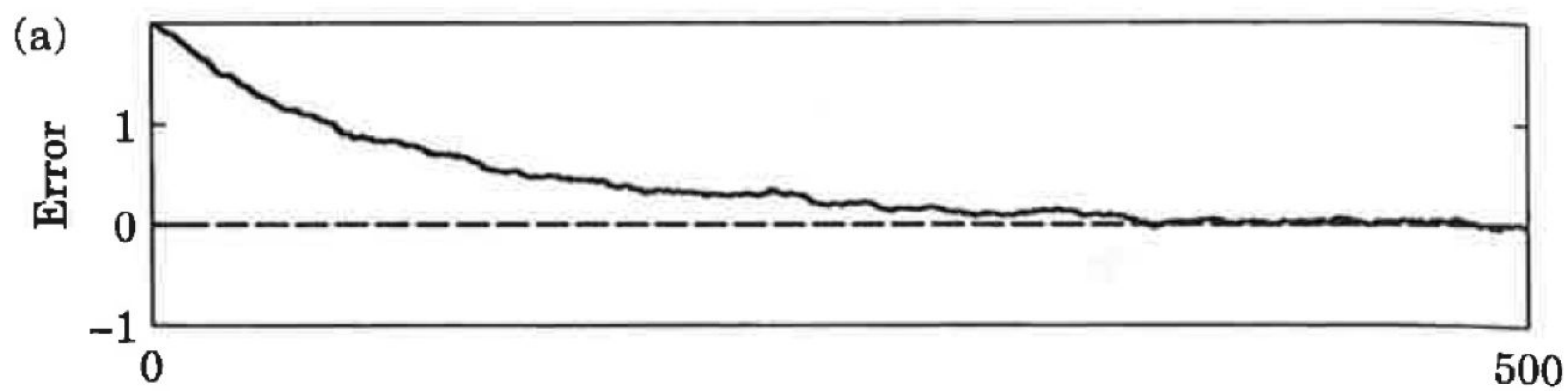
- We use the formulas given:

$$\hat{x}[k+1|k] = \hat{x}[k|k-1] + K[k](y[k] - \hat{x}[k|k-1])$$

$$K[k] = \frac{P[k]}{\sigma^2 + P[k]}$$

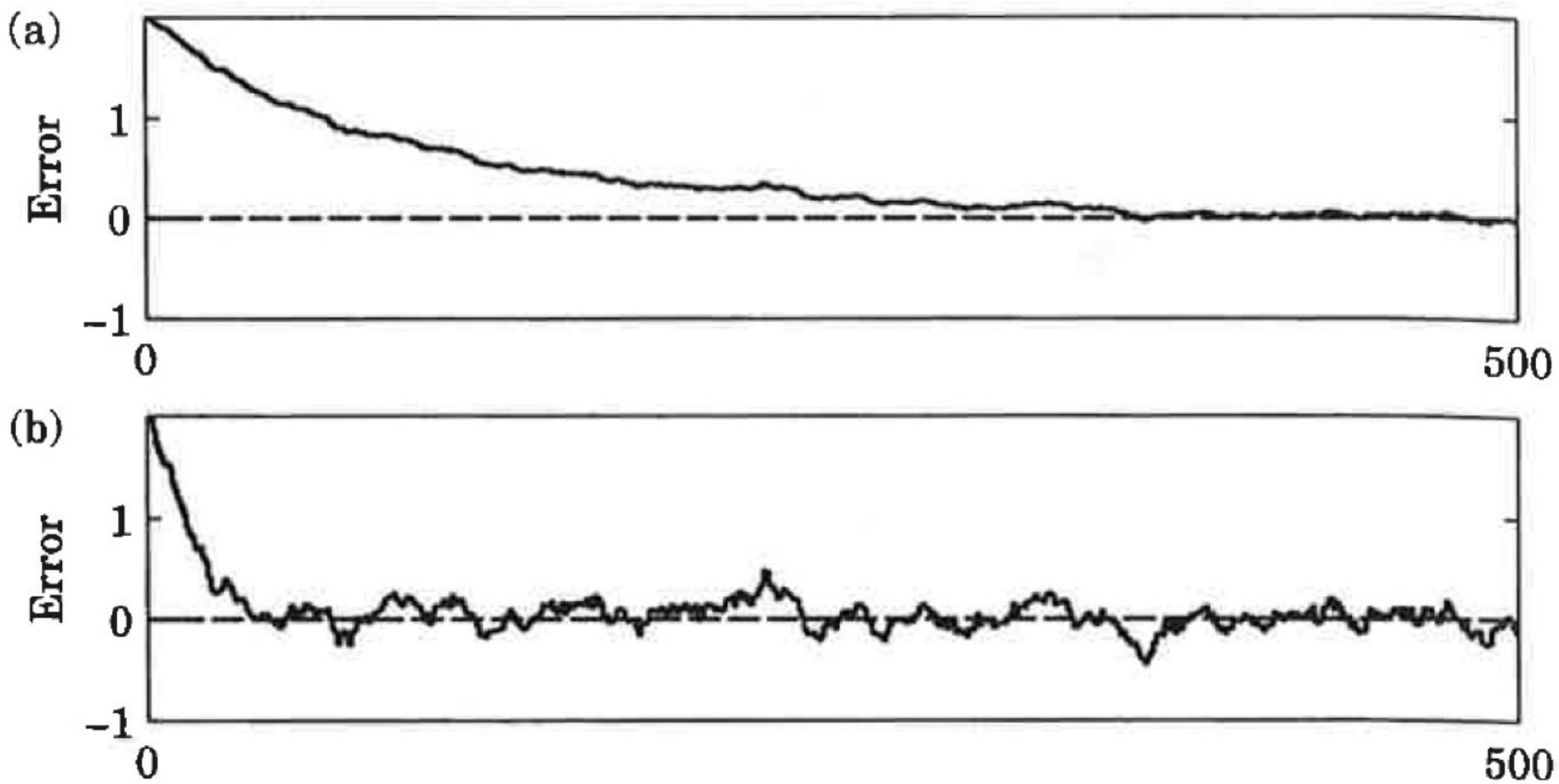
$$P[k+1] = \frac{\sigma^2 P[k]}{\sigma^2 + P[k]}$$

