Aalto University
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Engineering

# ELEC-E8740 — Extended and Unscented Kalman Filtering

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# Contents

# Intended Learning Outcomes

After this lecture, you will be able to:

- recognize the challenges for filtering in nonlinear state-space models,
- describe and employ the extended and unscented Kalman filters for nonlinear state-space models

# Recap: Filtering and the Kalman Filter

- The filtering approach iterates between two steps:
  1. Prediction: $\hat{\mathbf{x}}_{n-1|n-1}, \mathbf{P}_{n-1|n-1} \Rightarrow \hat{\mathbf{x}}_{n|n-1}, \mathbf{P}_{n|n-1}$
  2. Measurement update: $\hat{\mathbf{x}}_{n|n-1}, \mathbf{P}_{n|n-1} \Rightarrow \hat{\mathbf{x}}_{n|n}, \mathbf{P}_{n|n}$
- The Kalman filter is the optimal filter for linear state-space models
  1. Prediction:

$$
\begin{aligned}
\hat{\mathbf{x}}_{n|n-1} &= \mathbf{F}_n \hat{\mathbf{x}}_{n-1|n-1} \\
\mathbf{P}_{n|n-1} &= \mathbf{F}_n \mathbf{P}_{n-1|n-1} \mathbf{F}_n^\mathsf{T} + \mathbf{Q}_n
\end{aligned}
$$

  2. Measurement update:

$$
\begin{aligned}
\mathbf{K}_n &= \mathbf{P}_{n|n-1} \mathbf{G}_n^\mathsf{T} (\mathbf{G}_n \mathbf{P}_{n|n-1} \mathbf{G}_n^\mathsf{T} + \mathbf{R}_n)^{-1} \\
\hat{\mathbf{x}}_{n|n} &= \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n (\mathbf{y}_n - \mathbf{G}_n \hat{\mathbf{x}}_{n|n-1}) \\
\mathbf{P}_{n|n} &= \mathbf{P}_{n|n-1} - \mathbf{K}_n (\mathbf{G}_n \mathbf{P}_{n|n-1} \mathbf{G}_n^\mathsf{T} + \mathbf{R}_n) \mathbf{K}_n^\mathsf{T}
\end{aligned}
$$

# Discrete-Time Nonlinear State-Space Model

- Discrete-time nonlinear state-space model:

$$\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$
$$\mathbf{y}_n = \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n$$

- Process and measurement noises ($\mathbf{q}_n$ and $\mathbf{r}_n$):

$$\mathrm{E}\{\mathbf{q}_n\} = 0, \; \mathrm{Cov}\{\mathbf{q}_n\} = \mathbf{Q}_n$$
$$\mathrm{E}\{\mathbf{r}_n\} = 0, \; \mathrm{Cov}\{\mathbf{r}_n\} = \mathbf{R}_n$$

- Initial conditions:

$$\mathrm{E}\{\mathbf{x}_0\} = \mathbf{m}_0, \; \mathrm{Cov}\{\mathbf{x}_0\} = \mathbf{P}_0$$

# Filtering for Nonlinear Models

- For most nonlinear models, exact prediction and/or update steps can not be found
- Example: Prediction for general nonlinear model

$$\hat{\mathbf{x}}_{n|n-1} = \mathrm{E}\{\mathbf{x}_n \mid \mathbf{y}_{1:n-1}\}$$

Approximations to the exact solutions are required!

# Linearized Model: Prediction (1/2)

- State estimate from $t_{n-1}$: $\hat{\mathbf{x}}_{n-1|n-1}$, $\mathbf{P}_{n-1|n-1}$
- Linearization around $\hat{\mathbf{x}}_{n-1|n-1}$ (dynamic model):

$$
\begin{aligned}
\mathbf{x}_n &= \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n \\
&\approx \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x\,(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{q}_n
\end{aligned}
$$

*Note that* $\mathbf{F}_x = \mathbf{F}_x(\hat{\mathbf{x}}_{n-1|n-1})$.

