Aalto University
School of Engineering

Mechatronics Machine Design (MMD)

MEC-E5001
Lecture 2
On Jan 14, 2020

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6 week spurt, stay active!

2) Laplace transform, Transfer function, Impulse and step responses, Basics dynamic models, Preliminary exam deadline

Learning goals, this lecture, this week

Physics based design of mechatronic machines. Computational methods for machine design

Physical model creation, computation of specification

Other: Preliminary exam deadline, release of project work

Note: strong emphasis on dynamic systems analysis

Learning goals, exercises this week

Laplace transform

Transfer function

Impulse and step responses

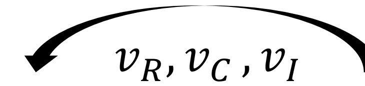
Basics of electric circuits

Time constant

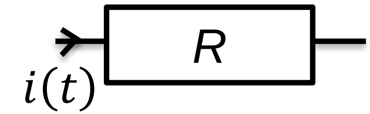
Moment of inertia, gearbox transmission

Basics of electric circuits and mechanical systems

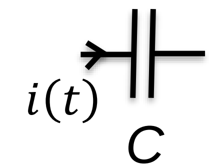
Basics of electric circuits



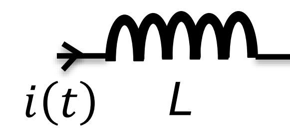
Resistor.......... $v_R(t) = Ri(t)$



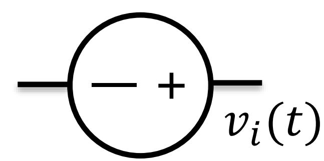
Capacitor.... $v_C(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v_C(t_0)$ $\qquad i(t) = \frac{Cdv_C}{dt}$

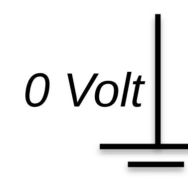


Inductor.......... $v_I(t) = \frac{Ldi}{dt}$



Voltage source, ground 0 V...

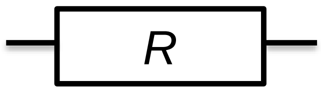




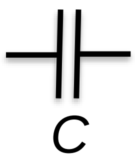
Impedance

Impedance (*Z*) extends the concept of resistance to AC circuits, and possesses both magnitude and phase

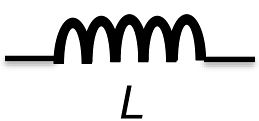
s: Laplace variable, coming later in this slide set



$$Z_R = R$$

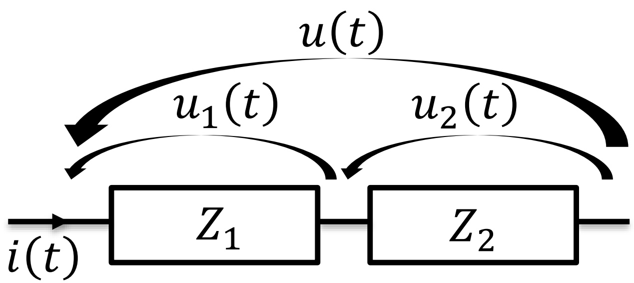


$$Z_C = \frac{1}{\mathrm{j}\omega C} = \frac{1}{sC}, \mathrm{j} = \sqrt{-1}$$



$$Z_L = \mathrm{j}\omega L = sL$$

Impedances in series, division of voltages



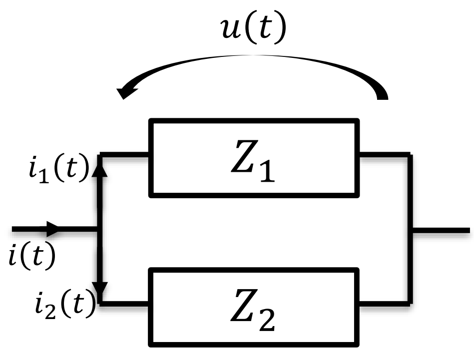
$$Z_{total} = Z_1 + Z_2$$

$$i(t) = \frac{u(t)}{Z_{total}}$$

$$u_1(t) = Z_1 i(t) = Z_1 \frac{u(t)}{Z_{total}}$$

$$= \frac{Z_1}{Z_1 + Z_2} u(t)$$

Impedances in parallel, division of currents



$$Z_{total} = \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)^{-1} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