Example



$K = 0.01$

Example



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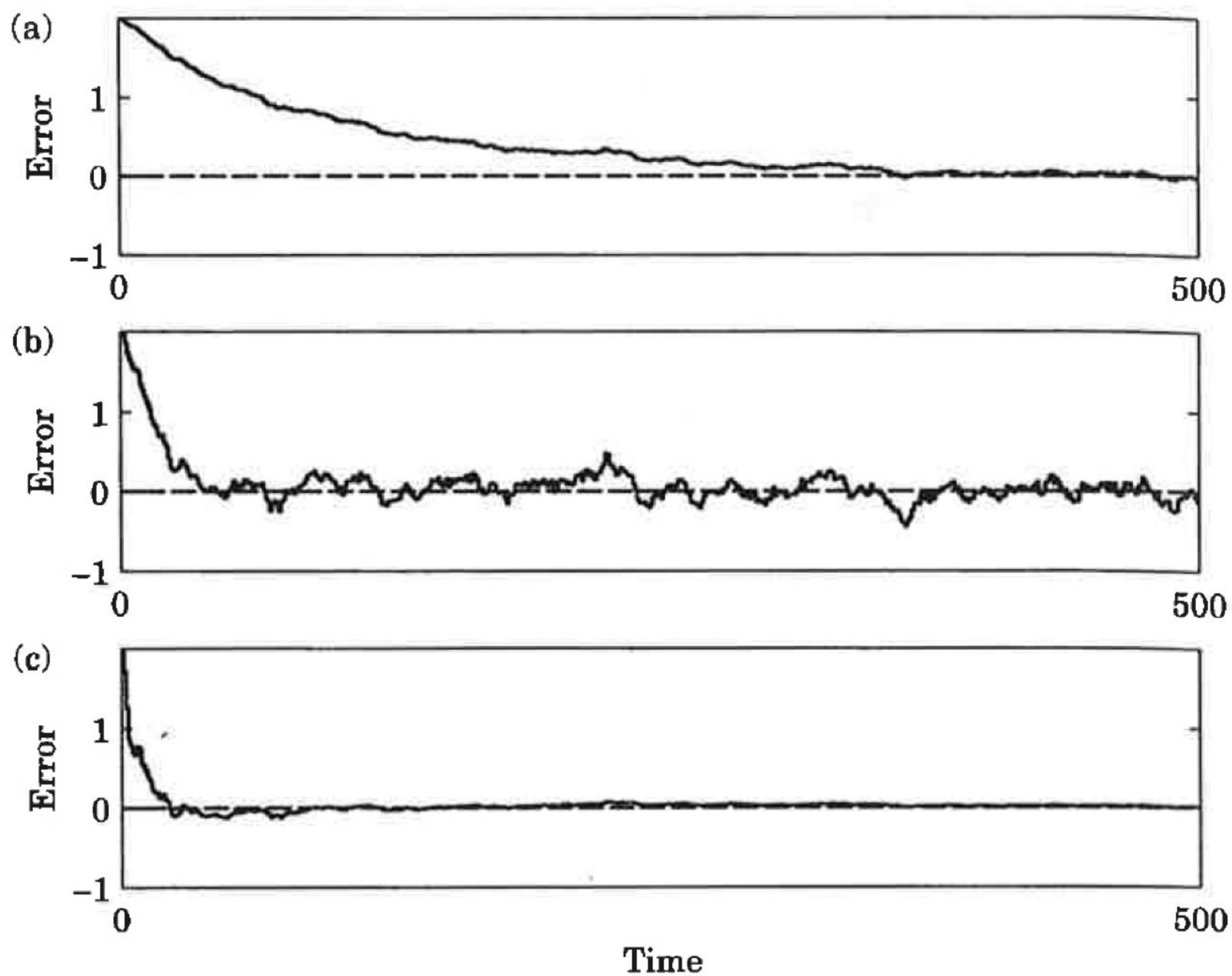
Example