- Predicted mean:

$$
\begin{aligned}
\hat{\mathbf{x}}_{n|n-1} &= \mathrm{E}\{\mathbf{x}_n \mid \mathbf{y}_{1:n-1}\} \\
&\approx \mathrm{E}\{\mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x\,(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{q}_n \mid \mathbf{y}_{1:n-1}\} \\
&= \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x\,\mathrm{E}\{\mathbf{x}_{n-1} \mid \mathbf{y}_{1:n-1}\} - \mathbf{F}_x\hat{\mathbf{x}}_{n-1|n-1} \\
&= \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x\hat{\mathbf{x}}_{n-1|n-1} - \mathbf{F}_x\hat{\mathbf{x}}_{n-1|n-1} \\
&= \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1})
\end{aligned}
$$

# Linearized Model: Prediction (2/2)

- State estimate from $t_{n-1}$: $\hat{\mathbf{x}}_{n-1|n-1}$, $\mathbf{P}_{n-1|n-1}$
- Linearization around $\hat{\mathbf{x}}_{n-1|n-1}$ (dynamic model):

$$\mathbf{x}_n \approx \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x \left(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1}\right) + \mathbf{q}_n,$$

- Covariance:

$$\begin{aligned}
\mathbf{P}_{n|n-1} &= \mathrm{E}\{(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})^\mathsf{T} \mid \mathbf{y}_{1:n-1}\} \\
&\approx \mathrm{E}\{[\mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{q}_n - \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1})] \\
&\qquad \times [\ldots]^\mathsf{T} \mid \mathbf{y}_{1:n-1}\} \\
&= \mathrm{E}\{[\mathbf{F}_x(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{q}_n][\ldots]^\mathsf{T} \mid \mathbf{y}_{1:n-1}\} \\
&= \mathbf{F}_x \, \mathrm{E}\{(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1})(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1})^\mathsf{T} \mid \mathbf{y}_{1:n-1}\}\mathbf{F}_x^\mathsf{T} \\
&\qquad + \mathrm{E}\{\mathbf{q}_n \mathbf{q}_n^\mathsf{T} \mid \mathbf{y}_{1:n-1}\} \\
&= \mathbf{F}_x \mathbf{P}_{n-1|n-1} \mathbf{F}_x^\mathsf{T} + \mathbf{Q}_n
\end{aligned}$$

# Linearized Model: Measurement Update (1/3)

- Prediction from $t_{n-1}$ to $t_n$: $\hat{\mathbf{x}}_{n|n-1}$, $\mathbf{P}_{n|n-1}$
- Linearization around $\hat{\mathbf{x}}_{n|n-1}$:

$$\begin{aligned}
\mathbf{y}_n &= \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n \\
&\approx \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) + \mathbf{G}_x(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}) + \mathbf{r}_n
\end{aligned}$$

- Regularized linear least squares:

$$\begin{aligned}
J_{\text{ReLS}}(\mathbf{x}_n) &= (\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) - \mathbf{G}_x(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}))^{\mathsf{T}} \mathbf{R}_n^{-1} \\
&\quad \times (\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) - \mathbf{G}_x(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})) \\
&\quad + (\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})^{\mathsf{T}} \mathbf{P}_{n|n-1}^{-1} (\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}) \\
\hat{\mathbf{x}}_{n|n} &= \operatorname*{argmin}_{\mathbf{x}_n} J_{\text{ReLS}}(\mathbf{x}_n)
\end{aligned}$$

# Linearized Model: Measurement Update (2/3)

- Regularized linear least squares:

$$
\begin{aligned}
J_{\text{ReLS}}(\mathbf{x}_n) = (\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) - \mathbf{G}_x(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}))^{\mathsf{T}} \mathbf{R}_n^{-1} \\
\times (\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) - \mathbf{G}_x(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})) \\
+ (\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})^{\mathsf{T}} \mathbf{P}_{n|n-1}^{-1} (\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})
\end{aligned}
$$