$$u(t) = Z_{total} i(t)$$

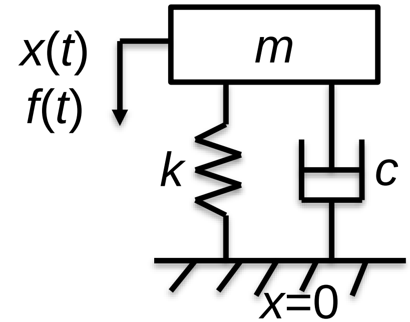
$$i(t) = i_1(t) + i_2(t)$$

$$i_2(t) = \frac{u(t)}{Z_2} = \frac{Z_{total} i(t)}{Z_2} = \frac{\frac{Z_1 Z_2}{Z_1 + Z_2} i(t)}{Z_2}$$

$$i_2(t) = \frac{Z_1}{Z_1 + Z_2} i(t)$$

Basics of mechanical systems: force – displacement, or torque – torsion angle

Spring $f_S(t) = kx(t), T_S(t) = k_T\theta(t)$



Damper $f_D(t) = c\dot{x}(t), T_D(t) = c_T\dot{\theta}(t)$

Mass, inertia $f_M(t) = m\ddot{x}(t), T_I(t) = J\ddot{\theta}(t)$

Force *f(t)*, displacement *x(t)*, ground *x*=0

Discussion

Where can you find electrical and mechanical circuits & systems?

Discuss with your pair and share after 1 min

Analysis of dynamic systems

Laplace transform, background

Definition:

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^{\infty} f(t)e^{-st}dt$$

Properties:

$$\mathcal{L}\{cf(t)\}(s) = cF(s)$$

$$\mathcal{L}\{C_1 f_1(t) + C_2 f_2(t)\}(s) = C_1 F_1(s) + C_2 F_2(s)$$

$$\mathcal{L}\{f_1(t) \cdot f_2(t)\}(s) \neq F_1(s) \cdot F_2(s)$$

$$\mathcal{L}\{f_1(t) * f_2(t)\}(s) = F_1(s) \cdot F_2(s)$$

convolution

Laplace transform, background

Use transformation table for reference (.pdf can be found at MyCourses), remember:

- **Derivative → multiply by *s***
- **Integral → divide by *s***
- **Matlab is good for checking**

$$\mathcal{L}\{f'(t)\}(s) = sF(s)$$

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{1}{s}F(s)$$

Example:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$$
$$\Rightarrow ms^2X(s) + csX(s) + kX(s) = F(s)$$

Laplace transform, usage in engineering

A convenient way to solve differential equations:

$$u'(t) = y(t)$$
$$sU(s) = Y(s)$$

Frequency domain analysis

$$s \to \mathrm{j}\omega,$$
$$\mathrm{j} = \sqrt{-1}$$

Explain physical model and how it behaves

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$$

Laplace transform, example 1

Transform: $f(t) = 4e^{-5t}$

$$\boldsymbol{M5}: \mathcal{L}\{e^{-at}\}(s) = \frac{1}{s+a}$$

$$\Rightarrow \mathcal{L}\{4e^{-5t}\}(s) = 4\mathcal{L}\{e^{-5t}\}(s) = \frac{4}{s+5}$$