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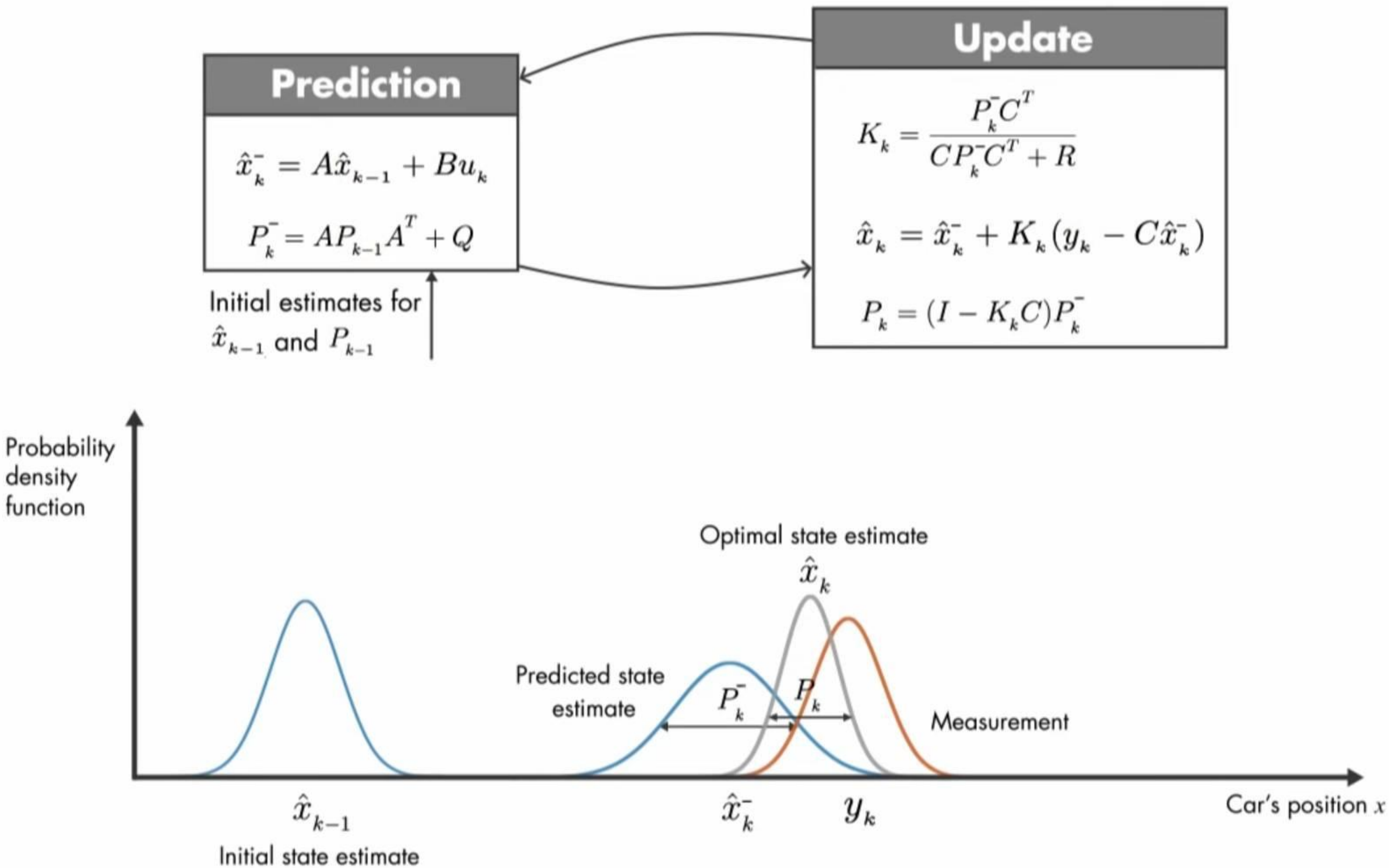
Kalman