- Change of variables: $\mathbf{z}_n = \mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) + \mathbf{G}_x \hat{\mathbf{x}}_{n|n-1}$:

$$
\begin{aligned}
J_{\text{ReLS}}(\mathbf{x}_n) = (\mathbf{z}_n - \mathbf{G}_x \mathbf{x}_n)^{\mathsf{T}} \mathbf{R}_n^{-1} (\mathbf{z}_n - \mathbf{G}_x \mathbf{x}_n) \\
+ (\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})^{\mathsf{T}} \mathbf{P}_{n|n-1}^{-1} (\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})
\end{aligned}
$$

- Solution (see Chapters 2.4, 5.2):

$$
\begin{aligned}
\hat{\mathbf{x}}_{n|n} &= \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n (\mathbf{z}_n - \mathbf{G}_x \hat{\mathbf{x}}_{n|n-1}) \\
\mathbf{K}_n &= \mathbf{P}_{n|n-1} \mathbf{G}_x^{\mathsf{T}} (\mathbf{G}_x \mathbf{P}_{n|n-1} \mathbf{G}_x^{\mathsf{T}} + \mathbf{R}_n)^{-1} \\
\mathbf{P}_{n|n} &\approx \mathbf{P}_{n|n-1} - \mathbf{K}_n (\mathbf{G}_x \mathbf{P}_{n|n-1} \mathbf{G}_x^{\mathsf{T}} + \mathbf{R}_n) \mathbf{K}_n^{\mathsf{T}}
\end{aligned}
$$

# Linearized Model: Measurement Update (3/3)

- Measurement update:

$$\hat{\mathbf{x}}_{n|n} = \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n(\mathbf{z}_n - \mathbf{G}_x \hat{\mathbf{x}}_{n|n-1})$$

- Substitution of $\mathbf{z}_n = \mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) + \mathbf{G}_x \hat{\mathbf{x}}_{n|n-1}$:

$$\begin{aligned}\hat{\mathbf{x}}_{n|n} &= \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n(\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) + \mathbf{G}_x \hat{\mathbf{x}}_{n|n-1} - \mathbf{G}_x \hat{\mathbf{x}}_{n|n-1}) \\ &= \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n(\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}))\end{aligned}$$

# Linearized Model: Summary

- Model approximation:

$$\mathbf{x}_n \approx \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{q}_n$$
$$\mathbf{y}_n \approx \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) + \mathbf{G}_x(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}) + \mathbf{r}_n$$

- Prediction:

$$\hat{\mathbf{x}}_{n|n-1} = \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}),$$
$$\mathbf{P}_{n|n-1} = \mathbf{F}_x \mathbf{P}_{n-1|n-1} \mathbf{F}_x^\mathsf{T} + \mathbf{Q}_n,$$

- Measurement update:

$$\mathbf{K}_n = \mathbf{P}_{n|n-1} \mathbf{G}_x^\mathsf{T} (\mathbf{G}_x \mathbf{P}_{n|n-1} \mathbf{G}_x^\mathsf{T} + \mathbf{R}_n)^{-1},$$
$$\hat{\mathbf{x}}_{n|n} = \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n(\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1})),$$
$$\mathbf{P}_{n|n} = \mathbf{P}_{n|n-1} - \mathbf{K}_n(\mathbf{G}_x \mathbf{P}_{n|n-1} \mathbf{G}_x^\mathsf{T} + \mathbf{R}_n)\mathbf{K}_n^\mathsf{T}.$$

# Extended Kalman Filter

**Algorithm 1** Extended Kalman Filter

1: Initialize $\hat{\mathbf{x}}_{0|0} = \mathbf{m}_0$, $\mathbf{P}_{0|0} = \mathbf{P}_0$
2: **for** $n = 1, 2, \ldots$ **do**
3: Prediction (time update):

$$\hat{\mathbf{x}}_{n|n-1} = \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1})$$
$$\mathbf{P}_{n|n-1} = \mathbf{F}_x \mathbf{P}_{n-1|n-1} \mathbf{F}_x^\mathsf{T} + \mathbf{Q}_n$$