Time constant: $\tau = \frac{1}{a}$

Inverse Laplace transform, example 2

Find time-domain counterpart: $F(s) = \frac{1}{s^2 + 6s + 8}$

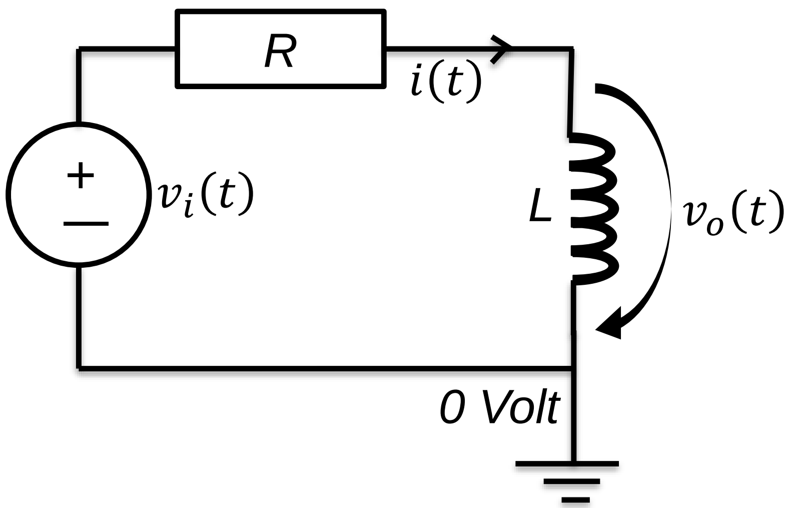
$$\boldsymbol{M9}: \mathcal{L}^{-\mathbf{1}}\left\{\frac{1}{(s+a)(s+b)}\right\}(t) = \frac{1}{a-b}\left(e^{-bt} - e^{-at}\right)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 6s + 8}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s+4)}\right\}(t)$$

$$= \frac{1}{2-4}(e^{-4t} - e^{-2t}) = -\frac{1}{2}(e^{-4t} - e^{-2t})$$

Transfer function example, with Laplace transforms (1/2)

Analyse an electric circuit with a voltage source, resistance, and inductance. Derive the transfer functions $\frac{I(s)}{V_i(s)}, \frac{V_o(s)}{V_i(s)}$



$$v_i(t) = Ri(t) + \frac{Ldi}{dt}$$

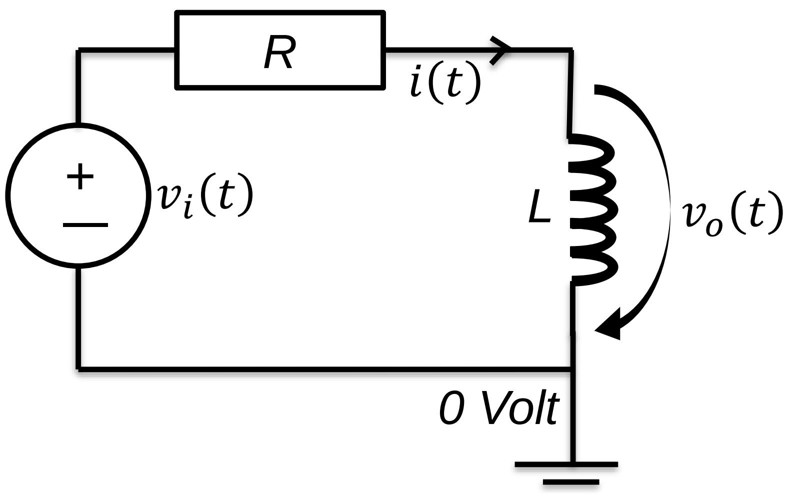
$$V_i(s) = RI(s) + sLI(s) = (sL + R)I(s)$$
$$\frac{I(s)}{V_i(s)} = \frac{1}{sL + R} = \frac{1}{L}\frac{1}{s + \frac{R}{L}}$$

Time constant: $\tau = \frac{L}{R}$

Transfer function example, with Laplace transforms (2/2)

Transfer function $\frac{V_o(s)}{V_i(s)}$

What can you say about transfer functions?



$$v_o(t) = \frac{Ldi}{dt}$$

$$V_o(s) = sLI(s)$$

$$\frac{V_o(s)}{I(s)}\frac{I(s)}{V_i(s)} = sL \cdot \frac{1}{L}\frac{1}{s+\frac{R}{L}}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{s}{s+\frac{R}{L}}$$

Impulse response and back to time domain with inverse Laplace transforms (1/2)

