Aalto University
School of Electrical Engineering

ELEC-E8101: Digital and Optimal Control

Lecture 12
Course
Summary

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Today

Let's start with:

- Exam information and practicalities

Then we will go through each lecture topic very briefly.

- Please ask questions related to the topics/lectures.

Exam

Practical information about the exam:

- Exam Wednesday December 7, 2pm-5pm.

- Exam on Dec 7 covers only the second half of the course.
  - All additional exams given afterwards will cover the entire course.

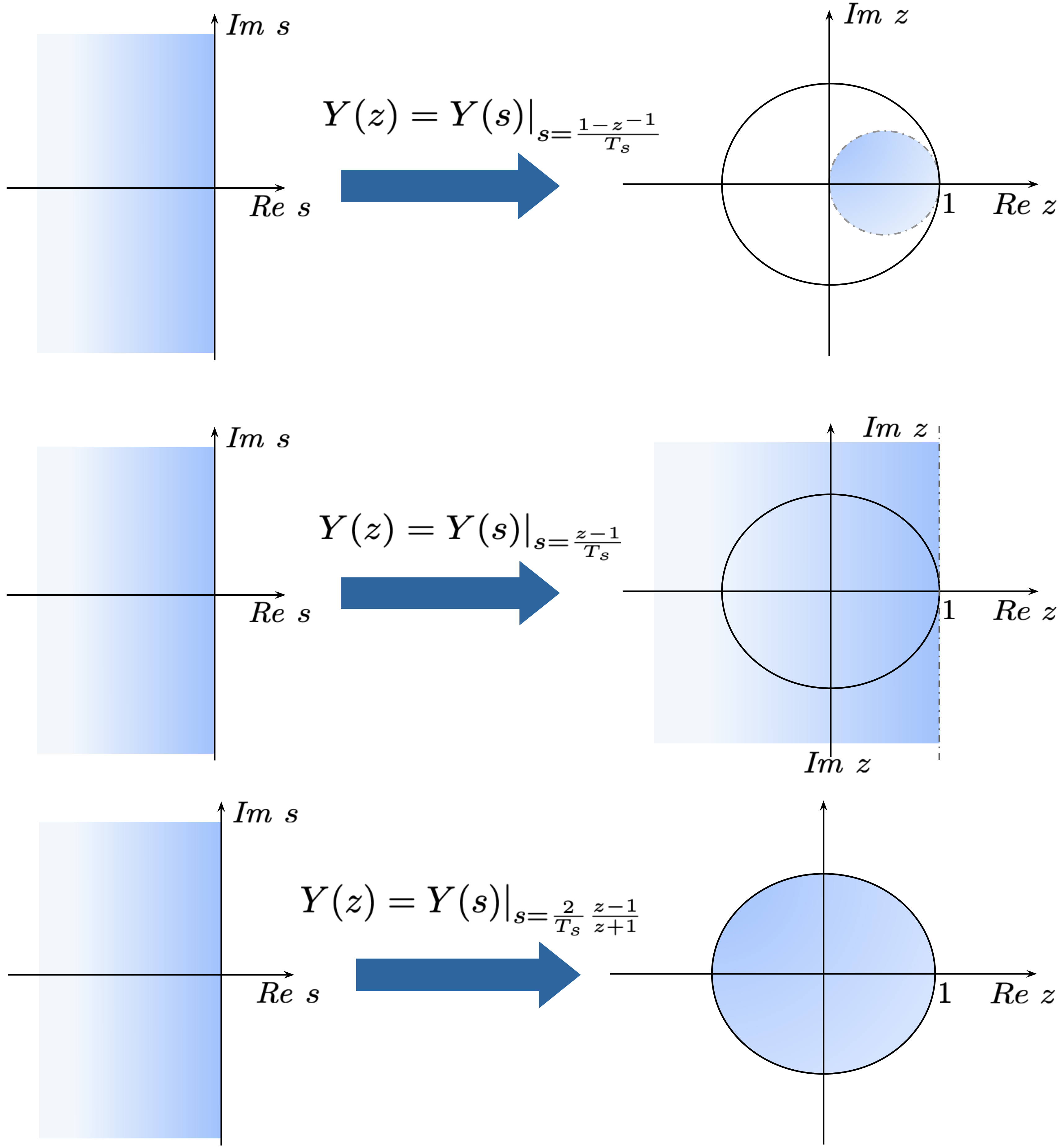
- Exam consists of 3 problems, with possible sub-problems. Each problem is worth 6 points, for a total of 18 points.
- Table of formulas provided in MyCourses is allowed in exam. Print your own copy (a limited number of copies available from course staff at exam). No other materials are allowed.
- A calculator is allowed. Only basic, non-symbolic calculations are allowed with the calculator
  - For example, no symbolic solving of equations, taking Laplace- or z-transformations.