4: Measurement update:

$$\mathbf{K}_n = \mathbf{P}_{n|n-1} \mathbf{G}_x^\mathsf{T} (\mathbf{G}_x \mathbf{P}_{n|n-1} \mathbf{G}_x^\mathsf{T} + \mathbf{R}_n)^{-1}$$
$$\hat{\mathbf{x}}_{n|n} = \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n(\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}))$$
$$\mathbf{P}_{n|n} = \mathbf{P}_{n|n-1} - \mathbf{K}_n (\mathbf{G}_x \mathbf{P}_{n|n-1} \mathbf{G}_x^\mathsf{T} + \mathbf{R}_n) \mathbf{K}_n^\mathsf{T}$$

5: **end for**

# Example: Object Tracking (1/3)

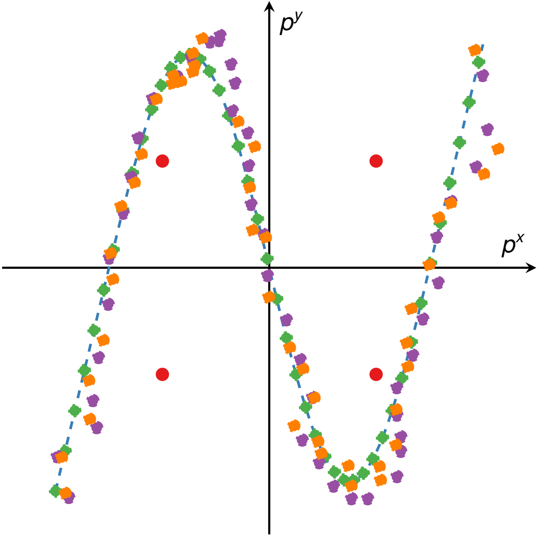
- Quasi-constant turn model:

$$\begin{bmatrix} \dot{p}^x(t) \\ \dot{p}^y(t) \\ \dot{v}(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} v(t)\cos(\varphi(t)) \\ v(t)\sin(\varphi(t)) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{w}(t)$$

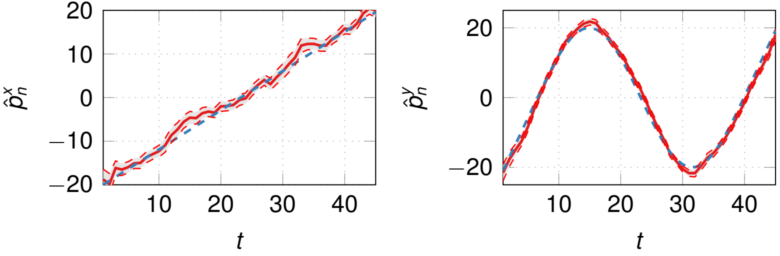
- We can use Euler–Maruyama to discretize this.
- Range (distance) measurements:

$$\mathbf{y}_n = \begin{bmatrix} |\mathbf{p}_n - \mathbf{p}^s_1| \\ |\mathbf{p}_n - \mathbf{p}^s_2| \\ \vdots \\ |\mathbf{p}_n - \mathbf{p}^s_K| \end{bmatrix} + \mathbf{r}_n$$

# Example: Object Tracking (2/3)



# Example: Object Tracking (3/3)

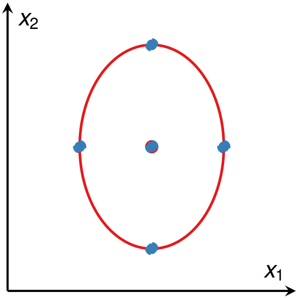



Position RMSE: 3.83 m

# Nonlinear Transformations of Random Variables (1/3)

- Given: Random variable $\mathbf{x}$ with mean $\mathbf{m}$ and covariance $\mathbf{P}$
- Choose points $\mathbf{x}^j$ and weights $w_m^j$, $w_P^j$ such that:

