The Navier-Stokes Equation and 1D Pipe Flow
Simulation of Shocks in a Closed Shock Tube

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Shock Waves in Experimental Fluid Dynamics
The Picture Below is a Schlieren Photograph of AK-47 Firing a Round
Shock Waves in 3D Computational Fluid Dynamics
Supersonic Jet in a Crossflow (Courtesy of Kawai et al.)
Non-Linear vs Linear Partial Differential Equations

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- **Example 1**: 1D flow of compressible gas in an exhaust pipe.
- **Example 2**: 3D turbulent mixing and combustion in a combustion engine.
- In this lecture we link the CD-equation to the compressible Navier-Stokes equation.
- Finally, the 1D Euler equation is presented and we discuss its numerical solution involving shock-waves in exhaust pipes of combustion engines.
The General Form of Any Transport Phenomenon
Also the Navier-Stokes Equation is of This Form

The Convection-Diffusion Equation for the Quantity $\phi$

\[
\frac{\partial \rho \phi}{\partial t} + \nabla \cdot (u \rho \phi) - \nabla \cdot (\rho \nu \nabla \phi) = S_\phi(\phi)
\]

- $\phi$ convects and diffuses in a fluid (e.g. air, water...).
- $\phi$ could represent e.g. concentration of soot particles in cigarette smoke or water temperature in ocean currents etc.
- $\rho$ is the density of the fluid and $u = u(x,y,z,t)$ the velocity field, and $\nu$ the diffusivity (e.g. heat diffusivity) of $\phi$. 
The Full, Compressible Navier-Stokes Equation

Motion of Gases at Sub- and Supersonic Flows

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} &= 0 \text{ (Conservation of Mass)} \\
\frac{\partial \rho u_i}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} &= \frac{\partial}{\partial x_j} (-p \delta_{ij} + \sigma_{ij}) \text{ (Conservation of Momentum)} \\
\frac{\partial \rho e}{\partial t} + \frac{\partial (u_j (\rho e + p))}{\partial x_j} &= \frac{\partial}{\partial x_j} (\sigma_{ij} u_i) + \frac{\partial}{\partial x_j} \left( \lambda \frac{\partial T}{\partial x_j} \right) \text{ (Conservation of Energy)}
\end{align*}
\]
Conservative vs Primitive Variables

**Definition**

The variables $\rho$, $\rho u_i$ and $\rho e$ are termed **conservative** because they are conserved.

Pressure, velocity and temperature ($p$, $u_i$ and $T$) are termed **primitive** variables.

Reconstruction of Primitive Variables from Conservative

\[ u_i = \frac{\rho u_i}{\rho} \]

\[ T = \frac{\rho e - 0.5 \rho (u_1^2 + u_2^2 + u_3^2)}{c_v \rho} \]

\[ p = \rho RT \]
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\[
p = \rho R T
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- $u_i = \rho u_i / \rho$
- $T = (\rho e - 0.5 \rho (u_1^2 + u_2^2 + u_3^2)) / (c_v \rho)$
- $p = \rho RT$
Conservation of Mass
In the Absence of Nuclear Reactions

Conservation of Mass

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0, \ j = 1, 2, 3
\]
Momentum is Conserved

Conservation of Momentum

\[
\frac{\partial \rho u_i}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} = \frac{\partial}{\partial x_j} \left( -p \delta_{ij} + \sigma_{ij} \right), \quad i,j = 1,2,3
\]
Energy is Conserved
But It May Transfer into Heat via Viscous Effects

Conservation of Energy

\[
\frac{\partial \rho e}{\partial t} + \frac{\partial (u_j (\rho e + p))}{\partial x_j} = \frac{\partial}{\partial x} (\sigma_{ij} u_i) + \frac{\partial}{\partial x} \left( \lambda \frac{\partial T}{\partial x} \right), \quad j = 1, 2, 3
\]

Note: The energy \( \rho e \) is the total energy i.e. \( \rho e = \rho (c_v T + 0.5|u|^2) \). Thus all the energy (at an arbitrary point) is expressed as a sum of kinetic and thermodynamic energies.
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Definitions (1)

Equation of State and Internal Energy of Gas

\[ p = \rho RT \quad \text{and} \quad e = c_v T + 1/2 u^2 \]

where
- \( c_v \) = specific heat,
- \( e(x, y, z, t) \) = total energy per unit volume,
- \( u^2 = u(x, y, z, t)^2 \) = velocity magnitude squared and
- \( R \) = molar gas constant.
Definitions (2)

Speed of Sound

\[ a = \sqrt{\gamma RT} \]

where \( a \) = speed of sound, \( R \) = molar gas constant, and \( \gamma = c_p/c_v \) = adiabatic constant.
Mach Number

\[ Ma = \frac{u}{a} \]

where \( u \) = local flow velocity. When \( Ma < 1 \) flow is \textit{subsonic}, when \( Ma = 1 \) flow is \textit{sonic} (e.g. in converging nozzle), and when \( Ma > 1 \) flow is \textit{supersonic}. 
Viscous Stress Tensor for Newtonian Fluids

\[ \sigma_{ij} = \mu \tau_{ij} - \frac{2}{3} \mu \nabla \cdot \mathbf{u}, \]

where \( \tau_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \). The dynamic viscosity \( \mu = \rho \nu \), where \( \nu \) is the kinematic viscosity.
The Euler Equations

The Inviscid NS-Equation ($\mu = 0$)

