Extra material, group voting

#1 Voting

a)

Table 3. Borda count for alternatives.

	Group 1				Group 2					Combined Group
Scores	$\succ_1$	$\succ_2$	$\succ_3$	Sum	$\succ_4$	$\succ_5$	$\succ_6$	$\succ_7$	Sum	Sum
A	2	0	1	3	2	2	0	1	5	8
B	1	2	0	3	1	1	2	2	6	9
C	0	1	2	3	0	0	1	0	1	4

Group 1: A $\sim_g$ B $\sim_g$ C (3=3=3)

Group 2: B $\succ_g$ A $\succ_g$ C (6>5>1)

Combined Group: B $\succ_g$ A $\succ_g$ C (9>8>4).

b)

In plurality voting, each DM has one vote that he or she casts to his or her most favourite alternative. The alternative that gets the most votes wins. In lecture slides, also the following run-off technique of plurality voting is presented: If no alternative gets over 50 % of the votes, the one with least votes is discarded and a new round of voting is performed. This routine is repeated until the winner of a given round gets over 50 % of the votes.

The plurality voting result can now be determined directly from the preference table presented in the exercise paper.

In group 1 all alternatives get 1 vote. Thus, no alternative can be selected or discarded.

In group 2, alternatives A and B receive 2 votes. C is discarded. On the second round both A and B still get two votes.

In the combined group, A and B get three votes, C one vote. C is discarded. On the second round, DM3, who voted for C, votes for A. Thus A is selected with four votes.

The plurality vote results can also be analyzed with the following table.

Table 4. Plurality vote.

Preference relations	Group 1				Group 2					Combined Group
	$\succ_1$	$\succ_2$	$\succ_3$	Group	$\succ_4$	$\succ_5$	$\succ_6$	$\succ_7$	Group	Group
A $\succ$ B	X		X	X	X	X			~	X
B $\succ$ C	X	X		X	X	X	X	X	X	X
C $\succ$ A		X	X	X			X			

If we take a look at the preference relations, 2 DMs in group 1 think that A $\succ$ B, 2 think that B $\succ$ C and 2 think that C $\succ$ A. As a group, they think that A$\succ_g$ B, B$\succ_g$ C, C $\succ_g$ A. Thus, none of the alternatives can be said to be better than another from the group's point of view.

For group 2: A $\sim_g$ B, B $\succ_g$ C, A $\succ_g$ C. Thus, C would be discarded after the first round, but no preference between A and B could be determined.

For the Combined Group: A $\succ_g$ B, B $\succ_g$ C, A $\succ_g$ C. Thus, C would be discarded after the first round, and A would win B on the second round.

Note how different methods yield different results. For example, for the Combined Group, when Borda count is used, alternative B is the winner, whereas when Plurality voting is used, A is the winner. Also note that there is an intransitive cycle in the preferences of Group 1, i.e. plurality voting violates the transitivity requirement of the social choice function.

#2 Aggregated group value

a) and b)

Let

$$V^A(x) = \sum_{i=1}^{2} w_i^A v_i^A(x_i) \tag{1}$$

be the normalized additive value function of decision maker *A*. The preference statements yield the equations

$$V^A(x_1 - c^A, x_2) - V^A(x_1, x_2) = V^A(x_1, x_2 + 1) - V^A(x_1, x_2),\ A = W, H, \tag{2}$$

where *W* and *H* refer to the wife and the husband, respectively. Substituting (1) into (2) gives now

$$w_1^A v_1^A(x_1 - c^A) - w_1^A v_1^A(x_1) = w_2^A v_2^A(x_2 + 1) - w_2^A v_2^A(x_2),\ A = W, H. \tag{3}$$

The attribute specific value functions are linear, i.e.,

$$v_i^A(x_i) = \frac{x_i - x_i^0}{x_i^* - x_i^0}, i = 1{,}2, \tag{4}$$

where x_i^*, x_i^0 are the best and worst attribute levels, respectively. Substituting (4) into (3) and manipulating the equation a bit yields

$$\frac{w_1^A}{w_2^A} = \frac{1/(x_2^* - x_2^0)}{-c^A/(x_1^* - x_1^0)} = \frac{(x_1^* - x_1^0)}{-c^A(x_2^* - x_2^0)}. \tag{5}$$

Using (5) and the normalization condition $w_1^A + w_2^A = 1$, one can solve the weights w_1^A, w_2^A. The result is

$$w_1^A = \frac{x_1^* - x_1^0}{x_1^* - x_1^0 - c^A(x_2^* - x_2^0)}, w_2^A = \frac{-c^A(x_2^* - x_2^0)}{x_1^* - x_1^0 - c^A(x_2^* - x_2^0)}. \tag{6}$$

Substituting (4), (6), as well as $c^W = 5, c^H = 1$ into (1) now yields the normalized additive value functions for the wife and the husband.

It turns out that the wife prefers Hotel 1 whereas the husband prefers Hotel 2; see the Excel file.

c) For the group $V^G(x) = g_1 V^W(x) + g_2 V^H(x)$, where $g_1 = g_2 = 0.5$. The couple as a group prefers Hotel 1; see the Excel file.

d) See the Excel file for a numerical solution.

Repeat the computations with the new value of $x_2^* = 40$. The wife still prefers Hotel 1 and the husband Hotel 2. With both of the DMs, the weight of the attribute *Quality* does increase as its range increases. But at the same time, with each of the DMs, the new overall values are only positive affine transformations of the original overall values (like we have learned to expect during this course).

However, with $x_2^* = 40$, the group prefers Hotel 2 instead of Hotel 1 that was the original group favorite. Why does the change of the scale in an attribute now lead to a change in the most preferred alternative? This is related to the group weights staying intact; $g_1 = g_2 = 0.5$. The weight g_i represent the increase in the overall value experienced by the group when things move from worst to best from the point of view of the group member i, relative to an increase in the overall value experienced by the group when things move from worst to best from the point of view of some other group member. The problem here is that we do not have any preference statements that would describe these relations for different attribute scales. Thus, with the updated scale of *Quality*, the initial weights $g_1 = g_2 = 0.5$ are not necessarily valid anymore. Furthermore, it would be difficult to formulate preference statements that would allow us to define the group weights sensibly. This jeopardizes the use of the weights g_1 and g_2.