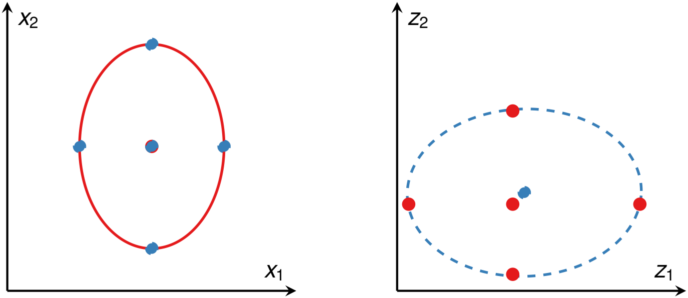
$$\mathbf{m} = \sum_{j=0}^{J-1} w_m^j \mathbf{x}^j, \mathbf{P} = \sum_{j=0}^{J-1} w_P^j (\mathbf{x}^j - \mathbf{m})(\mathbf{x}^j - \mathbf{m})^\mathsf{T},$$



# Nonlinear Transformations of Random Variables (2/3)

- Given: Points $\mathbf{x}^j$ and weights $w_m^j$, $w_P^j$
- Nonlinear transformation: $\mathbf{z} = \mathbf{h}(\mathbf{x})$
- Transformed points:

$$\mathbf{z}^j = \mathbf{h}(\mathbf{x}^j)$$



# Nonlinear Transformations of Random Variables (3/3)

- Given: Points $\mathbf{x}^j$ and weights $w_m^j$, $w_P^j$
- Nonlinear transformation: $\mathbf{z} = \mathbf{h}(\mathbf{x})$
- Transformed points:

$$\mathbf{z}^j = \mathbf{h}(\mathbf{x}^j)$$

- Moments of the transformed variable

$$\mathrm{E}\{\mathbf{z}\} \approx \sum_{j=1}^{J} w_m^j \mathbf{z}^j$$

$$\mathrm{Cov}\{\mathbf{z}\} \approx \sum_{j=1}^{J} w_P^j (\mathbf{z}^j - \mathrm{E}\{\mathbf{z}\})(\mathbf{z}^j - \mathrm{E}\{\mathbf{z}\})^\mathsf{T}$$

$$\mathrm{Cov}\{\mathbf{x}, \mathbf{z}\} \approx \sum_{j=1}^{J} w_P^j (\mathbf{x}^j - \mathbf{m})(\mathbf{z}^j - \mathrm{E}\{\mathbf{z}\})^\mathsf{T}$$

# Unscented Transform

- Unscented Transform: One way of choosing $\mathbf{x}^j$, $w_m^j$ and $w_P^j$, uses $2L+1$ points
- Location of the sigma-points:

$$
\begin{aligned}
\mathbf{x}^0 &= \mathbf{m} \\
\mathbf{x}^j &= \mathbf{m} + \sqrt{L+\lambda}\,[\sqrt{\mathbf{P}}]_j, & j &= 1,\ldots,L \\
\mathbf{x}^j &= \mathbf{m} - \sqrt{L+\lambda}\,[\sqrt{\mathbf{P}}]_{(j-L)}, & j &= L+1,\ldots,2L
\end{aligned}
$$

- Weights of the sigma-points:

$$
\begin{aligned}
w_m^0 &= \frac{\lambda}{L+\lambda} \\
w_P^0 &= \frac{\lambda}{L+\lambda} + (1-\alpha^2+\beta) \\
w_m^j &= w_P^j = \frac{1}{2(L+\lambda)}, & j &= 1,\ldots,2L
\end{aligned}
$$

# Unscented Transform: Prediction (1/2)

- Dynamic model:

$$\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

- Sigma-points with $\mathbf{m} = \hat{\mathbf{x}}_{n-1|n-1}$, $\mathbf{P} = \mathbf{P}_{n-1|n-1}$:

$$\begin{aligned}
\mathbf{x}_{n-1}^0 &= \hat{\mathbf{x}}_{n-1|n-1} \\
\mathbf{x}_{n-1}^j &= \hat{\mathbf{x}}_{n-1|n-1} + \sqrt{L+\lambda}\left[\sqrt{\mathbf{P}_{n-1|n-1}}\right]_j, \qquad j = 1, \ldots, L \\
\mathbf{x}_{n-1}^j &= \hat{\mathbf{x}}_{n-1|n-1} - \sqrt{L+\lambda}\left[\sqrt{\mathbf{P}_{n-1|n-1}}\right]_{(j-L)}, \; j = L+1, \ldots, 2L
\end{aligned}$$

