ELEC-E8101 Digital and Optimal Control

Exercise 11 - Solutions

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1. A constant variable x is measured through two different sensors. However, the measurements are noisy and have different accuracy. Assume the system is described by

$$x[k+1] = x[k]$$
$$y[k] = Cx[k] + e[k]$$

where $C^T = [1 \quad 1]$ and $e[k]$ is zero-mean, white-noise vector with the covariance matrix

$$R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$$

Estimate x as

$$\hat{x}[k] = a_1 y_1[k] + a_2 y_2[k]$$

Determine constant a_1 and a_2 such that the mean value of the prediction error is zero and such that the variance of the prediction error is as low as possible. Compare the minimum variance with the cases when only one of the measurements is used.

Solution. First, we want the mean value of the prediction error to be zero. This means that the correct mean value should be predicted

$$\mathbb{E}\{x[k]\} - \mathbb{E}\{\hat{x}[k]\} = 0 \Rightarrow \mathbb{E}\{x[k]\} = a_1 C\mathbb{E}\{x[k]\} + a_2 C\mathbb{E}\{x[k]\} \Rightarrow a_1 + a_2 = 1 \qquad (1)$$

Next, we want to minimize the variance which is given by ($e[k] = [e_1[k] \quad e_2[k]]^T$)

$$\begin{aligned} V &= \mathbb{E}\{(x[k] - \hat{x}[k])^2\} = \mathbb{E}\{(x[k] - a_1 x[k] - a_1 e_1[k] - a_2 x[k] - a_2 e_2[k])^2\} \\ &= \mathbb{E}\{\underbrace{(1 - a_1 - a_2)^2}_{=0} x - 2(1 - a_1 - a_2)x(-a_1 \underbrace{e_1}_{\mathbb{E}\{\}=0} - a_2 \underbrace{e_2}_{\mathbb{E}\{\}=0}) + (a_1^2 e_1^2 + a_2^2 e_2^2 + 2a_1 a_2 \underbrace{e_1 e_2}_{\substack{\mathbb{E}\{\}=0 \\ \text{since } R_{12}=0}})\} \\ &= a_1^2 + 9a_2^2 = a_1^2 + 9(1 - a_1)^2 = 10a_1^2 - 18a_1 + 9 \end{aligned}$$

This can also be obtained more easily by using the properties of variance. For time-varying and $x[k]$ and $y[k]$ and a constant a we have

$$\begin{aligned} \text{var}\{x[k] + y[k]\} &= \text{var}\{x[k]\} + \text{var}\{y[k]\} \\ \text{var}\{ax[k]\} &= a^2 \text{var}\{x[k]\} \\ \text{var}\{a\} &= 0 \end{aligned}$$

The variance of the estimation error is

$$\begin{aligned} V &= \text{var}\{x[k] - \hat{x}[k]\} = \text{var}\{x[k] - a_1 x[k] - a_1 e_1[k] - a_2 x[k] - a_2 e_2[k]\} \\ &= \text{var}\{(1 - a_1 - a_2)x[k]\} + a_1^2 \text{var}\{e_1\} + a_2^2 \text{var}\{e_2\} \\ &= a_1^2 + 9a_2^2 = a_1^2 + 9(1 - a_1)^2 = 10a_1^2 - 18a_1 + 9 \end{aligned}$$

Note that from the system equations we have that $x[k]$ is also constant and thus the first term in the second line would have been zero even if $1 - a_1 - a_2$ was non-zero.

To find a_1 which minimizes this we take the derivative of V with respect to a_1 and set it to zero

$$20a_1 - 18 = 0 \Rightarrow a_1 = 0.9$$

and from (1) we get $a_2 = 0.1$. As a result, the estimator is

$$\hat{x}[k] = 0.9y_1[k] + 0.1y_2[k]$$

and the minimum variance is 0.9. Using only y_1 gives the variance 1 and only y_2 results in variance 9. Therefore, using the combination of both measurements gives a lower variance.

2. A stochastic process is generated as

$$x[k+1] = 0.5x[k] + v[k]$$
$$y[k] = x[k] + e[k]$$

where v and e are uncorrelated white-noise processes with covariances r_1 and r_2, respectively. Further, $x[0]$ is normally distributed with zero mean and standard deviation σ.

a) Determine the Kalman filter for the system.

b) What is the gain in steady state?

Solution. Consider the following process:

$$x[k+1] = \Phi x[k] + \Gamma u[k] + v[k]$$
$$y[k] = Cx[k] + e[k]$$

where v and e are discrete-time Guassian white-noise processes with zero mean and

$$\mathbb{E}\{v[k]v^T[k]\} = R_1$$
$$\mathbb{E}\{e[k]v^T[k]\} = R_{12}$$
$$\mathbb{E}\{e[k]e^T[k]\} = R_2$$

Furthermore assume that the initial state $x[0] \sim \mathcal{N}(m_0, R_0)$, i.e., $x[0]$ is Gaussian distributed with

$$\mathbb{E}\{x[0]\} = m_0 \qquad \mathbb{E}\{x[0]x^T[0]\} = R_0$$

Let the estimator have the form

$$\hat{x}[k+1] = (\Phi - K[k]C)\hat{x}[k] + \Gamma u[k] + K[k]y[k]$$

where the estimator's gain is time-varying. Then, the reconstruction error $\tilde{x} = x - \hat{x}$ is governed by

$$\tilde{x}[k+1] = (\Phi - K[k]C)\tilde{x}[k] + v[k] - K[k]e[k]$$

With Kalman filter, $K[k]$ is determined such that the variance of the estimation error, i.e., $P(k) = \mathbb{E}\left\{(\tilde{x}[k] - \mathbb{E}\{\tilde{x}[k]\})(\tilde{x}[k] - \mathbb{E}\{\tilde{x}[k]\})^T\right\}$, is minimized and the equations for calculating $K[k]$ and the resulting $P[k]$ are given in the lecture.

Here, v and e are uncorrelated which means that $R_{12} = 0$. As a result,

$$K[k] = \Phi P[k]C^T(R_2 + CP[k]C^T)^{-1} \tag{2}$$
$$P[k+1] = \Phi P[k]\Phi^T + R_1 - \Phi P[k]C^T(R_2 + CP[k]C^T)^{-1}CP[k]\Phi^T \tag{3}$$

a) We are dealing with a scalar system with $R_1 = r_1$ and $R_2 = r_2$. Direct substitution in (2) gives

$$K[k] = (0.5)P[k](1)^T(r_2 + (1)P[k](1)^T)^{-1} = \frac{0.5P[k]}{r_2 + P[k]} \tag{4}$$

where $P[k]$ is given by (3)

$$P[k+1] = (0.5)^2 P[k] + r_1 - (0.5)^2 P^2[k](r_2 + P[k])^{-1} = \frac{r_1 r_2 + (0.25 r_2 + r_1)P[k]}{r_2 + P[k]} \quad (5)$$

where the filter is initiated with $P[0] = R_0$ which in this case is $P[0] = \sigma^2$.

b) In steady state, from (5) we have

$$P = \frac{r_1 r_2 + (0.25 r_2 + r_1)P}{r_2 + P}$$

This leads to a quadratic equation ($P^2 + (0.75 r_2 - r_1)P - r_1 r_2 = 0$) which could be solved if r_1 and r_2 were known and the positive solution is the steady-state error covariance but that's not what we are interested in here. The corresponding steady-state gain from (4) is found to be

$$K = \frac{0.5P}{r_2 + P}$$