A voltage impulse occurs at t=1 s: $v_i(t) = \delta(t-a), a = 1$

$$\boldsymbol{T4}: \mathcal{L}\left\{\begin{cases} 0 & ,t \le a \\ f(t-a) & ,t > a \end{cases}\right\}(s) = e^{-as}F(s)$$

$$\boldsymbol{M1}: \mathcal{L}\{\delta(t)\}(s) = 1$$

$$(\boldsymbol{M1}\ \&\ \boldsymbol{T4}) \Rightarrow V_i(s) = \mathcal{L}\{\delta(t-a)\}(s) = e^{-as}\cdot 1$$

$$\frac{I(s)}{V_i(s)} = \frac{1}{L}\frac{1}{s+\frac{R}{L}} \Rightarrow I(s) = \frac{1}{L}\cdot\frac{e^{-as}}{s+\frac{R}{L}}$$

$$\boldsymbol{M5}: L^{-1}\left\{\frac{1}{s+a}\right\}(t) = e^{-at}$$

$$(\boldsymbol{M5}\ \&\ \boldsymbol{T4}) \Rightarrow i(t) = \mathcal{L}^{-1}\left\{\frac{1}{L}\cdot\frac{e^{-as}}{s+\frac{R}{L}}\right\}(t) = \frac{1}{L}\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s+\frac{R}{L}}\right\}(t) = \begin{cases} 0 & ,t \le a \\ \frac{1}{L}e^{-\frac{R}{L}(t-a)} & ,t > a \end{cases}$$

Impulse response and back to time domain with inverse Laplace transforms (2/2)

A voltage impulse occurs at t=1 s: $v_i(t) = \delta(t-a), a = 1$

$$V_i(s) = e^{-as} \cdot 1$$

$$\frac{V_o(s)}{V_i(s)} = \frac{s}{s + \frac{R}{L}}$$

$$V_o(s) = s\frac{e^{-as}}{s + \frac{R}{L}}$$

$$\boldsymbol{M15}\text{: } L^{-1}\left\{\frac{s+a}{s+b}\right\}(t) = \delta(t) + (a-b)e^{-bt}$$

$$(\boldsymbol{M15}\ \&\ \boldsymbol{T4}) \Rightarrow v_o(t) = \mathcal{L}^{-1}\left\{e^{-as}\frac{s}{s+\frac{R}{L}}\right\}(t)$$

$$= \begin{cases} 0 & , t \leq a \\ \delta(t-a) - \frac{R}{L}e^{-\frac{R}{L}(t-a)} & , t > a \end{cases}$$

Impulse response $I(s)/V_i(s)$ – Matlab check

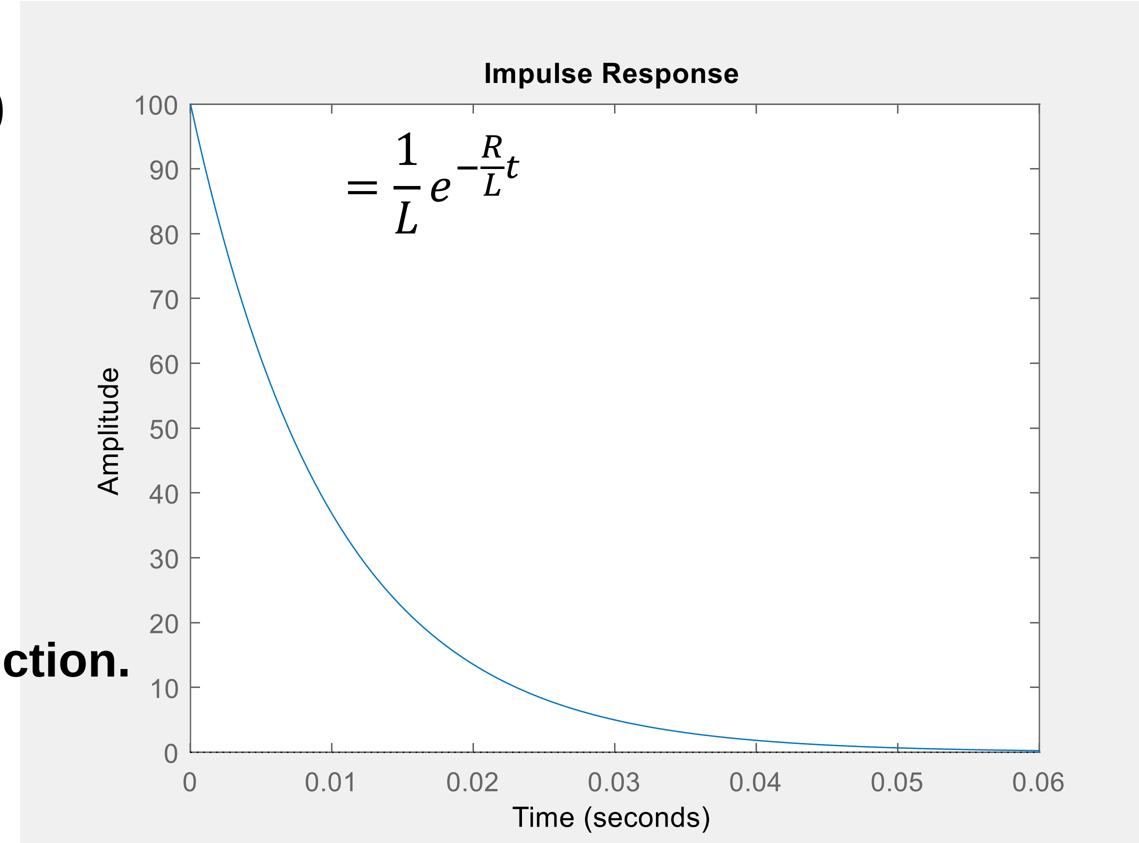
```
>> sys_I=tf([1/0.01],[1 1/0.01])

sys_I =

   100
  -------
  s + 100

Continuous-time transfer function.

>> impulse(sys_I)
```