# Unscented Transform: Prediction (2/2)

- Dynamic model:

$$\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

- Transformed points:

$$\mathbf{x}_n^j = \mathbf{f}(\mathbf{x}_{n-1}^j), \qquad j = 0, \ldots, 2L$$

- Moments of the prediction:

$$\hat{\mathbf{x}}_{n|n-1} = \sum_{j=0}^{2L} w_m^j \mathbf{x}_n^j$$

$$\mathbf{P}_{n|n-1} = \sum_{j=0}^{2L} w_c^j (\mathbf{x}_n^j - \hat{\mathbf{x}}_{n|n-1})(\mathbf{x}_n^j - \hat{\mathbf{x}}_{n|n-1})^\mathsf{T} + \mathbf{Q}_n$$

# Unscented Transform: Measurement Update (1/2)

- Measurement model:

$$\mathbf{y}_n = \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n$$

- Recall: Alternative form of measurement update:

$$\begin{aligned}
\mathbf{K}_n &= \mathrm{Cov}\{\mathbf{x}_n, \mathbf{y}_n \mid \mathbf{y}_{1:n-1}\} \, \mathrm{Cov}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}^{-1}, \\
\hat{\mathbf{x}}_{n|n} &= \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n (\mathbf{y}_n - \mathrm{E}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}), \\
\mathbf{P}_{n|n} &= \mathbf{P}_{n|n-1} - \mathbf{K}_n \, \mathrm{Cov}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\} \mathbf{K}_n^{\mathsf{T}}.
\end{aligned}$$

- We can calculate $\mathrm{E}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}$, $\mathrm{Cov}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}$, and $\mathrm{Cov}\{\mathbf{x}_n, \mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}$ using the unscented transform
- Sigma-points based on $\hat{\mathbf{x}}_{n|n-1}$, $\mathbf{P}_{n|n-1}$:

$$\begin{aligned}
\mathbf{x}_n^0 &= \hat{\mathbf{x}}_{n|n-1} \\
\mathbf{x}_n^j &= \hat{\mathbf{x}}_{n|n-1} + \sqrt{L+\lambda} \left[\sqrt{\mathbf{P}_{n|n-1}}\right]_j, \qquad j = 1, \ldots, L \\
\mathbf{x}_n^j &= \hat{\mathbf{x}}_{n|n-1} - \sqrt{L+\lambda} \left[\sqrt{\mathbf{P}_{n|n-1}}\right]_{(j-L)}, \; j = L+1, \ldots, 2L
\end{aligned}$$

# Unscented Transform: Measurement Update (2/2)

- Measurement model:

$$\mathbf{y}_n = \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n$$

- Transformed sigma-points:

$$\mathbf{y}_n^j = \mathbf{g}(\mathbf{x}_n^j), \qquad j = 0, \ldots, 2L$$

- Moments of the predicted $\mathbf{y}_n$:

$$\mathrm{E}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\} = \sum_{j=0}^{2L} w_m^j \mathbf{y}_n^j$$

$$\mathrm{Cov}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\} = \sum_{j=0}^{2L} w_P^j (\mathbf{y}_n^j - \mathrm{E}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}) \times (\mathbf{y}_n^j - \mathrm{E}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\})^\mathsf{T} + \mathbf{R}_n$$