Syllabus

Topics for the second half of the course:

- Lecture 7 Discretization (28.10.)
- Lecture 8 Discrete PID (4.11.)
- Lecture 9 Disturbances (11.11.)
- Lecture 10 Optimal control in state space (18.11., Gökhan)
- Lecture 11 Introduction to stochastic optimal control (25.11., Gökhan)
- Lecture 12 Summary (2.12.)

Lecture 7



Im s
Re s
$Y(z) = Y(s)|_{s=\frac{1-z^{-1}}{T_s}}$
Im z
1
Re z
Im s
Re s
$Y(z) = Y(s)|_{s=\frac{z-1}{T_s}}$
Im z
1
Re z
Im z
Im s
Re s
$Y(z) = Y(s)|_{s=\frac{2}{T_s}\frac{z-1}{z+1}}$
1
Re z

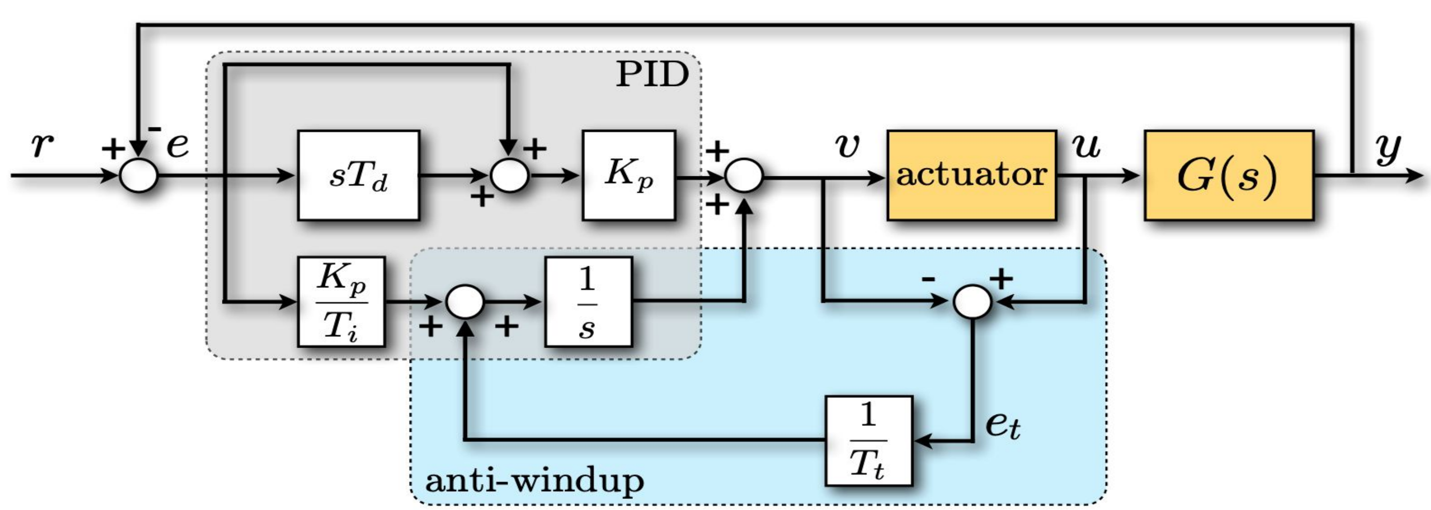
Lecture 8

$$u(kh) = K_p e(kh) + K_i h \sum_{n=-\infty}^{k-1} e(nh) + \frac{K_d}{h} \Delta e(kh)$$