How to use the Kalman filter



Remarks

- Note that this was an *algebraic derivation* of the Kalman filter (predictor case)

- There are other approaches (based on Bayesian analysis, using the orthogonality principle, etc.), which give more insights on the problem.

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- As we have seen, the distribution of the estimation error does not depend on the inputs. Then, the problems of optimal control and optimal estimation can be decoupled. This is known as the **separation principle**.

- In LQG, the optimal solution is a combination of the optimal LQ control and optimal prediction, i.e.,

$$\mathbf{u}^*[k] = -L[k]\hat{\mathbf{x}}[k|k-1]$$

where $L[k]$ is given by the Riccati equation of the LQ problem and the state estimate is obtained by the Kalman filter

Separation principle

- Assume we have a linear, time-invariant **completely controllable** and **completely observable** system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

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- Estimate *x* using an observer including the input *u*

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

$$e = x - \hat{x}$$

$$\implies \dot{e} = (A - LC)e$$

Separation principle

- We want to stabilize both x and e! Let's analyze their joint dynamics

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- Let's rewrite them together

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

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- Stabilization of both x and e is possible if and only if this system is asymptotically stable.

Separation principle

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} \overset{M1}{\boxed{A - BK}} & BK \\ 0 & {}_{M2}\boxed{A - LC} \end{bmatrix}}_{M} \begin{bmatrix} x \\ e \end{bmatrix}$$

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- M is an upper triangular block-matrix and its eigenvalues are given by the eigenvalues of the diagonal blocks M1 and M2.