Impulse response $V_o(s)/V_i(s)$ – Matlab check

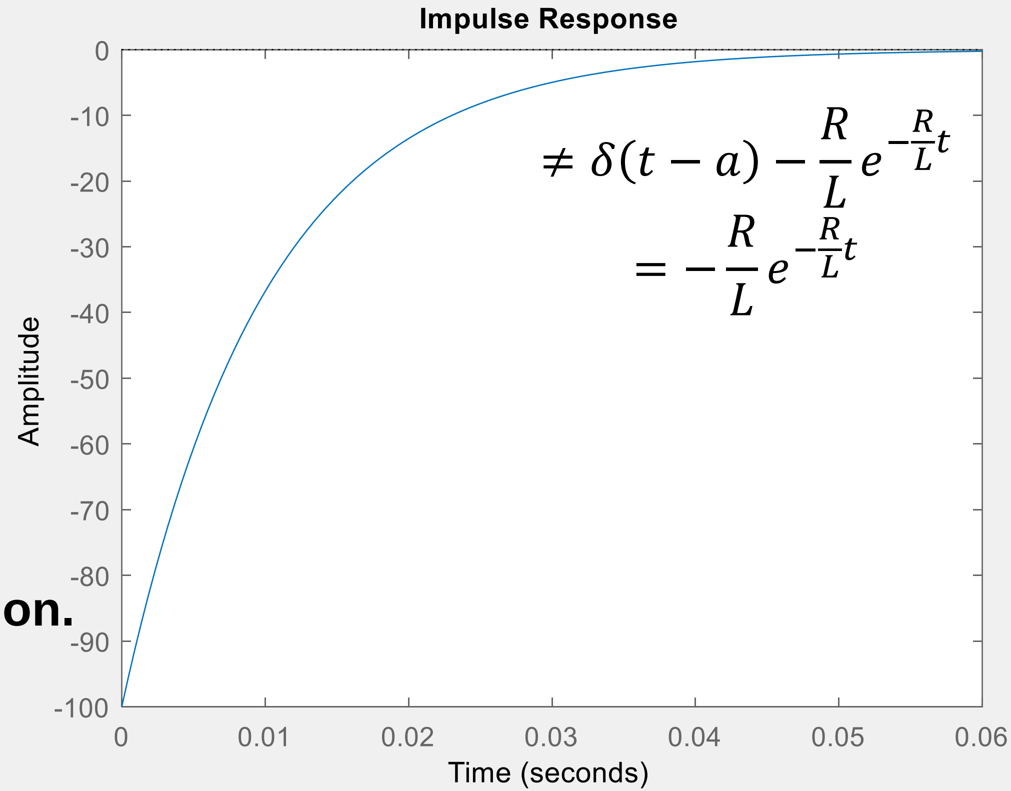
```
>> sys_V=tf([1 0],[1 1/0.01])

sys_V =

    s
  -------
  s + 100

Continuous-time transfer function.

>> impulse(sys_V)
```



$$\neq \delta(t-a) - \frac{R}{L}e^{-\frac{R}{L}t} = -\frac{R}{L}e^{-\frac{R}{L}t}$$