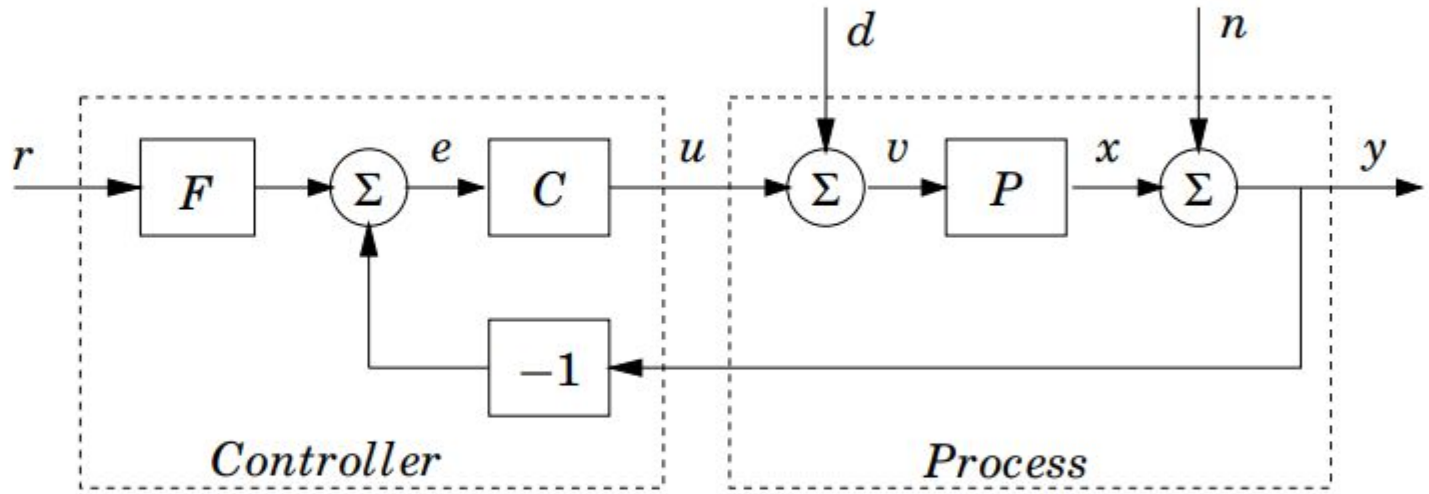
$$U(z) = \underbrace{\left(K_p + \frac{K_i h}{z-1} + \frac{K_d}{h} \frac{z-1}{z} \right)}_{H_{\mathrm{PID}}} E(z)$$

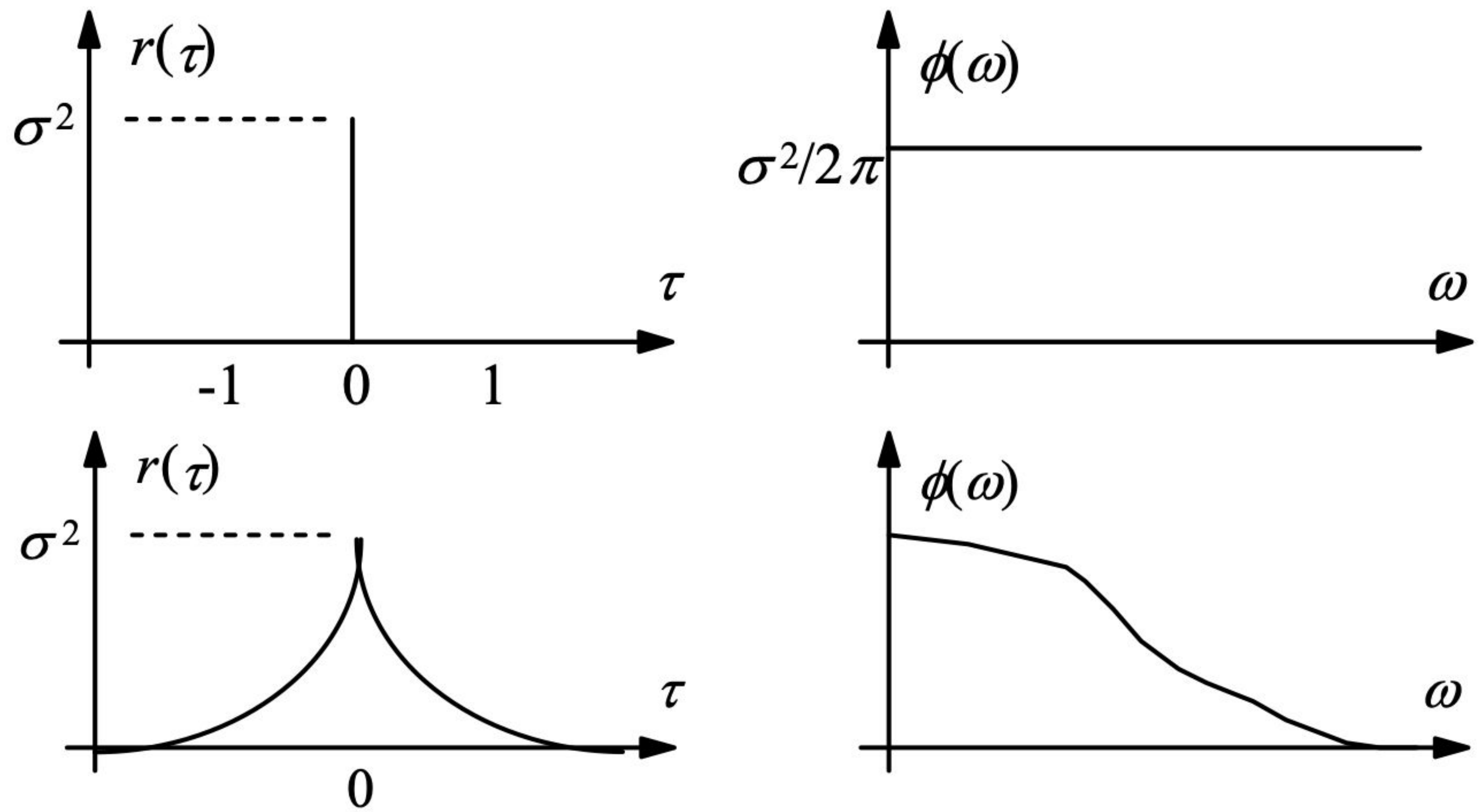
Effects of *increasing* a parameter independently