- So, the characteristic equation would be as follows

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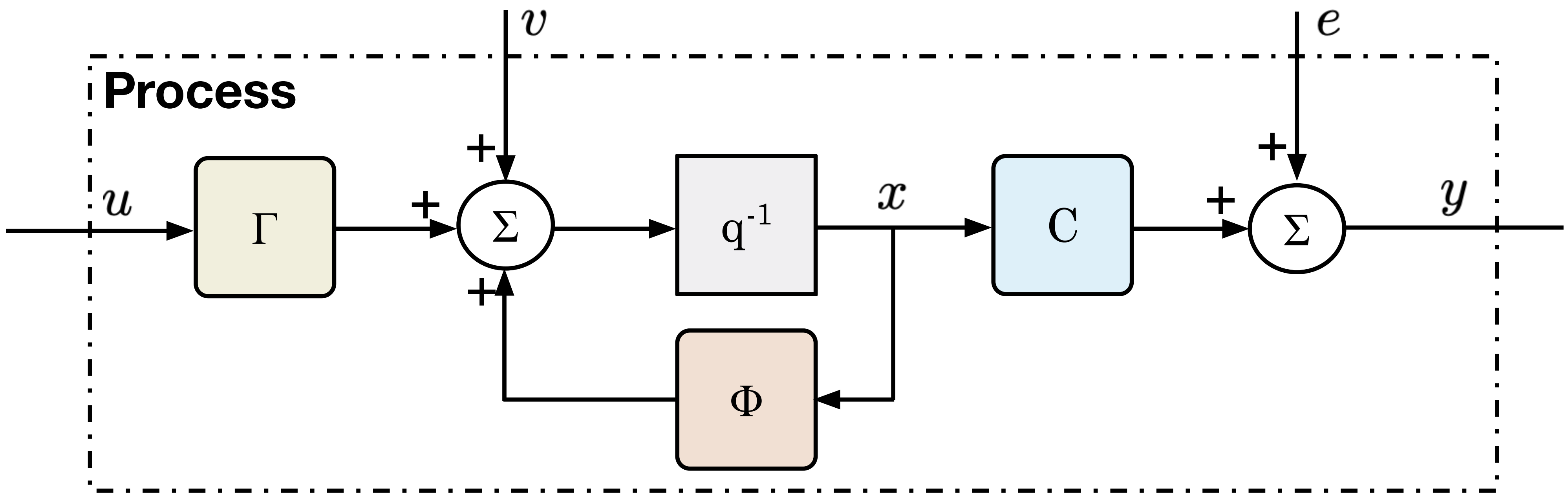
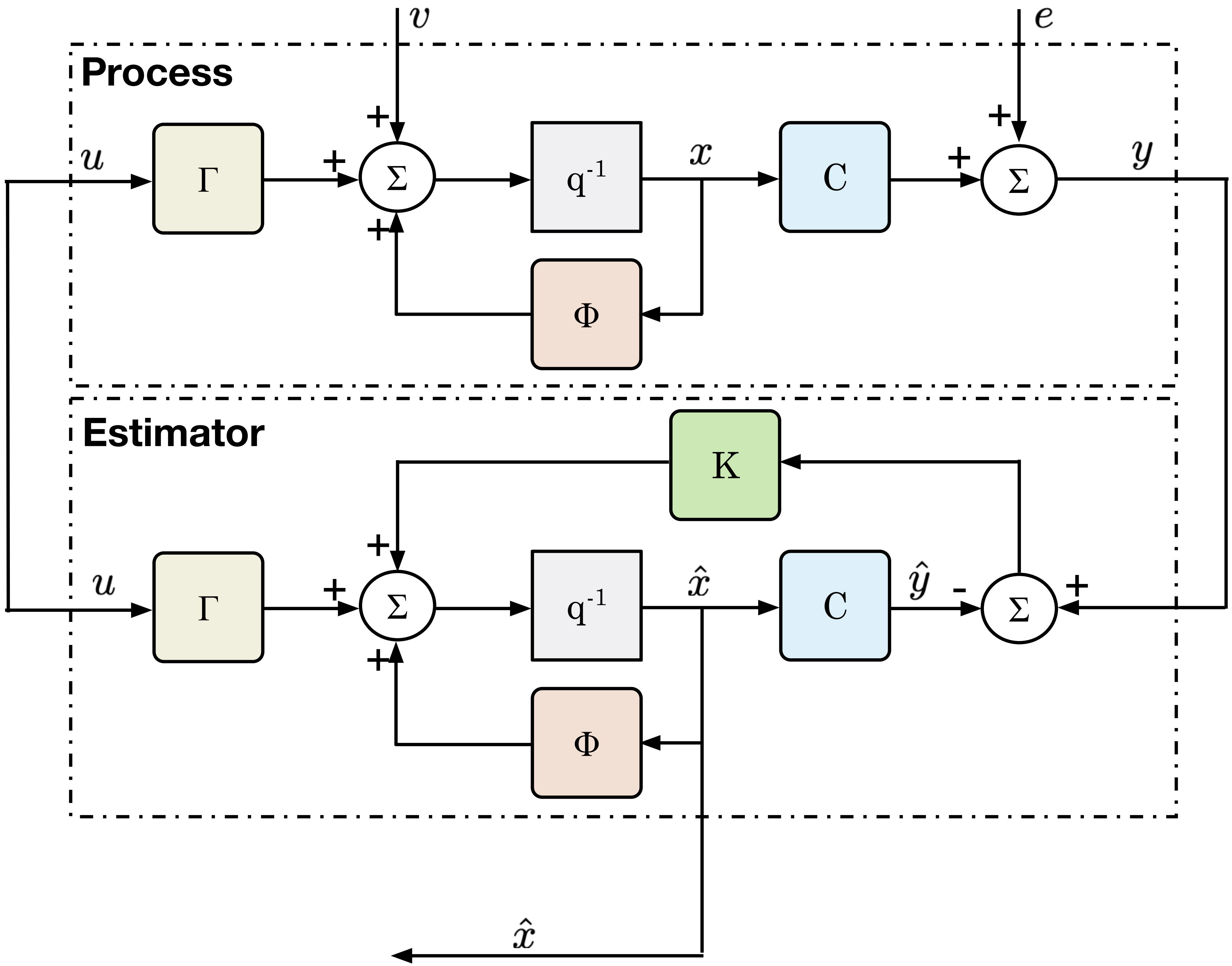
- If M1 and M2 are separately stable, then M is stable as well!
- This fact is known as the **Separation Principle!**