Note: computing V_o/V_i is somewhat code dependent! The plot above represents Ldi/dt

Step response and back to time domain with inverse Laplace transforms (1/2)

A voltage step occurs at t=1 s: $v_i(t) = h(t-a), a = 1$

$$\boldsymbol{M2}: L\{1\}(s) = \mathcal{L}\{h(t)\}(s) = \frac{1}{s}$$

$$(\boldsymbol{M2} \,\&\, \boldsymbol{T4}) \Rightarrow V_i(s) = \mathcal{L}\{h(t-a)\}(s)$$

$$= e^{-as} \cdot \frac{1}{s}$$

$$\frac{I(s)}{V_i(s)} = \frac{1}{L}\frac{1}{s+\frac{R}{L}} \Rightarrow I(s) = \frac{1}{s} \cdot \frac{1}{L} \cdot \frac{e^{-as}}{s+\frac{R}{L}}$$

$$\boldsymbol{M8}: \mathcal{L}^{-1}\left\{\frac{1}{s(s+a)}\right\}(t) = \frac{1}{a}(1-e^{-at})$$

$$(\boldsymbol{M8} \,\&\, \boldsymbol{T4}) \Rightarrow i(t) = \frac{1}{L}L^{-1}\left\{\frac{e^{-as}}{s\left(s+\frac{R}{L}\right)}\right\}$$

$$= \begin{cases} 0 & ,t \leq a \\ \frac{1}{L} \cdot \frac{1}{\frac{R}{L}}\left(1 - e^{-\frac{R}{L}(t-a)}\right) & ,t > a \end{cases}$$

$$= \begin{cases} 0 & ,t \leq a \\ \frac{1}{R}\left(1 - e^{-\frac{R}{L}(t-a)}\right) & ,t > a \end{cases}$$

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What is a fundamental difference between step responses I/V_i and V_o/V_i?

Step response and back to time domain with inverse Laplace transforms (2/2)

A voltage step occurs at, t=1 s: $v_i(t) = h(t-a), a = 1$

$$V_i(s) = e^{-as} \cdot \frac{1}{s}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{s}{s + \frac{R}{L}}$$

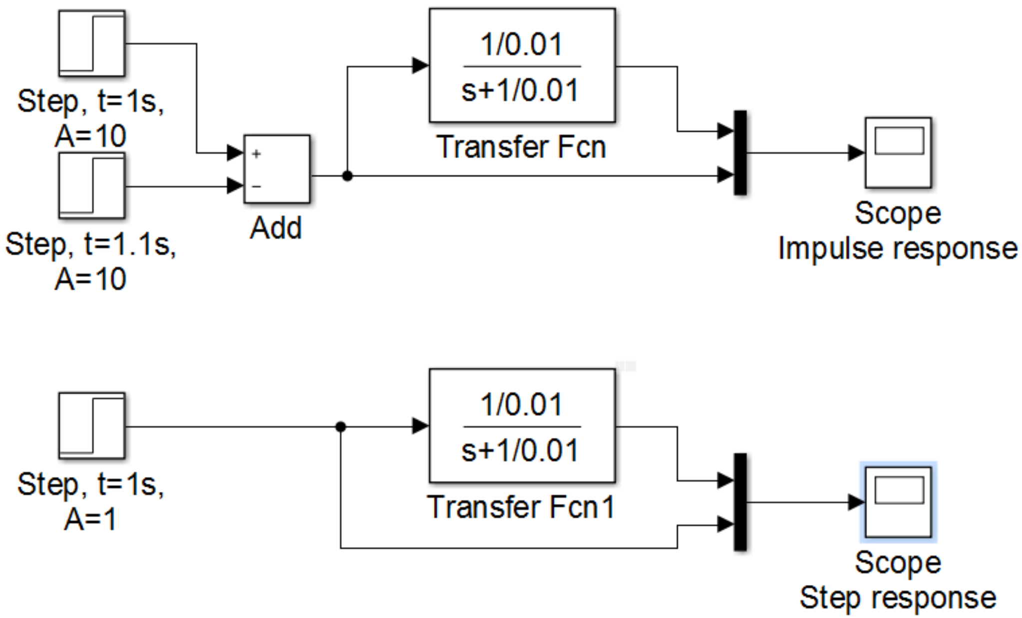
$$\Rightarrow V_o(s) = s \frac{e^{-as}}{s + \frac{R}{L}} \cdot \frac{1}{s} = \frac{e^{-as}}{s + \frac{R}{L}}$$