### The Euler Equations in 1D

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0
\]

\[
\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} = 0
\]

\[
\frac{\partial \rho e}{\partial t} + \frac{\partial (u(\rho e + p))}{\partial x} = 0
\]

\[p = \rho RT \text{ and } e = c_v T + 1/2u^2\]
### The Euler Equations in 1D

- The Euler equations are a very good model for 1D (and 3D!) flow simulations where viscous effects are small which is often the case.
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- The Euler eqs can be analyzed by eigenvalue analysis and it is seen that information can propagate at velocities $\lambda_1 = u - a$, $\lambda_2 = u$ and $\lambda_3 = u + a$. 
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- For example in a fully supersonic flows $\lambda_i > 0$, $i = 1, 2, 3$ so that information can not propagate upstream the flow.
Pressure and T in a 1D Closed Shock Tube Using Lax-Wendroff (1st Order vs 2nd Order) Method
CD=contact discontinuity; SD=shock discontinuity
Pressure Field in 1D Closed Shock Tube vs Time
Temperature Field in 1D Closed Shock Tube vs Time
Numerical Solution of the Euler Equations

The Simplest Approach: the Explicit Euler Time-Integration Method

\[
\frac{\partial \rho}{\partial t} \approx \frac{\rho^{k}_{n+1} - \rho^{k}_{n}}{\Delta t}
\]

\[
\frac{\partial \rho u}{\partial t} \approx \frac{(\rho u)^{k}_{n+1} - (\rho u)^{k}_{n}}{\Delta t}
\]

\[
\frac{\partial \rho e}{\partial t} \approx \frac{(\rho e)^{k}_{n+1} - (\rho e)^{k}_{n}}{\Delta t}
\]

- **Note:** again \(k\) is the space index and \(n\) the time index.
- **Note:** do not confuse the Euler time integration method with Euler equations although ’Euler’ refers to the same scientist.
Numerical Solution of the Euler Equations
Central Differences for the Spatial Derivatives

\[
\frac{\partial \rho u}{\partial x} \approx \frac{(\rho u)^{k+1}_n - (\rho u)^{k-1}_n}{2\Delta x}
\]

\[
\frac{\partial (\rho u^2 + p)}{\partial x} \approx \frac{(\rho u^2 + p)^{k+1}_n - (\rho u^2 + p)^{k-1}_n}{2\Delta x}
\]

\[
\frac{\partial u(\rho e + p)}{\partial x} \approx \frac{(u(\rho e + p))^{k+1}_n - (u(\rho e + p))^{k-1}_n}{2\Delta x}
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The Euler Equations Have to Be Stabilized with Numerical Viscosity

We Add Artificial Viscosity Terms to RHS of the Equations

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} &= \gamma \frac{\partial^2 \rho}{\partial x^2} \\
\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} &= \gamma \frac{\partial^2 \rho u}{\partial x^2} \\
\frac{\partial \rho e}{\partial t} + \frac{\partial (u(\rho e + p))}{\partial x} &= \gamma \frac{\partial^2 \rho e}{\partial x^2} \\
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- The artificial viscosity $\gamma > 0$ and it smoothes sharp shock fronts and allows numerical stability.
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- The practical role of extra terms is low-pass filtering of the high-frequency components (i.e. small wiggles) from numerical solution.

If $\gamma = 0$ a simulation will not run for many timesteps. The small wiggles tend to grow exponentially and blow up the solution.

It is easy to show that 'good' limits for $\gamma$ are (roughly)

$$\Delta x^2 / 10 \pi^2 \Delta t \leq \gamma \leq \Delta x^2 \pi^2 \Delta t$$

where larger $\gamma$ gives smoother solutions ($\Delta x =$ grid spacing and $\Delta t =$ timestep size).
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  ($\Delta x =$grid spacing and $\Delta t =$ timestep size).
The Second Derivative (1D Laplacian) is Discretized as Follows

Example: the Momentum Term

\[
\frac{\partial^2 \rho u}{\partial x^2} \approx \frac{(\rho u)^{k+1}_n - 2(\rho u)^k_n + (\rho u)^{k-1}_n}{\Delta x^2}
\]
Boundary Conditions (BCs)

- Modeling of boundaries is a crucial part of CFD-simulations e.g. wall, inflow, outflow.

To simplify a bit, here we explain one way of modeling a wall. Say, in a 1D pipe flow with closed ends we first set zero derivative (gradient) for $p$ and $T$ by e.g. $p(1)=p(2)$ and $T(1)=T(2)$.

Then, e.g. $\rho(1)=\frac{p(1)}{R*T(1)}$.

For a wall the velocity 'reflects back' so e.g. $U(1)=-U(2)$.

Thus, e.g. $\rho U(1)=\rho*U(1)$ and similarly for $\rho e(1) = \rho(1)*(Cv*T(1) + 0.5*U(1)*U(1))$.

From this kind of BCs a back and forth bouncing pulse results.
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Matlab Implementation

- The folder `/demos2012/shocktube/` shows demo files in which the Explicit Euler method is applied for the Euler equations.
- `SolveEulerEquation1D.m` is the solver which can be executed by command `»SolveEulerEquation1D`.
- The files `PlottingFigure.m`, `SurfaceAnimation.m` and `DrawingSurface.m` show how graphs can be drawn, surface motion be animated etc. Look at the end of these files to see how `.png` pictures can be saved.
An introduction to Navier-Stokes equation was given.

The Euler equations are a simpler case of the NS-eqs ($\mu = 0$), the Euler equations have discontinuities solutions e.g. shocks. We tested how increased viscosity allows numerical capturing of shock waves via 'low pass filtering of wiggles i.e. small wavelengths'. Again, the demonstrated Euler time integration method is not used in practice but it is a copy-paste type effort to extend the method to Runge-Kutta methods which are very robust and much used also in 3D realistic flows.
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The End

Thank you

Questions?