Parameter	Rise time	Overshoot	Settling time	Steady-state error	Stability
K_p	Decrease	Increase	Small change	Decrease	Degrade
K_i	Decrease	Increase	Increase	Eliminate	Degrade
K_d	Minor change	Decrease	Decrease	No effect in theory	Improve if small



Lecture 9



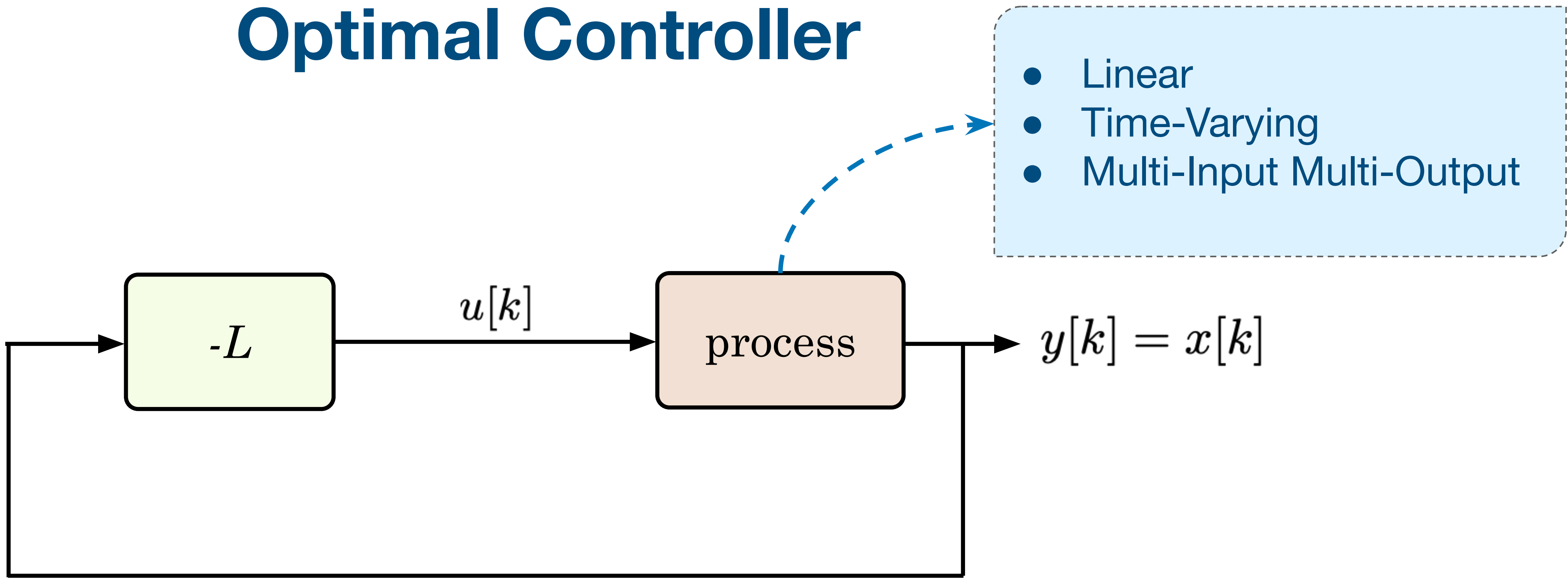


$$\mathbf{m}[k] \triangleq E\{\mathbf{x}[k]\}$$

$$\begin{aligned}\mathrm{var}\{\mathbf{x}[k]\} &\triangleq E\{(\mathbf{x}[k]-\mathbf{m}[k])(\mathbf{x}[k]-\mathbf{m}[k])^T\} \\ &= E\Big\{\big(\mathbf{x}[k]-E\{\mathbf{x}[k]\}\big)\big(\mathbf{x}[k]-E\{\mathbf{x}[k]\}\big)^T\Big\}\end{aligned}$$

Lecture 10

Optimal Controller



- LQR optimal control, with the gain to change over time based on the cost

$$J = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} \{ x_k^T Q x_k + u_k^T R u_k \}$$

which gives:

$$L_k = (\Gamma^T S_{k+1} \Gamma + R)^{-1} \Gamma^T S_{k+1} \Phi$$

$$u_k^* = -L_k x_k$$

$$S_k = (\Phi - \Gamma L_k)^T S_{k+1} (\Phi - \Gamma L_k) + Q + L_k^T R L_k$$

$$J_k^* = \frac{1}{2} x_k^T S_k x_k$$

Lecture 11

System corrupted with process and measurement noise

- The full LQG model assumes **linear dynamics**, **quadratic costs** and **Gaussian noise**. Imperfect observation is the most important point. The model is:

$$\mathbf{x}[k+1] = \Phi\mathbf{x}[k] + \Gamma\mathbf{u}[k] + \mathbf{v}[k],$$
$$\mathbf{y}[k] = C\mathbf{x}[k] + \mathbf{e}[k]$$

where $\mathbf{v}$ and $\mathbf{e}$ are discrete-time Gaussian white noise processes with zero-mean value and

$$\left.\begin{aligned} E\{\mathbf{v}\mathbf{v}^T\} &= R_1 \\ E\{\mathbf{v}\mathbf{e}^T\} &= R_{12} \\ E\{\mathbf{e}\mathbf{e}^T\} &= R_2 \end{aligned}\right\} \Rightarrow \operatorname{cov}\left\{\begin{bmatrix}\mathbf{v}\\ \mathbf{e}\end{bmatrix}\right\} = E\left\{\begin{bmatrix}\mathbf{v}\\ \mathbf{e}\end{bmatrix}\begin{bmatrix}\mathbf{v}\\ \mathbf{e}\end{bmatrix}^T\right\} = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

- The initial state $\mathbf{x}[0]$ is assumed to be Gaussian distributed with

$$E\{\mathbf{x}[0]\} = m_0 \quad \operatorname{cov}\{\mathbf{x}[0]\} = R_0$$

Using standard notation from the literature, we can write $\mathbf{x}[0] \sim \mathcal{N}(m_0, R_0)$

Kalman filter

- Recall the approach using the observer/estimator

$$\hat{\mathbf{x}}[k+1|k] = \Phi\hat{\mathbf{x}}[k|k-1] + \Gamma u[k] + K(y[k] - C\hat{\mathbf{x}}[k|k-1])$$
$$= (\Phi - KC)\hat{\mathbf{x}}[k|k-1] + \Gamma u[k] + Ky[k]$$

Matrix was chosen such that the eigenvalues of Φ_o are at desired places in the complex plane.