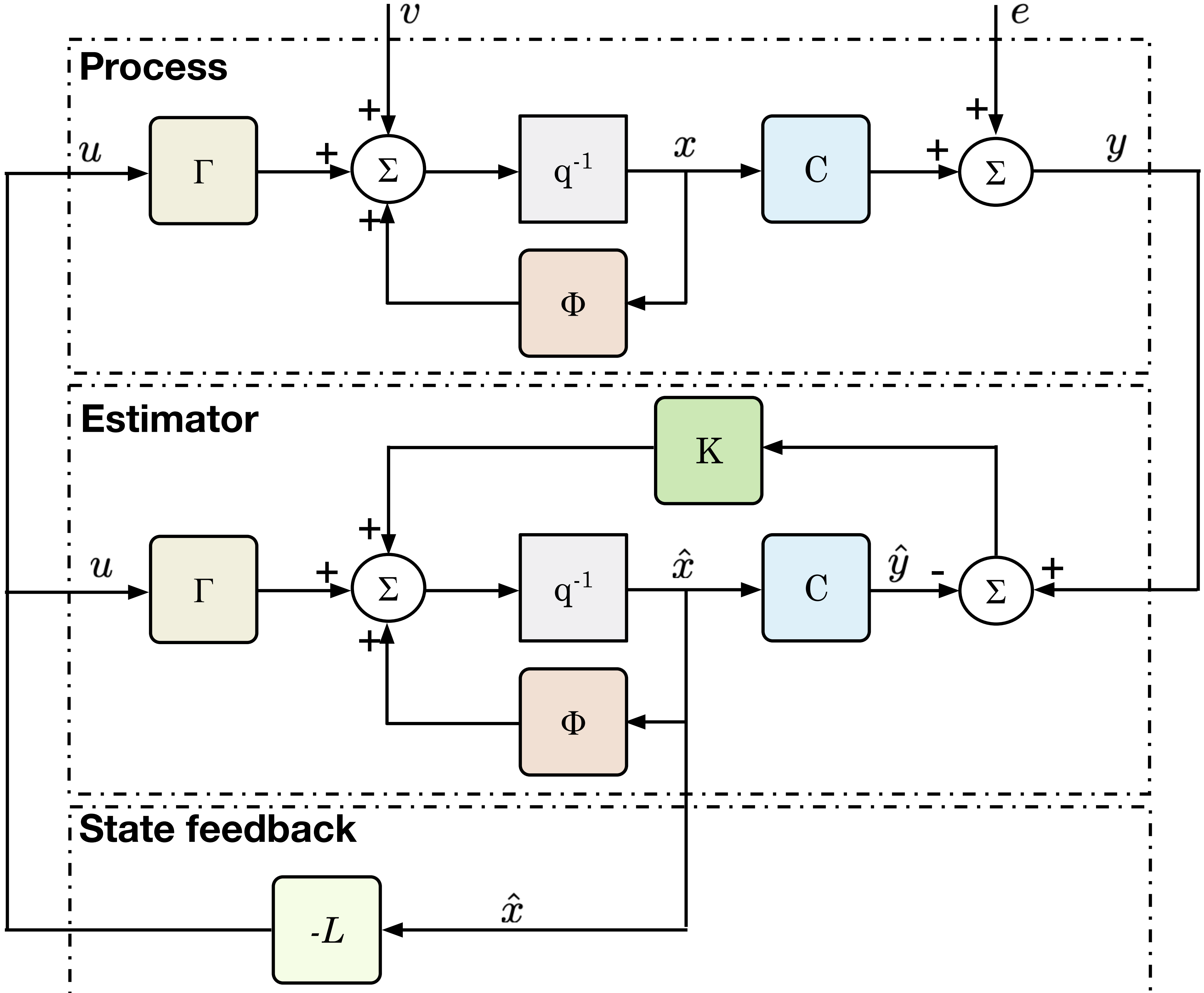
Structure of LQG control



Structure of LQG control



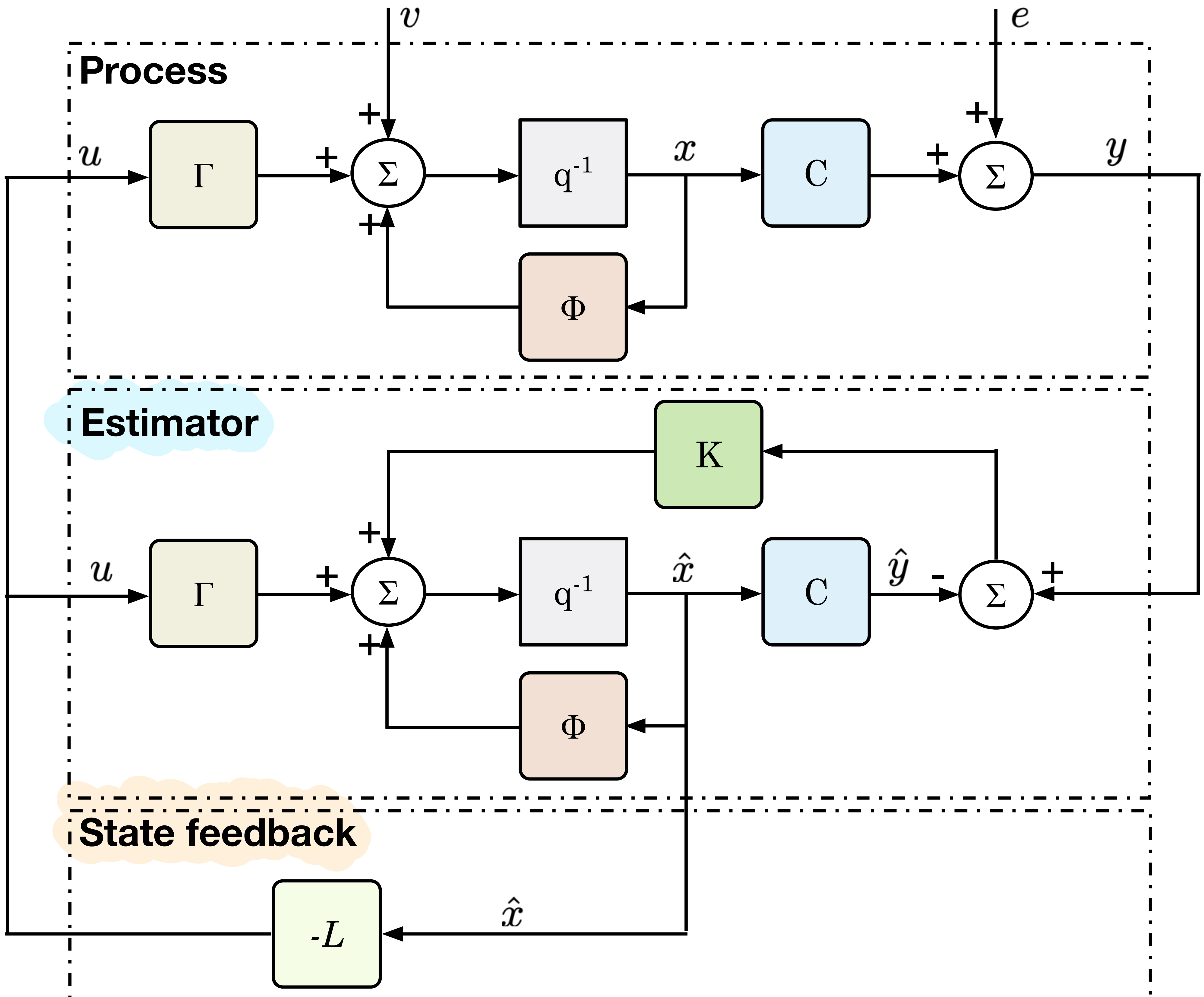
Structure of LQG control



Structure of LQG control

Kalman filter

LQR



Learning outcomes

By the end of *this* lecture, you should be able to:

- Compute the mean and covariance matrices of dynamical processes
- Understand the separation principle
- Use Kalman filter and optimal control to tackle various estimation and control problems for linear systems