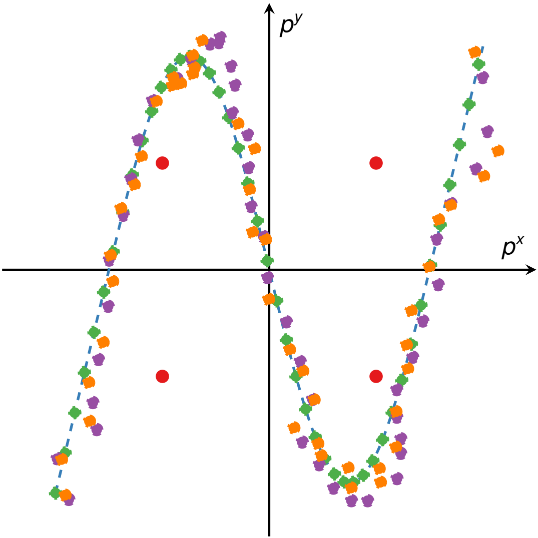
$$\mathrm{Cov}\{\mathbf{x}_n, \mathbf{y}_n \mid \mathbf{y}_{1:n-1}\} = \sum_{j=0}^{2L} w_P^j (\mathbf{x}_n^j - \hat{\mathbf{x}}_{n|n-1})(\mathbf{y}_n^j - \mathrm{E}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\})^\mathsf{T}$$

# Unscented Kalman Filter

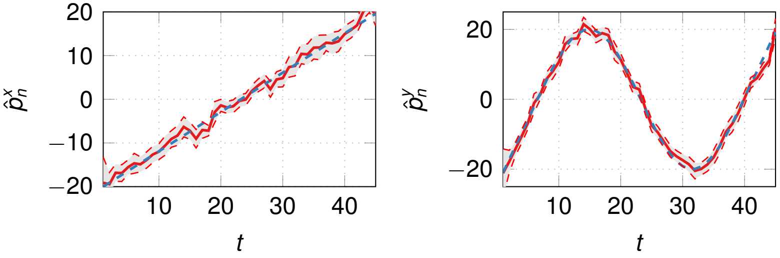
- Prediction:
  - Calculate the sigma-points using $\hat{\mathbf{x}}_{n-1|n-1}$ and $\mathbf{P}_{n-1|n-1}$
  - Propagate the sigma-points $\mathbf{x}_n^j = \mathbf{f}(\mathbf{x}_{n-1}^j)$
  - Calculate the mean and covariance $\hat{\mathbf{x}}_{n|n-1}$, $\mathbf{P}_{n|n-1}$
- Measurement update:
  - Calculate the sigma-points using $\hat{\mathbf{x}}_{n|n-1}$ and $\mathbf{P}_{n|n-1}$
  - Propagate the sigma-points $\mathbf{y}_n^j = \mathbf{g}(\mathbf{x}_n^j)$
  - Calculate the mean and covariance $\mathrm{E}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}$, $\mathrm{Cov}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}$, $\mathrm{Cov}\{\mathbf{x}_n, \mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}$
  - Perform the Kalman filter measurement update:

$$
\begin{aligned}
\mathbf{K}_n &= \mathrm{Cov}\{\mathbf{x}_n, \mathbf{y}_n \mid \mathbf{y}_{1:n-1}\} \, \mathrm{Cov}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}^{-1}, \\
\hat{\mathbf{x}}_{n|n} &= \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n(\mathbf{y}_n - \mathrm{E}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}), \\
\mathbf{P}_{n|n} &= \mathbf{P}_{n|n-1} - \mathbf{K}_n \, \mathrm{Cov}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\} \mathbf{K}_n^{\mathsf{T}}.
\end{aligned}
$$

# Example: Object Tracking (1/2)



# Example: Object Tracking (2/2)




Position RMSE: 1.45 m

# Unscented Transform: Choice of Parameters

- The parameter $\lambda$ is actually:

$$\lambda = \alpha^2 (L + \kappa) - L$$

- $\alpha$, $\beta$, and $\kappa$ are tuning parameters
- $\kappa$ is usually set to 0
- $\alpha$ controls the spread of the sigma-points:

$$\sqrt{L + \lambda} = \sqrt{L + \alpha^2 (L + \kappa) - L} = \alpha \sqrt{L}.$$

- Suggestions vary, e.g., $\alpha = 1 \times 10^{-3}$
- $\beta$ only affects the covariance weight, a good starting point is $\beta = 2$

# Summary

- Nonlinear state-space models require approximative solutions
- The extended Kalman filter uses a linearization of the dynamic and measurement models
- The unscented Kalman filter uses a set of deterministic sigma-points (samples) to calculate the means and covariances