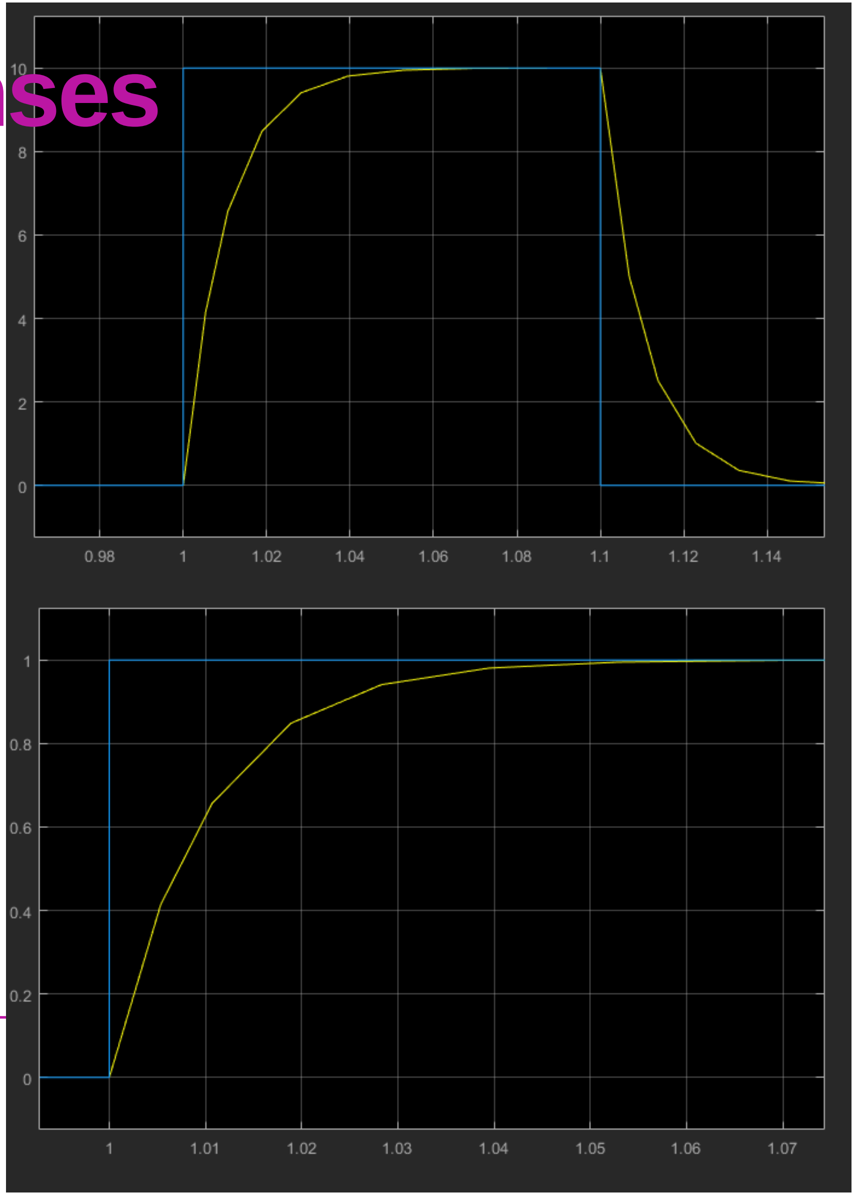
$$(\boldsymbol{M5} \;\&\; \boldsymbol{T4}) \Rightarrow v_o(t) = \mathcal{L}^{-1}\left\{\frac{e^{-as}}{s + \frac{R}{L}}\right\}(t)$$