- No Bias
- Low-Variance

- The Kalman filter:

$$\hat{\mathbf{x}}[k+1|k] = \Phi\hat{\mathbf{x}}[k|k-1] + \Gamma u[k] + K[k](y[k] - C\hat{\mathbf{x}}[k|k-1])$$

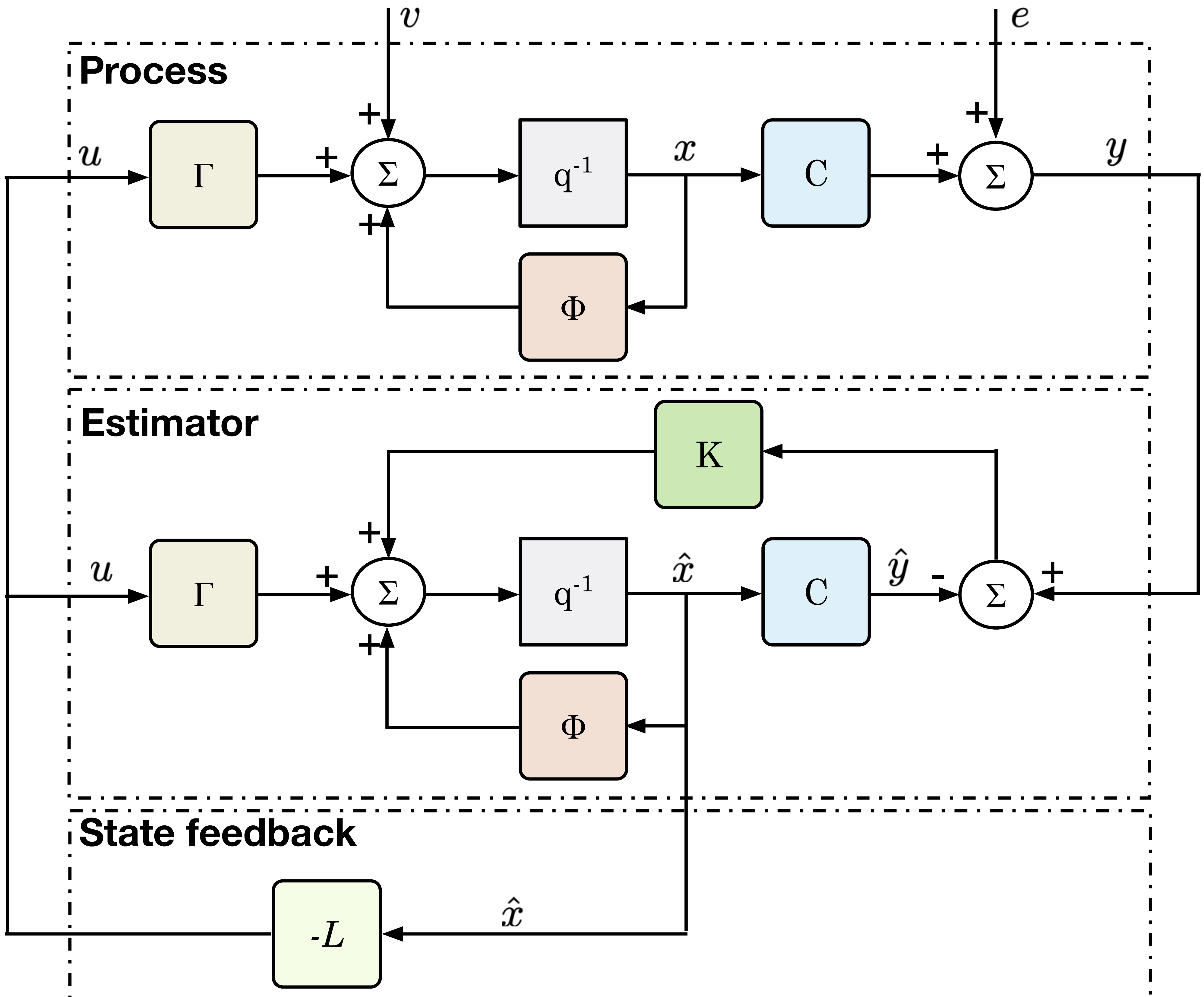
$$K[k] = \left(\Phi P[k]C^T + R_{12}\right)\left(CP[k]C^T + R_2\right)^{-1}$$

where

$$P[k+1] = \Phi P[k]\Phi^T + R_1 - \left(\Phi P[k]C^T + R_{12}\right)\left(CP[k]C^T + R_2\right)^{-1}\left(CP[k]\Phi^T + R_{12}\right)$$

$$P[0] = R_0$$

Structure of LQG control



Kalman Filter

LQR

Thank you!

Please give course feedback! Extra point awarded!