$$= \mathcal{L}^{-1}\left\{\frac{e^{-as}}{s + \frac{R}{L}}\right\}(t)$$

$$= \begin{cases} 0 & , t \le a \\ e^{-\frac{R}{L}(t-a)} & , t > a \end{cases}$$

Impulse & step responses
Simulink check



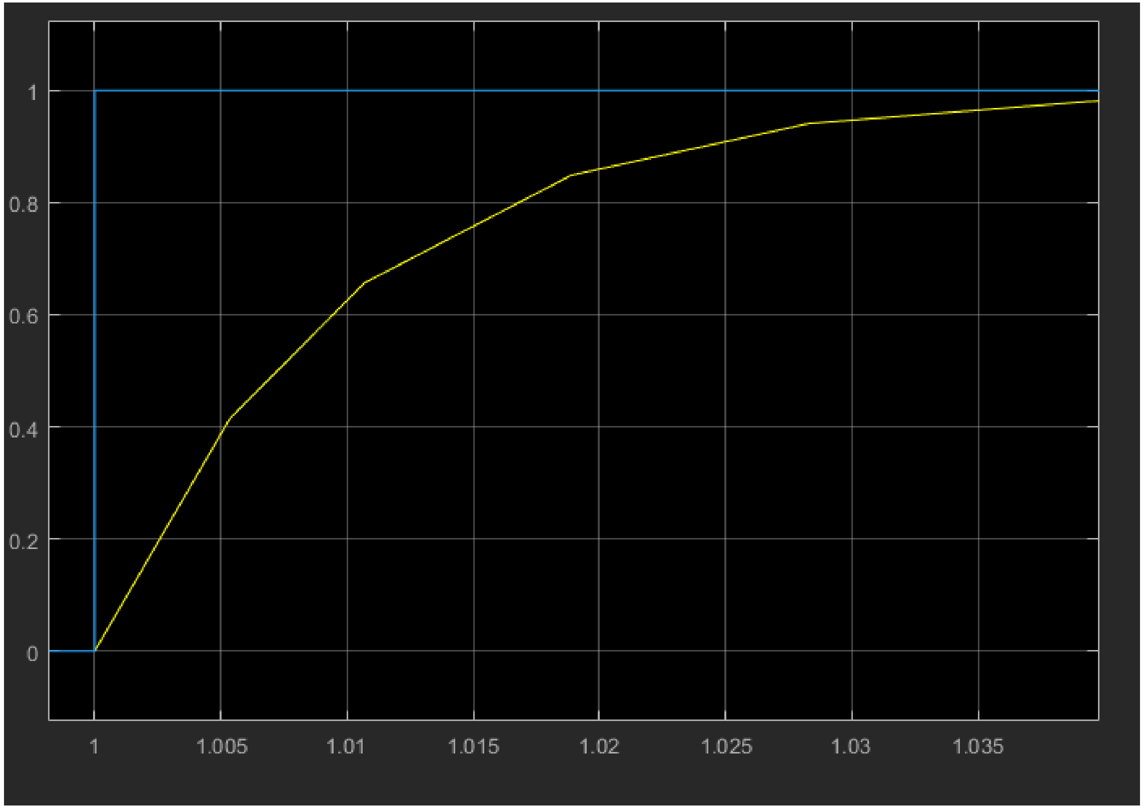


Time constant (and zoom of the previous step response)

Exponential decay
Asymptotic growth

$$A(t) = A_0[1 - e^{-\frac{t}{\tau}}]$$

$$A(t = 0) = 0$$
$$A(t = \tau) = A_0[1 - e^{-1}] \sim 0.63A_0$$
$$A(t \to \infty) \to A_0$$



Time constant in pic?

Group work (and lecture quiz)

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Group work & lecture quiz 2

Discuss with your pair. Write down your answers and use them to answer lecture quiz today.

1. Derive the transfer function *X*(*s*)/*F*(*s*) by Laplace transforming the following equation of motion (1 point): $m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$

2. Analyse the transfer function *X*(*s*)/*F*(*s*) behaviour at: a) low frequencies (0 rad/s), and b) high frequencies (→∞ rad/s) (1 point).

3. Analyse and explain how the moment of inertia *J* is "seen" over a reduction gear (gear ratio *i*) at the input side. Derive the *i*^2 relationship. Use the variables given in the picture (1 point).